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GEOMETRICAL PROPERTIES OF SUBCLASS OF ANALYTIC FUNCTION WITH ODD DEGREE

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ABSTRACT. The objective of the paper is to study the geometrical properties of the class $\mathcal{B}(\lambda, t)$. For which we have proved that the radius $\frac{\sqrt{3-u_3}}{3}$ is optimal ,(i.e) the number $\frac{\sqrt{3-u_3}}{3}$ cannot be replaced by a larger one. Additionally, the graphs for various values of t and λ are compared in order to study the sharpness of the coefficient bounds.

Key words and phrases: Analytic functions; Locally univalent odd function; Close to convex functions; Radius of convexity.

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1. INTRODUCTION

Let $\mathcal{A}$ be the family of all normalized analytic function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the unit disk $\mathbb{U}$ and $\mathcal{S}$ denote the class of all analytic univalent functions belonging to the class $\mathcal{A}$. Let the $n-th$ section of $f \in \mathcal{A}$ is defined as

$$s_n(z) = z + \sum_{m=2}^{n} a_m z^m. \tag{1.2}$$

If a function f belonging to the class $\mathcal{A}$ is univalent in some neighbourhood of a point $z_0 \in \mathbb{U}$, then it is $locally\ univalent$ at that point, equivalently $f'(z_0) \neq 0$. A function $f \in \mathcal{A}$ is said to be $convex$ with respect to the origin if it maps $\mathbb{U}$ onto a convex domain with respect to the origin and it is denoted by $\mathcal{C}$. Necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{C}$

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ (z \in \mathbb{U}).$$

Definition 1.1. If f is of the form $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}$ and is a locally univalent odd function, then $f \in \mathcal{B}(\lambda, t)$ if it satisfies the condition that

$$\Re\left(\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda) z^2 f''(z) + zf'(z)]}{\lambda z^2[f''(z) - t^2 f''(tz)] + z[f'(z) - tf'(tz)]}\right) > -\frac{1}{2} \tag{1.3}$$

for $0 \leq \lambda \leq 1$ and $|t| \leq 1,\ t \neq 1$

Remark 1.1. (i) For $\lambda = 0$ and $t = 0$, the class $\mathcal{B}(\lambda, t)$ reduces to $\mathcal{L}$ (See [1]) which satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \tag{1.4}$$

(ii) For $t = -\frac{1}{2}$, the class $\mathcal{B}(\lambda, t)$ reduces to $\mathcal{B}(\lambda, -\frac{1}{2})$ which satisfies the condition

$$\Re\left(\frac{6[\lambda z^3 f'''(z) + (1+2\lambda) z^2 f''(z) + zf'(z)]}{\lambda z^2[4f''(z) - f''(-\frac{z}{2})] + 2z[2f'(z) + f'(-\frac{z}{2})]}\right) > -\frac{1}{2}$$

(iii) For $t = \frac{1}{2}$, the class $\mathcal{B}(\lambda, t)$ reduces to $\mathcal{B}(\lambda, \frac{1}{2})$ which satisfies the condition

$$\Re\left(\frac{2[\lambda z^3 f'''(z) + (1+2\lambda) z^2 f''(z) + zf'(z)]}{\lambda z^2[4f''(z) - f''(\frac{z}{2})] + 2z[2f'(z) - f'(\frac{z}{2})]}\right) > -\frac{1}{2}$$

To investigate the problems in the theory of odd univalent functions, such as inverse functions, coefficient bounds, etc. is intriguing. In fact, a Cauchy-Schwarz inequality application demonstrates that the Robertson conjecture (1936) , which states that

$$1 + |c_3|^2 + |c_5|^2 + ... + |c_{2n-1}|^2 \leq n, n \geq 2,$$

for each odd function $f(z) = z + c_3 z^3 + c_5 z^5 + ...$ of $\mathcal{S}$ implies the well-known Bieberbach conjecture [see [2]]. In 1928, Szegö initiated the problem of finding the radius of univalence of sections of $f \in \mathcal{S}$. As a result of the Szegö theorem [3], every section $s_n(z)$ of a function $f \in \mathcal{S}$ is univalent in the disk $|z| < 1/4$ (See [14]) . The radius $\frac{1}{4}$ is best possible and reader can use second partial sum of köebe function to verify the radius. Robertson determined the radius of starlikeness of the section $s_n(z)$ of $f \in \mathcal{S}^*$ in [8] [also refer [12]] as shown below:

Theorem 1.1. [8] *If $f \in \mathcal{S}$ is either starlike, convex, typically-real, or convex in the direction of imaginary axis, then there is an N such that, for $n \geq N$, the partial sum $s_n(z)$ has the same property in $\mathbb{U}_r := z \in \mathbb{U} : |z| < r$, where $r \geq 1 - 3(logn)/n$.*

Ruscheweyh [11] proved a more profound result by showing that the partial sums of functions f are starlike in $\mathbb{U}_{1/4}$ for not only functions belonging to $\mathcal{S}$, but also functions belonging to the closed convex hull of $\mathcal{S}$. Moreover, Robertson [8] demonstrated that the constant 3 cannot be replaced by a smaller constant in the Koebe function $k(z)$. As a result of a known theorem by Ruscheweyh and Sheil-Small [10] on convolutions, we can automatically show that if f belongs to $\mathcal{C}$, $\mathcal{S}^*$, or $\mathcal{K}$, then its $n-th$ section is respectively convex, starlike, or close-to-convex in the disk $|z| < 1 - 3(logn)/n$, for $n \geq 5$. For different subclasses of $\mathcal{S}$, many authors have solved various problems related to sections (see [4, 8, 9, 13]) and the articles [5, 7]. Ponnusamy et.al [6] considered the subclass $\mathcal{F}$ of the class $\mathcal{K}$, close to convex function which consists of locally univalent function $f \in \mathcal{A}$ which satisfies the condition (1.4). Sarit Agarwal et al. [1]considered functions from $\mathcal{F}$ that have odd degrees. Whereas as an extension we use generalized sakaguchi kind function of odd degree. The ultimate aim of this paper is to find the disk in which every section $s_{2n-1}(z) = z + \sum_{m=2}^{n} a_{2m-1} z_{2k-1}$, of $f \in \mathcal{B}(\lambda, t)$, is convex; that is, s_{2n-1} satisfies

$$\Re\left(\frac{(1-t)[\lambda z^3 s'''_{2n-1}(z) + (1+2\lambda) z^2 s''_{2n-1}(z) + z s'_{2n-1}(z)]}{\lambda z^2 [s''_{2n-1}(z) - t^2 s''_{2n-1}(tz)] + z[s'_{2n-1}(z) - t s'_{2n-1}(tz)]}]\right) > 0$$

2. MAIN RESULT

Theorem 2.1. *Every section of a function in $\mathcal{B}(\lambda, t)$ is convex in the disk $|z| < \frac{\sqrt{3-u_3}}{3}$. The radius $\frac{\sqrt{3-u_3}}{3}$ cannot be replaced by a greater one.*

Remark 2.1. Radius of convexity is shown in the below table for different classes:

Class	Radius ($\lvert z\rvert < r$)
$\mathcal{L}$	$r = \frac{\sqrt{2}}{3}$
$\mathcal{B}(\lambda, \frac{1}{2})$	$r = \frac{\sqrt{1.25}}{3}$
$\mathcal{B}(\lambda, -\frac{1}{2})$	$r = \frac{\sqrt{2.25}}{3}$

3. PRE-REQUISITE

Lemma 3.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{B}(\lambda, t)$, then we arrive at the following results:*

$$(a)\ |a_{2n-1}| \leq \frac{1}{(2n-1)\gamma_n} \left[\frac{\prod_{m=1}^{n-1} 2u_{2m-1} + 2m - 1}{\prod_{m=2}^{n} 2m - 1 - u_{2m-1}} \right] \tag{3.1}$$

for $n \geq 2$, where $\gamma_n = 1 + 2(n-1)\lambda$; $u_n = \frac{1-t^n}{1-t}$.

$$(b) \left| \frac{\lambda(1-t)z^2 f'''(z) + (1+\lambda-t-2\lambda t) z f''(z) + \lambda t^2 z f''(tz) + t[f'(tz) - f'(z)]}{z\lambda[f''(z) - t^2 f''(tz)] + [f'(z) - t f'(tz)]} \right| \prec \frac{3r^2}{1-r^2}$$

for $|z| = r < 1$. Sharpness is studied using geometrical representation.

(c) If $f(z) = s_{2n-1}(z) + \beta_{2n-1}(z)$, where $\beta_{2n-1}(z) = \sum_{m=n+1}^{\infty} a_{2m-1} z^{2m-1}$

then for $|z| = r < 1$ *we have*

$$|z\lambda[\beta''_{2n-1}(z) - t^2\beta''_{2n-1}(tz)] + [\beta'_{2n-1}(z) - t\beta'_{2n-1}(tz)]| \leq \tau(n,r)$$

and

$$|\lambda(1-t)z^2\beta'''_{2n-1}(z)+(1+\lambda-t-2\lambda t)z\beta''_{2n-1}(z)+\lambda t^2 z\beta''_{2n-1}(tz)+t[\beta'_{2n-1}(tz)-\beta'_{2n-1}(z)]| \leq \eta(n,r)$$

where

$$\tau(n,r) = \sum_{m=n+1}^{\infty}(1-t^{2m-1})\left[\frac{\prod_{k=1}^{m-1} 2u_{2k-1}+2k-1}{\prod_{k=2}^{m} 2k-1-u_{2k-1}}\right]r^{2m-2}$$

and

$$\eta(n,r) = \sum_{m=n+1}^{\infty}[(2m-2)(1-t)+t^{2m-1}-t]\left[\frac{\prod_{k=1}^{m-1} 2u_{2k-1}+2k-1}{\prod_{k=2}^{m} 2k-1-u_{2k-1}}\right]r^{2m-2}.$$

All the series above are convergent according to the ratio test.

Proof. (a) Let

$$\frac{3}{2}p(z) - \frac{1}{2} = \frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z^2[f''(z) - t^2 f''(tz)] + z[f'(z) - tf'(tz)]} \tag{3.2}$$

Evidently, $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ is analytic in $\mathbb{U}$ and $\Re\,(p(z)) > 0$ there and so by Carathéodory lemma , we get $|p_n| \leq 2$ for all $n \geq 1$. Substituting the expansion of $f'(z), f''(z), f'''(z)$ and $p(z)$ in (3.2) we obtain

$$\sum_{n=2}^{\infty}(2n-1)(2n-1-u_{2n-1})\gamma_n a_{2n-1}z^{2n-1}$$

$$= \frac{3}{2}\sum_{n=2}^{\infty}\left[\sum_{m=1}^{n-1}(2n-2m-1)\gamma_{n-m}p_{2m-1}u_{2n-2m-1}a_{2n-2m-1}\right]z^{2n-2}$$

$$+\frac{3}{2}\sum_{n=2}^{n-1}\left[\sum_{m=1}^{\infty}(2n-2m-1)\gamma_{n-m}p_{2m}u_{2n-2m-1}a_{2n-2m-1}\right]z^{2n-1}$$

Comparing z^{2n-1} and z^{2n-2} terms we get

$$\sum_{m=1}^{\infty}(2n-2m-1)\gamma_{n-m}p_{2m-1}u_{2n-2m-1}a_{2n-2m-1} = 0$$

$$(2n-1)(2n-1-u_{2n-1})\gamma_n a_{2n-1} = \frac{3}{2}\sum_{m=1}^{n-1}(2n-2m-1)\gamma_{n-m}p_{2m}u_{2n-2m-1}a_{2n-2m-1} \tag{3.3}$$

for all $n \geq 2$.
Therefore ,

$$|a_{2n-1}| \leq \frac{3}{(2n-1)(2n-1-u_{2n-1})\gamma_n}\sum_{m=1}^{n-1}(2m-1)\gamma_m u_{2m-1}|a_{2m-1}|.$$

For $n = 2$, we obtain

$$|a_3| \leq \frac{1}{(3-u_3)\gamma_2}$$

For $n = 3$, we get

$$|a_5| \leq \frac{3}{5(5-u_5)\gamma_3}\Big[1 + 3\gamma_2 u_3|a_3|\Big] \leq \frac{3(2u_3+3)}{5(3-u_3)(5-u_5)\gamma_3}$$

In general we can write ,

$$|a_{2n-1}| \leq \frac{1}{(2n-1)\gamma_n}\left[\frac{\prod_{m=1}^{n-1} 2u_{2m-1}+2m-1}{\prod_{m=2}^{n} 2m-1-u_{2m-1}}\right]$$

where $\gamma_n = 1 + 2(n-1)\lambda$.
For equality , it can easily be seen that

$$f_0(z) = z + \sum_{n=2}^{\infty} \frac{1}{(2n-1)\gamma_n}\left[\frac{\prod_{m=1}^{n-1} 2u_{2m-1}+2m-1}{\prod_{m=2}^{n} 2m-1-u_{2m-1}}\right] z^{2n-1} \tag{3.4}$$

the above equation belongs to $\mathcal{B}(\lambda, t)$.

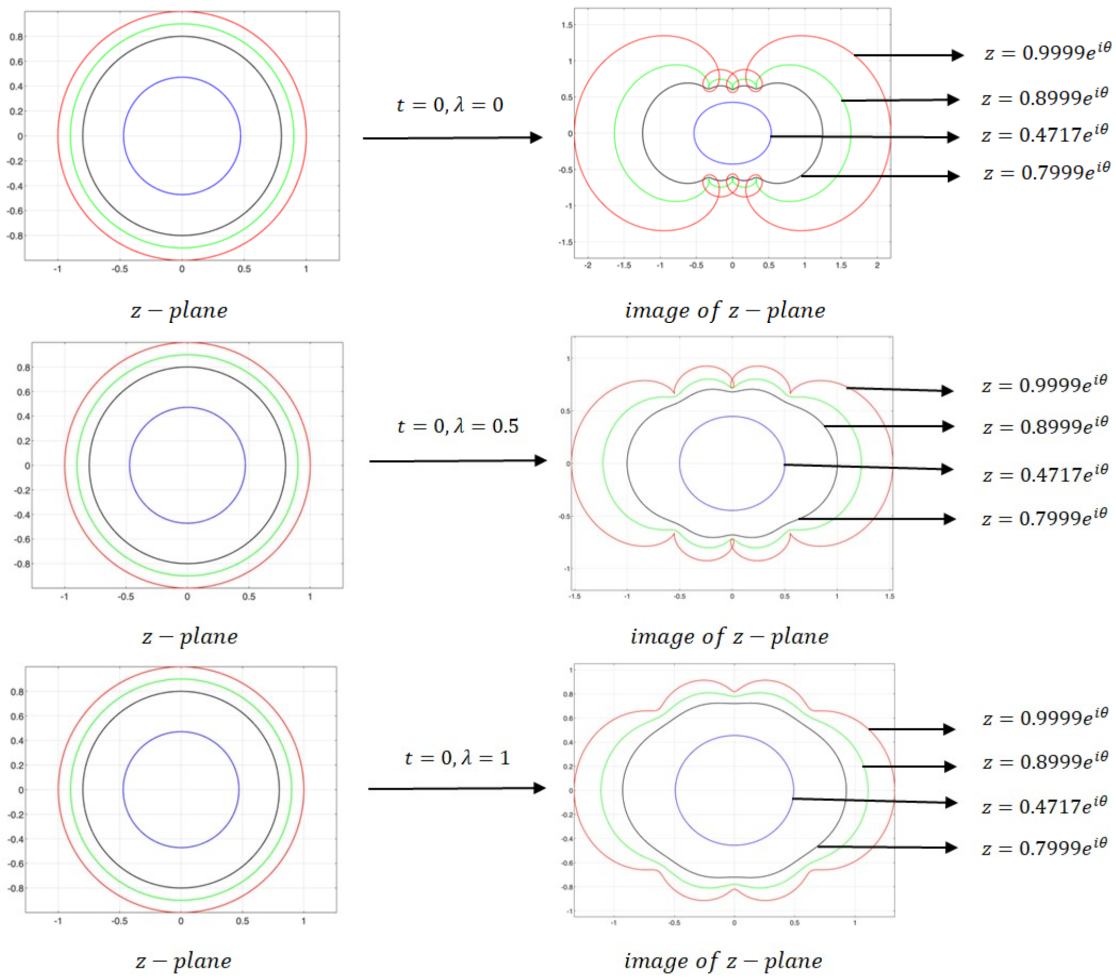


Fig 1.Graphical representation of $\mathcal{B}(\lambda, t)$ when $t = 0, \lambda = 0, 0.5, 1$

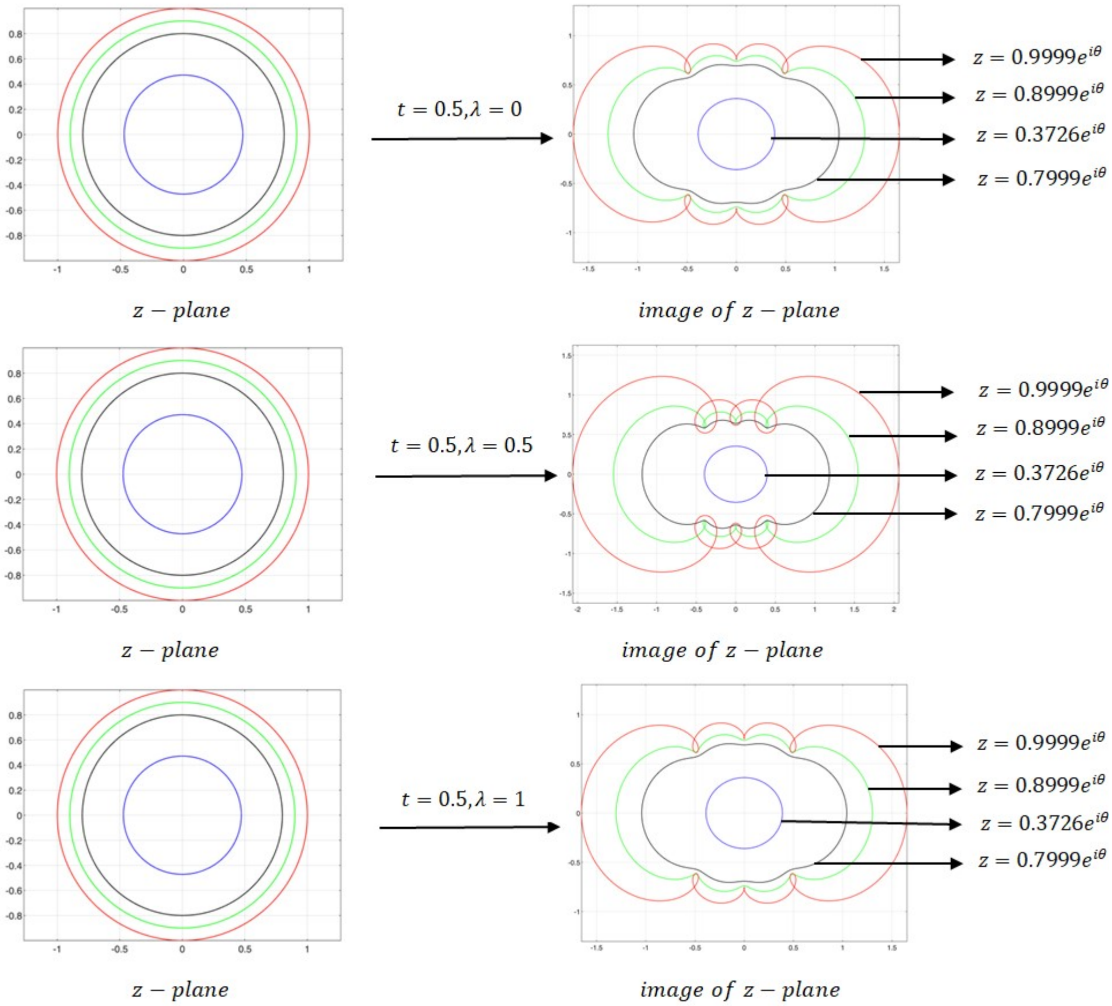
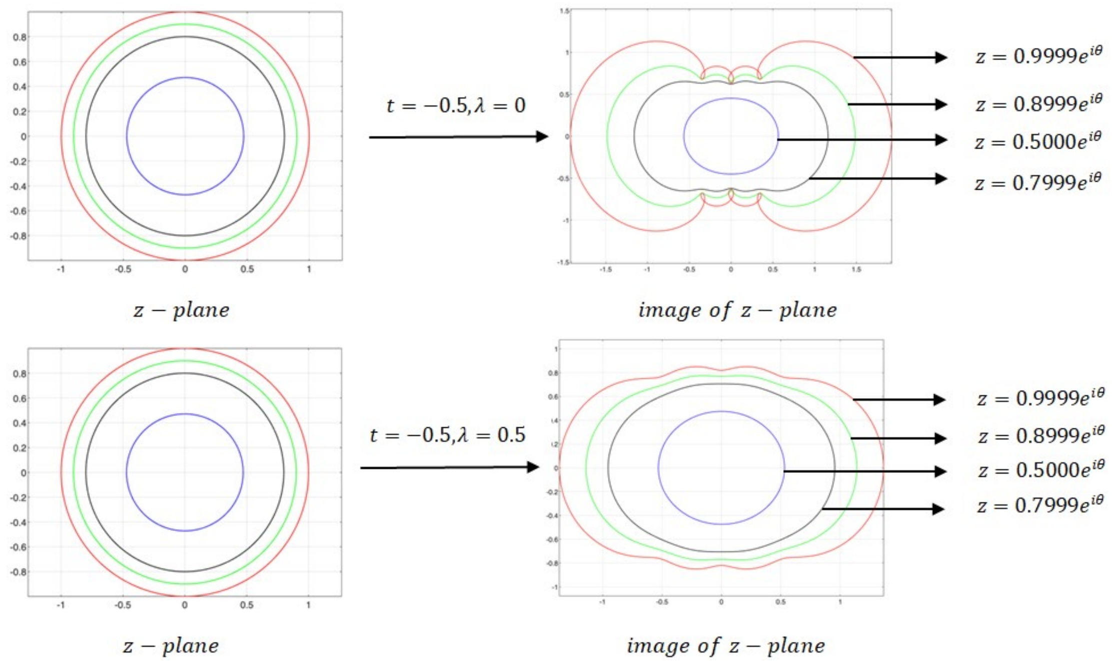


Fig 2.Graphical representation of $\mathcal{B}(\lambda, t)$ when $t = 0.5, \lambda = 0, 0.5, 1$



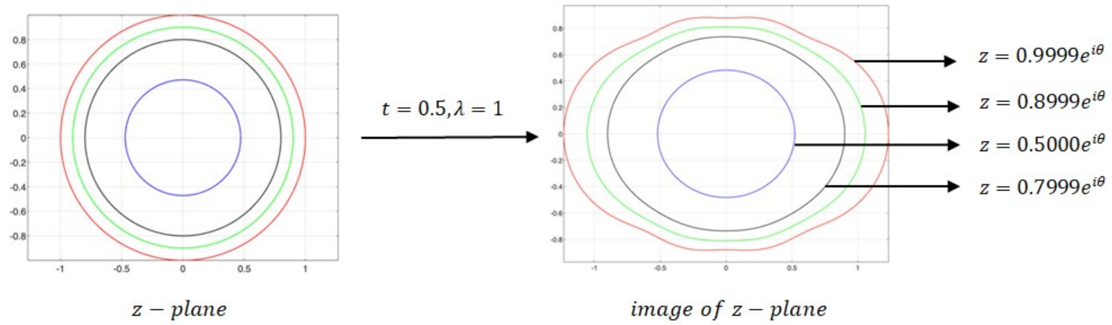


Fig 3.Graphical representation of $\mathcal{B}(\lambda,t)$ when $t=-0.5,\lambda=0,0.5,1$

(b) According to the definition of $\mathcal{B}(\lambda,t)$,

$$\frac{(1-t)[\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)]}{\lambda z^2[f''(z)-t^2 f''(tz)]+z[f'(z)-tf'(tz)]}\prec\frac{1+2z^2}{1-z^2}$$

$$(i.e),\ \frac{\lambda(1-t)z^2 f'''(z)+(1+\lambda-t-2\lambda t)zf''(z)+\lambda t^2 zf''(tz)+t[f'(tz)-f'(z)]}{z\lambda[f''(z)-t^2 f''(tz)]+[f'(z)-tf'(tz)]}\prec\frac{3r^2}{1-r^2}=:g(z)$$

here $\prec$ denotes subordination.

(c) From (a) we see that ,

$$|z\lambda[\beta''_{2n-1}(z)-t^2\beta''_{2n-1}(tz)]+\beta'_{2n-1}(z)-t\beta'_{2n-1}(tz)|$$
$$\le\sum_{m=n+1}^{\infty}(2m-1)[1+(m-1)2\lambda](1-t^{2m-1})|a_{2m-1}|r^{2m-2}\le\tau(n,r)$$

$$|\lambda(1-t)z^2\beta'''_{2n-1}(z)+(1+\lambda-t-2\lambda t)z\beta''_{2n-1}(z)+\lambda t^2 z\beta''_{2n-1}(tz)+t[\beta'_{2n-1}(tz)-\beta'_{2n-1}(z)]|$$
$$\le\sum_{m=n+1}^{\infty}(2m-1)[(2k-2)(1-t)+t^{2m-1}-t][1+(m-1)2\lambda]|a_{2m-1}|r^{2m-2}\le\eta(n,r)$$

We have proved our lemma. ∎

4. PROOF OF MAIN RESULT

For an arbitrary $f(z)=z+\sum_{n=2}^{\infty}a_{2n-1}z^{2n-1}\in\mathcal{B}(\lambda,t)$, we first consider its third section $s_3(z)=z+a_3z^3$ of f. Simple calculation shows

$$\frac{(1-t)[\lambda z^3 s_3'''(z)+(1+2\lambda)z^2 s_3''(z)+zs_3'(z)]}{\lambda z^2[s_3''(z)-t^2 s_3''(tz)]+z[s_3'(z)-ts_3'(tz)]}=1+\frac{3(3-u_3)\gamma_2 a_3 z^2}{1+3u_3\gamma_2 a_3 z^2}$$

By using (3.1) we have $|a_3|\le\frac{1}{(3-u_3)\gamma_2}$ and hence

$$\Re\left(\frac{(1-t)[\lambda z^3 s_3'''(z)+(1+2\lambda)z^2 s_3''(z)+zs_3'(z)]}{\lambda z^2[s_3''(z)-t^2 s_3''(tz)]+z[s_3'(z)-ts_3'(tz)]}\right)\ge 1-\frac{3(3-u_3)\gamma_2 a_3 z^2}{1-3u_3\gamma_2 a_3 z^2}\ge 1-\frac{3|z|^2}{1-\frac{3u_3}{3-u_3}|z|^2}$$

which is positive for $|z|<\frac{\sqrt{3-u_3}}{3}$. Thus $s_3(z)$ is convex in the disk $|z|<\frac{\sqrt{3-u_3}}{3}$.
For the purpose of proving the constant $|z|<\frac{\sqrt{3-u_3}}{3}$ is the best possible, we will consider the function $f_0(z)$ defined in (3.4). Then we define the third partial sum of $f_0(z)$ as $s_{3,0}(z)$ such that $s_{3,0}(z)=z+\left(\frac{1}{(3-u_3)\gamma_2}\right)z^3$ and hence, we find

$$\frac{(1-t)[\lambda z^3 s_{3,0}'''(z)+(1+2\lambda)z^2 s_{3,0}''(z)+zs_{3,0}'(z)]}{\lambda z^2[s_{3,0}''(z)-t^2 s_{3,0}''(tz)]+z[s_{3,0}'(z)-ts_{3,0}'(tz)]}=\frac{3-u_3+9z^2}{3-u_3+3u_3z^2}.$$

This shows that
$$\Re\left(\frac{(1-t)[\lambda z^3 s'''_{3,0}(z)+(1+2\lambda)z^2 s''_{3,0}(z)+zs'_{3,0}(z)]}{\lambda z^2[s''_{3,0}(z)-t^2 s''_{3,0}(tz)]+z[s'_{3,0}(z)-ts'_{3,0}(tz)]}\right)=0$$
when $z^2=-(\frac{3-u_3}{9})$ or $|z|=(\frac{\sqrt{3-u_3}}{3})$. Therefore, the equality holds.
Next, let's examine the case $n=3$. In this case we aim to show that

$$\Re\left(\frac{(1-t)[\lambda z^3 s'''_5(z)+(1+2\lambda)z^2 s''_5(z)+zs'_5(z)]}{\lambda z^2[s''_5(z)-t^2 s''_5(tz)]+z[s'_5(z)-ts'_5(tz)]}\right)=\Re\left(\frac{1+9\gamma_2 a_3 z^2+25\gamma_3 a_5 z^4}{1+3\gamma_2 u_3 a_3 z^2+5\gamma_3 u_5 a_5 z^4}\right)>0$$
for $|z|<\frac{\sqrt{3-u_3}}{3}$. Since the real part is harmonic in $|z|<\frac{\sqrt{3-u_3}}{3}$, it suffices to check that
$$\Re\left(\frac{1+9\gamma_2 a_3 z^2+25\gamma_3 a_5 z^4}{1+3\gamma_2 u_3 a_3 z^2+5\gamma_3 u_5 a_5 z^4}\right)>0$$
for $|z|<\frac{\sqrt{3-u_3}}{3}$. Also we see that,

$$\begin{aligned}\Re\left(\frac{1+9\gamma_2 a_3 z^2+25\gamma_3 a_5 z^4}{1+3\gamma_2 u_3 a_3 z^2+5\gamma_3 u_5 a_5 z^4}\right)&=\frac{3}{u_3}-\Re\left(\frac{3-u_3-5\gamma_3(5u_3-3u_5)a_5 z^2}{u_3(1+3\gamma_2 u_3 a_3 z^2+5\gamma_3 u_5 a_5 z^4)}\right)\\&\geq\frac{3}{u_3}-\frac{1}{u_3}\left|\frac{3-u_3-5\gamma_3(5u_3-3u_5)a_5 z^2}{1+3\gamma_2 u_3 a_3 z^2+5\gamma_3 u_5 a_5 z^4}\right|\end{aligned}$$
thus by considering suitable rotation of f(z), the proof reduces to $|z|<\frac{\sqrt{3-u_3}}{3}$ which means that it is enough to prove
$$\frac{3}{3-u_3}>\left|\frac{81-5\gamma_3(5u_3-3u_5)(3-u_3)a_5}{81+27\gamma_2 u_3(3-u_3)a_3+5\gamma_3 u_5(3-u_3)^2 a_5}\right|$$
From (3.3), we have
$$a_3=\frac{p_2}{2\gamma_2(3-u_3)}\ \ ;\ \ a_5=\frac{3}{10\gamma_3(5-u_5)}\left[\frac{3u_3p_2^2}{2(3-u_3)}+p_4\right]$$
Since $|p_2|\leq 2$ and $|p_4|\leq 2$, the last two equations can be rewritten as follows:
$$a_3=\frac{\delta}{\gamma_2(3-u_3)}\ \ ;\ \ a_5=\frac{3}{5\gamma_3(5-u_5)}\left[\frac{3u_3\delta^2}{3-u_3}+\mu\right]$$
for some $|\delta|\leq 1$ and $|\mu|\leq 1$.
Substituting the values of a_3 and a_5 and applying the maximum principle in the last inequality, it suffices to show that,

$$\frac{3}{3-u_3}\left|81+27u_3\delta+\frac{9u_3u_5(3-u_3)\delta^2}{5-u_5}+\frac{3u_5(3-u_3)^2\mu}{5-u_5}\right|>\left|81-\frac{9u_3(5u_3-3u_5)\delta^2}{5-u_5}-\frac{3(3-u_3)(5u_3-3u_5)\mu}{5-u_5}\right|$$
for $|\delta|=1=|\mu|$. By triangle inequality we get,
$$9\left|9+3u_3\delta+\frac{u_3u_5(3-u_3)\delta^2}{5-u_5}\right|-3(3-u_3)\left|9-\frac{u_3(5u_3-3u_5)\delta^2}{5-u_5}\right|>\frac{5u_3(3-u_3)^2}{5-u_5}$$
which is equivalent to
$$9\left|9\bar{\delta}+3u_3+\frac{u_3u_5(3-u_3)\delta}{5-u_5}\right|-3(3-u_3)\left|9\bar{\delta}-\frac{u_3(5u_3-3u_5)\delta}{5-u_5}\right|>\frac{5u_3(3-u_3)^2}{5-u_5}$$
As $|\delta|=1$. Write $Re(\delta)=x$ and for convenience let us take $A_n=n-u_n$, $B=5u_3-3u_5$. As a result,it remains to show that

$$T(x):=9\sqrt{81+9u_3^2+\frac{u_3^2u_5^2A_3^2}{A_5^2}-\frac{18u_3u_5A_3}{A_5}+\left[54u_3+\frac{6u_3^2u_5A_3}{A_5}\right]x+\frac{36u_3u_5A_3}{A_5}x^2}>\frac{5u_3A_3^2}{A_5}+3A_3\sqrt{81+\frac{u_3^2B^2}{A_5^2}+\frac{18B}{A_5}-\frac{36B}{A_5}x^2}$$

for $-1 \leq x \leq 1$.

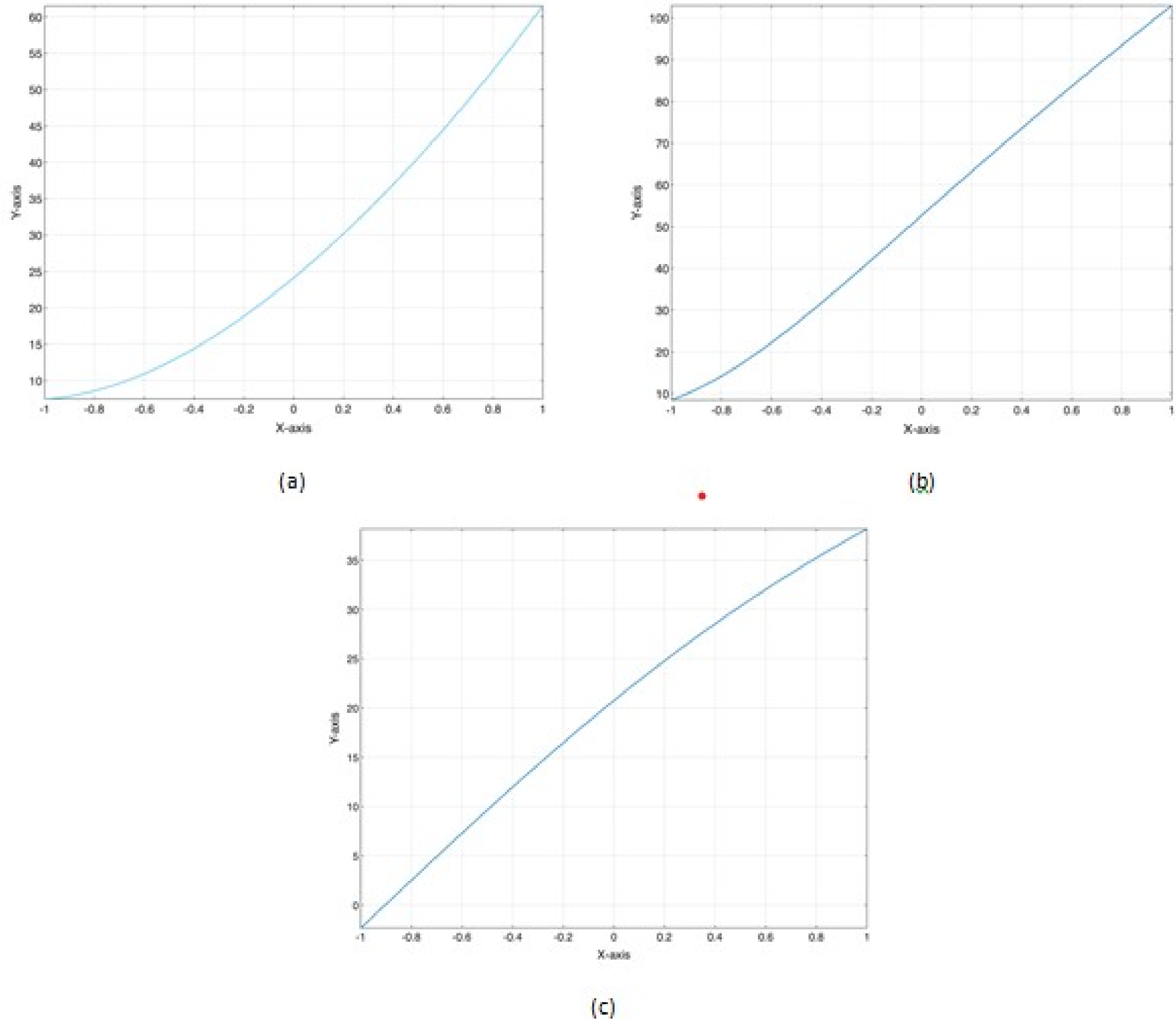


Fig 4. Graph of $T(x)$ when (a) $t = 0$, (b) $t = 0.5$ and (c) $t = -0.5$

Squaring both sides twice results in

$$\chi(x) = ax^4 + bx^3 + cx^2 + dx + e$$

where

$$a = \frac{104976A_3^2}{A_5^2}\left[81u_3^2u_5^2 + 18A_3B + A_3^2B^2\right]$$

$$b = \frac{314928u_3A_3}{A_5}\left[81u_3u_5 + 9A_3B + \frac{9u_3^2u_5^2A_3 + u_3u_5A_3^2B}{A_5}\right]$$

$$\begin{aligned} c = 236196&\left[81u_3^2 + \frac{162u_3u_5A_3 + 36u_3^3u_5A_3 + 18(A_3^2B - u_3u_5A_3^3) + 2(u_3^2A_3^2B - A_3^4B)}{A_5}\right] \\ &+ 26244\left[\frac{9u_3^4u_5^2A_3^2 - 4A_3^4B^2 - 72u_3u_5A_3^3B - 324u_3^2u_5^2A_3^2}{A_5^2} + \frac{2(u_3^2u_5^2A_3^4B - u_3^3u_5A_3^3B^2)}{A_5^3}\right] \\ &- 8100\left[\frac{18u_3^3u_5A_3^5 + 2u_3^2A_3^6B}{A_5^3}\right] - 8\left[\frac{729(u_3^2A_3^4B^3 + 81u_3^3u_5^3A_3^3) - 4050u_3^2A_3^6B}{A_5^3}\right] \end{aligned}$$

$$d = 78732\Bigg[729u_3 + 81u_3^3 - 81u_3A_3^2 + \frac{81u_3^2u_5A_3 + 9u_3^4u_5A_3 - 9u_3^2u_5A_3^3 - 162u_3^2u_5A_3 - 18u_3A_3^2B}{A_5}$$
$$+ \frac{9u_3^3u_5^2A_3^2 - u_3^3A_3^2B^2}{A_5^2}\Bigg] - 8748\Bigg[\frac{18u_3^2u_5A_3^3B + 162u_3^3u_5^2A_3^2}{A_5^2} + \frac{u_3^4u_5A_3^3B^2 - 9u_3^4u_5^3A_3^3}{A_5^3}\Bigg]$$
$$- 24300\Bigg[\frac{9u_3^3A_3^4}{A_5^2} + \frac{u_3^4u_5A_3^5}{A_5^3}\Bigg]$$

$$e = 531441\Bigg[81 + u_3^4 + A_3^4 - 2u_3^2A_3^2 + 18(u_3^2 - A_3^2) - \frac{36u_3u_5A_3}{A_5} + \frac{2u_3^2u_5^2A_3^2 + 4u_3^2u_5^2A_3^2}{A_5^2}\Bigg]$$
$$+ 6561\Bigg[\frac{324(u_3u_5A_3^3 - u_3^3u_5A_3 - A_3^2B)}{A_5} + \frac{2(u_3^2A_3^4B^2 - u_3^4A_3^2B^2) + 18(u_3^4u_5^2A_3^2 - u_3^2u_5^2A_3^4)}{A_5^2}$$
$$+ \frac{u_3^4u_5^4A_3^4}{A_5^4}\Bigg] - 900\Bigg[\frac{81u_3^2A_3^6}{A_5^2} + \frac{18u_3^2A_3^6B}{A_5^3} + \frac{u_3^4A_3^6B^2}{A_5^4}\Bigg] + 1458\Bigg[\frac{162(A_3^4B - u_3^2A_3^2B)}{A_5} + \frac{2u_3^2A_3^4B^3}{A_5^3}$$
$$+ \frac{81(4u_3u_5A_3^3B - u_3^2A_3^2B^2)}{A_5^2} + \frac{18(u_3^3u_5A_3^3B^2 - u_3^2u_5^2A_3^4B) - 162u_3^3u_5^3A_3^3}{A_5^3} - \frac{u_3^4u_5^2A_3^4B^2}{A_5^4}\Bigg]$$
$$+ 450\Bigg[\frac{81(u_3^2A_3^6 - u_3^4A_3^4) - 729u_3^2A_3^4}{A_5^2} + \frac{162u_3^3u_5A_3^5 + 18u_3^2A_3^6B}{A_5^3} + \frac{u_3^4A_3^6B^2 - 9u_3^4u_5^2A_3^6}{A_5^4}\Bigg]$$
$$+ 81\Bigg[\frac{324A_3^4B^2}{A_5^2} + \frac{u_3^4A_3^4B^4}{A_5^4}\Bigg] + \frac{625u_3^4A_3^8}{A_5^4}$$

By using long computation, for every order of differentiation , $\chi(x)$ is increasing in $-1 \leq x \leq 1$. Thus $n = 3$ is proved.
We next consider the general case for $n \geq 4$. It suffices to show that

$$\Re\left(1 + \frac{\lambda(1-t)z^2s'''_{2n-1}(z)+(1+\lambda-t-2\lambda t)zs''_{2n-1}(z)+\lambda t^2zs''_{2n-1}(tz)+t[s'_{2n-1}(tz)-s'_{2n-1}(z)]}{z\lambda[s''_{2n-1}(z)-t^2s''_{2n-1}(tz)]+[s'_{2n-1}(z)-ts'_{2n-1}(tz)]}\right) > 0$$

for $|z| = r = \frac{\sqrt{3-u_3}}{3}$. Therefore, it remains to find the largest r so that the last inequality holds for all for $n \geq 4$.

$$1 + \frac{\lambda(1-t)z^2s'''_{2n-1}(z) + (1+\lambda-t-2\lambda t)zs''_{2n-1}(z) + \lambda t^2zs''_{2n-1}(tz) + t[s'_{2n-1}(tz) - s'_{2n-1}(z)]}{z\lambda[s''_{2n-1}(z) - t^2s''_{2n-1}(tz)] + [s'_{2n-1}(z) - ts'_{2n-1}(tz)]}$$

By applying $s_{2n-1}(z) = f(z) - \beta_{2n-1}(z)$ and by simplification we get

$$= 1 + \frac{G}{H} + \frac{\frac{G}{H}N - M}{H - N}$$

where

$$G = \lambda(1-t)z^2f'''(z) + (1+\lambda-t-2\lambda t)zf''(z) + \lambda t^2zf''(tz) + t[f'(tz) - f'(z)]$$

$$H = z\lambda[f''(z) - t^2f''(tz)] + [f'(z) - tf'(tz)]$$

$$M = \lambda(1-t)z^2\beta'''_{2n-1}(z) + (1+\lambda-t-2\lambda t)z\beta''_{2n-1}(z) + \lambda t^2z\beta''_{2n-1}(tz) + t[\beta'_{2n-1}(tz) - \beta'_{2n-1}(z)]$$

$$N = z\lambda[\beta''_{2n-1}(z) - t^2\beta''_{2n-1}(tz)] + [\beta'_{2n-1}(z) - t\beta'_{2n-1}(tz)]$$

$$\Re\left(1 + \frac{\lambda(1-t)z^2s'''_{2n-1}(z) + (1+\lambda-t-2\lambda t)zs''_{2n-1}(z) + \lambda t^2zs''_{2n-1}(tz) + t[s'_{2n-1}(tz) - s'_{2n-1}(z)]}{z\lambda[s''_{2n-1}(z) - t^2s''_{2n-1}(tz)] + [s'_{2n-1}(z) - ts'_{2n-1}(tz)]}\right)$$

$$\geq 1 - \left|\frac{G}{H}\right| - \frac{|\frac{G}{H}||N| + |M|}{|H| - |N|}$$

By using Lemma (3.1) we attain ,

$$\Re\left(1+\frac{\lambda(1-t)z^2 s'''_{2n-1}(z) + (1+\lambda-t-2\lambda t)zs''_{2n-1}(z) + \lambda t^2 zs''_{2n-1}(tz) + t[s'_{2n-1}(tz) - s'_{2n-1}(z)]}{z\lambda[s''_{2n-1}(z) - t^2 s''_{2n-1}(tz)] + [s'_{2n-1}(z) - ts'_{2n-1}(tz)]}\right)$$

$$\geq 1 - \frac{3r^2}{1-r^2} - \frac{\frac{3r^2}{1-r^2}\tau(n,r) + \eta(n,r)}{\frac{1}{(1+r^2)^{\frac{3}{2}}} - \tau(n,r)}$$

Thus

$$\Re\left(1+\frac{\lambda(1-t)z^2 s'''_{2n-1}(z)+(1+\lambda-t-2\lambda t)zs''_{2n-1}(z)+\lambda t^2 zs''_{2n-1}(tz)+t[s'_{2n-1}(tz)-s'_{2n-1}(z)]}{z\lambda[s''_{2n-1}(z)-t^2 s''_{2n-1}(tz)]+[s'_{2n-1}(z)-ts'_{2n-1}(tz)]}\right) > 0$$

on condition that

$$\left(\frac{1-4r^2}{1-r^2} - \frac{(1+r^2)^{\frac{3}{2}}}{1-r^2}\frac{3r^2\tau(n,r) - \eta(n,r)}{1-(1+r^2)^{\frac{3}{2}}\tau(n,r)}\right) > 0$$

or equivalently ,

$$(1+r^2)^{\frac{3}{2}}\left(\frac{3r^2\tau(n,r) + (1-r^2)\eta(n,r)}{1-(1+r^2)^{\frac{3}{2}}\tau(n,r)}\right) < 1-4r^2$$

Next we show that the above equation holds for all $n \geq 4$ with $r = \frac{\sqrt{3-u3}}{3}$. By choosing $r = \frac{\sqrt{3-u3}}{3}$ the last inequality becomes

$$\left(\frac{12-u_3}{9}\right)^{\frac{3}{2}}\left(\frac{(\frac{3-u_3}{3})\tau(n,\frac{\sqrt{3-u_3}}{3}) + (\frac{6+u_3}{9})\eta(n,\frac{\sqrt{3-u_3}}{3})}{1-(\frac{12-u_3}{9})^{\frac{3}{2}}\tau(n,\frac{\sqrt{3-u_3}}{3})}\right) < \frac{4u_3-3}{9}$$

Let

$$\mu\left(n,\frac{\sqrt{3-u_3}}{3}\right) := 1 - \left(\frac{12-u_3}{9}\right)^{\frac{3}{2}}\tau\left(n,\frac{\sqrt{3-u_3}}{3}\right)$$

Let us prove that $\mu\left(n,\frac{\sqrt{3-u_3}}{3}\right) > 0$ for $n \geq 4$ (i.e),

$$\tau\left(n,\frac{\sqrt{3-u_3}}{3}\right) < \frac{27}{(12-u_3)^{\frac{3}{2}}}$$

$$\tau\left(n,\frac{\sqrt{3-u_3}}{3}\right) + \eta\left(n,\frac{\sqrt{3-u_3}}{3}\right) < \frac{27}{(12-u_3)^{\frac{3}{2}}}\left[\frac{4u_3-3}{6+u_3}\right] \quad for\ n \geq 4.$$

Proving the last inequality is sufficient , because if the last inequality is proven, then that implies that the previous inequality is also true.

$$\begin{aligned}
\tau\left(n,\frac{\sqrt{3-u_3}}{3}\right) + \eta\left(n,\frac{\sqrt{3-u_3}}{3}\right) &< \sum_{m=n+1}^{\infty}(1-t)(2m-1)\left[\frac{\prod_{k=1}^{m-1}2u_{2k-1}+2k-1}{\prod_{k=2}^{m}2k-1-u_{2k-1}}\right]r^{2m-2} \\
&\leq \sum_{m=5}^{\infty}(1-t)(2m-1)\left[\frac{\prod_{k=1}^{m-1}2u_{2k-1}+2k-1}{\prod_{k=2}^{m}2k-1-u_{2k-1}}\right]r^{2m-2} \\
&= \sum_{m=1}^{\infty}(1-t)(2m-1)\left[\frac{\prod_{k=1}^{m-1}2u_{2k-1}+2k-1}{\prod_{k=2}^{m}2k-1-u_{2k-1}}\right]r^{2m-2} \\
&\quad - \sum_{m=1}^{4}(1-t)(2m-1)\left[\frac{\prod_{k=1}^{m-1}2u_{2k-1}+2k-1}{\prod_{k=2}^{m}2k-1-u_{2k-1}}\right]r^{2m-2}
\end{aligned}$$

by substituting $r = \frac{\sqrt{3-u_3}}{3}$ it is evident that

$$\tau\left(n, \frac{\sqrt{3-u_3}}{3}\right) + \eta\left(n, \frac{\sqrt{3-u_3}}{3}\right) < \frac{27}{(12-u_3)^{\frac{3}{2}}}\left[\frac{4u_3-3}{6+u_3}\right]$$

This completes the proof.

5. CONCLUSION

In this article, we have studied the class $\mathcal{B}(\lambda, t)$ and proved that the radius $|z| < \frac{\sqrt{3-u_3}}{3}$ is best possible. For clear understanding we have varied the values of t and λ and projected in the graphical representation.

Data Availability
No data were used to support this study
Conflicts of Interest
The authors declare that there is no conflict of interests regarding the publication of this paper.

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