Aust. J. Math. Anal. Appl.
Vol. **20** (2023), No. 1, Art. 9, 7 pp.
AJMAA

ON THE CLASS OF TOTALLY POLYNOMIALLY POSINORMAL OPERATORS

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Received 25 July, 2022; accepted 17 February, 2023; published 28 March, 2023.

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ABSTRACT. In this paper, we proved that if $T \in \mathcal{B}(\mathcal{H})$ is totally P-posinormal operator with $P(z) = z^n + \sum_{j=1}^{n-1} c_j z^j, c_1 > 0$, then $ker(T - zI) \subseteq ker(T - zI)^*$. Moreover, we study spectral continuity and range kernel orthogonality of these class of operators.

Key words and phrases: Totally P-posinormal operator, Riesz Projection, Finite operator.

2010 *Mathematics Subject Classification.* Primary 47A10, 47B20.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the nullspace and range of T are denoted as $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$, *M-hyponormal* if $\|(T-zI)^*x\| \leq M\|(T-zI)x\|$ for all $z \in \mathbb{C}$ and for all $x \in \mathcal{H}$, and said to be *dominant* if for each $z \in \mathbb{C}$, there exist a constant $M(z) \geq 0$ such that $\|(T-zI)^*x\| \leq M(z)\|(T-zI)x\|$ for all $x \in \mathcal{H}$. It is well known that all the M-hyponormal operators are dominant.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *posinormal* if $\lambda^2 T^*T \geq TT^*$, for some $\lambda \geq 0$ ([12]). $T \in \mathcal{B}(\mathcal{H})$ is said to be *polynomially* (P)*-posinormal* if $\lambda^2 T^*T \geq P(T)P(T^*)$, where $P(z)$ is a polynomial with zero constant term and for some $\lambda \geq 0$ ([11]). If $P(z) = z$, then all the posinormal operator are polynomially (P)-posinormal. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *totally* P*-posinormal* if $\|(P(T-zI))^*x\| \leq M(z)\|(T-zI)x\|$ for all $x \in \mathcal{H}$, where $P(z)$ is a polynomial with zero constant term and $M(z)$ is bounded on compact sets of $\mathbb{C}$ ([11]). In general,

$$hyponormal \subset M-hyponormal \subset totally\ P-posinormal.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is dominant if and only if $T - zI$ is posinormal for all $z \in \mathbb{C}$ ([12]).

2. PROPERTIES OF TOTALLY P-POSINORMAL OPERATORS

Now, we prove that part of a totally P-posinormal operator on a closed subspace is again a totally P-posinormal operator.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. If T is totally P-posinormal operator, then $T|_{\mathcal{M}}$ is totally P-posinormal.*

Proof. Let $P(z) = z^n + \sum_{j=1}^{n-1} c_j z^j$. Let $x \in \mathcal{M}$ and Q be an orthogonal projection on to $\mathcal{M}$. Since $QT^*|_{\mathcal{M}} = (T|_{\mathcal{M}})^*$,

$$(T|_{\mathcal{M}} - zI)^*x = Q(T-zI)^*x.$$

$Q(T^*)^2|_{\mathcal{M}} = (T^2|_{\mathcal{M}})^*$, $\left((T|_{\mathcal{M}} - zI)^2\right)^* x = Q(T-zI)^{*2}x$.
Hence, $\left((T|_{\mathcal{M}} - zI)^n\right)^* x = Q(T-zI)^{*n}x$ for all $n \in \mathbb{N}$.
Thus, $\left(P(T|_{\mathcal{M}} - zI)\right)^* x = Q\left(P(T-zI)\right)^* x$.
Since T is totally P-posinormal, we have

$$\begin{aligned}\|\left(P(T|_{\mathcal{M}} - zI)\right)^* x\| &= \|Q\left(P(T-zI)\right)^* x\| \\ &\leq M(z)\|(T-zI)x\| \\ &= M(z)\|(T|_{\mathcal{M}} - zI)x\|.\end{aligned}$$

This completes the proof. ∎

Let $\mathcal{PB}$ denotes the collection of all totally P-posinormal operators, where $P(z) = z^n + \sum_{j=1}^{n-1} c_j z^j, c_1 > 0$.

Theorem 2.2. *If $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$, then $\mathcal{N}(T-zI) \subseteq \mathcal{N}(T-zI)^*$.*

Proof. Since T is totally P-posinormal operator, we have

$$\left(P(T-zI)\right)\left(P(T-zI)\right)^* \leq M(z)^2 (T-zI)^*(T-zI) \tag{2.1}$$

Let $x \in \mathcal{N}(T-zI)$. From equation (2.1), we have

$$\left(P(T-zI)\right)\left(P(T-zI)\right)^* x = 0.$$

Therefore, $\| (P(T-zI))^* x\|^2 = 0$. Hence, $x \in \mathcal{N}((P(T-zI))^*)$.
Thus, $\overline{c_1}(T-zI)^*x = -(T-zI)^{*n}x + \sum_{j=2}^{n-1} -\overline{c_j}(T-zI)^{*j}x$.
Hence,

$$\begin{aligned}\|\overline{c_1}(T-zI)^*x\| &\leq \| (P(T-zI))^* x\| \\ &\leq M(z)\|(T-zI)x\|.\end{aligned}$$

Since $x \in \mathcal{N}(T-zI)$, we have $\overline{c_1}(T-zI)^*x = 0$. As $c_1 > 0$, we have $(T-zI)^*x = 0$. Hence, $\mathcal{N}(T-zI) \subseteq \mathcal{N}(T-zI)^*$. ∎

Let $T \in \mathcal{B}(\mathcal{H})$ and λ be an isolated point of $\sigma(T)$. Then there exist $D_\lambda = \{z \in \mathbb{C} : |z-\lambda| \leq r\}$ with $D_\lambda \cap \sigma(T) = \{\lambda\}$. The operator defined by

$$E_\lambda = \frac{1}{2\pi i}\int_{\partial D_\lambda} (zI-T)^{-1}dz$$

is called *Riesz projection* of T with respect to λ, where ∂D_λ denotes the boundary of D_λ. It is well known that the Riesz projection E_λ satisfies the properties $E_\lambda^2 = E_\lambda, E_\lambda T = TE_\lambda,\ N(T-\lambda I) \subseteq R(E_\lambda)$ ([2]).

$T \in \mathcal{B}(\mathcal{H})$ is said to satisfy the property $H(q)$, if $H_0(T-\lambda I) = \mathcal{N}(T-\lambda I)^q$ for all $\lambda \in \mathbb{C}$ and for some integer $q \geq 1$, where $H_0(T) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$. It is well known that totally P-posinormal operators satisfy the property $H(q)$. Hence the following theorem holds for bounded totally P-posinormal operators by ([6]).

Theorem 2.3. *(*[6]*) Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$ and $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

In ([2]), M Cho and Y M Han proved that if $T \in \mathcal{B}(\mathcal{H})$ is a M-hyponormal operator, then $\mathcal{N}(E_\lambda) = \mathcal{R}(T-\lambda I)$. Now we prove this result holds for bounded totally P-posinormal operators also. For proving the result we use the following.

Theorem 2.4. *(*[9]*) Suppose $T \in \mathcal{B}(\mathcal{H})$ and E_λ be the Riesz projection with respect to an isolated eigen value λ. Then*
(1) E_λ is a projection.
(2) $\mathcal{R}(E_\lambda)$ and $\mathcal{N}(E_\lambda)$ are invariant under T.
(3) $\sigma(T|_{\mathcal{R}(E_\lambda)}) = \{\lambda\}$ and $\sigma(T|_{\mathcal{N}(E_\lambda)}) = \sigma(T) \setminus \{\lambda\}$.
(4) $\mathcal{N}(T-\lambda I) \subseteq \mathcal{R}(E_\lambda)$.

Theorem 2.5. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is a totally P-posinormal operator and λ is an isolated point of $\sigma(T)$. Then $\mathcal{N}(T-\lambda I) = \mathcal{R}(E_\lambda)$.*

Proof. From Theorem 2.4, we have $\mathcal{N}(T-\lambda I) \subseteq \mathcal{R}(E_\lambda)$.
Restriction $T|_{\mathcal{R}(E_\lambda)}$ is totally P-posinormal, by Theorem 2.1. Since λ is an isolated eigen value of T, we have $\sigma(T|_{\mathcal{R}(E_\lambda)}) = \{\lambda\}$, by Theorem 2.4.
If $\lambda = 0$, then $\sigma(T|_{\mathcal{R}(E_\lambda)}) = \{0\}$. From Theorem 2.3, we have $T|_{\mathcal{R}(E_\lambda)} = 0$. Hence, $\mathcal{R}(E_\lambda) \subseteq \mathcal{N}(T)$. If $\lambda \neq 0$, then $\sigma(T|_{\mathcal{R}(E_\lambda)}) = \{\lambda\}$. Thus $\sigma(T|_{\mathcal{R}(E_\lambda)} - \lambda I|_{\mathcal{R}(E_\lambda)}) = \{0\}$. From Theorem 2.3, we have $(T-\lambda I)|_{\mathcal{R}(E_\lambda)} = 0$. Hence, $\mathcal{R}(E_\lambda) \subseteq \mathcal{N}(T-\lambda I)$. ∎

For $T \in \mathcal{B}(\mathcal{H})$, let $\sigma_p(T)$ and $\sigma_a(T)$ denotes the point spectrum and approximate point spectrum of T. If $\lambda \in \sigma_p(T)$ and $\overline{\lambda} \in \sigma_p(T^*)$, then λ is in the joint point spectrum, $\sigma_{jp}(T)$. If $\lambda \in \sigma_a(T)$ and $\overline{\lambda} \in \sigma_a(T^*)$, then we say that λ is in the joint approximate point spectrum, $\sigma_{ja}(T)$.

Theorem 2.6. [1] *Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ satisfying the following properties for every $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$.*

(1) $\phi(A^*) = \phi(A)^*,\ \phi(I_{\mathcal{H}}) = I_{\mathcal{K}},\ \phi(\alpha A + \beta B) = \alpha\phi(A) + \beta\phi(B),$
$\phi(AB) = \phi(A)\phi(B),\ \|\phi(A)\| = \|A\|,\ \phi(A) \leq \phi(B)$ if A $\leq$ B
(2) $\phi(A) \geq 0$ if A ≥ 0
(3) $\sigma_a(A) = \sigma_a(\phi(A)) = \sigma_p(\phi(A))$.
(4) $\sigma_{ja}(A) = \sigma_{jp}(\phi(A))$.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$, then $\sigma_a(T) = \sigma_{ja}(T)$.*

Proof. Since T is totally P-posinormal,

$$M(z)^2(T - zI)^*(T - zI) - (P(T - zI))\,(P(T - zI))^* \geq 0 \tag{2.2}$$

Hence from Theorem 2.6, we have
$M(z)^2(\phi(T) - zI)^*(\phi(T) - zI) - (P(\phi(T) - zI))\,(P(\phi(T) - zI))^*$
$= M(z)^2\phi((T - zI)^*)\phi(T - zI) - \phi\,(P(T - zI))\,\phi\,(P(T - zI)^*)$
$= \phi\,(M(z)^2(T - zI)^*(T - zI) - (P(T - zI))\,(P(T - zI))^*)\,.$
Also from equation (2.2) and Theorem 2.6, we have

$$\phi\left(M(z)^2(T - zI)^*(T - zI) - (P(T - zI))\,(P(T - zI))^*\right) \geq 0.$$

Hence $\phi(T)$ is totally P-posinormal.

From Theorem 2.6, we have $\sigma_a(T) = \sigma_p(\phi(T))$. Since $\phi(T)$ is totally P-posinormal, we have $\mathcal{N}(\phi(T) - zI) \subset \mathcal{N}(\phi(T) - zI)^*$ (from Theorem 2.2). Hence, $\sigma_p(\phi(T)) = \sigma_{jp}(\phi(T))$. From Theorem 2.6, $\sigma_{jp}(\phi(T)) = \sigma_{ja}(T)$. Hence $\sigma_a(T) = \sigma_{ja}(T)$. ∎

Theorem 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$, then*

(1) *If $\sigma(T) = \{0\}$, then T is nilpotent.*
(2) *The matrix representation of T on $\mathcal{H} = N(T - \lambda I) \oplus (N(T - \lambda I))^\perp$ is*

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix},$$

where λ is a nonzero eigen value of T. Also $\lambda \notin \sigma_p(B)$ for some operator B and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.

Proof. Since $\sigma(T) = \{0\}$, it follows from Theorem 2.3 that $T = 0$. Hence T is nilpotent.
Let λ be a nonzero eigen value of T. Since $T \in \mathcal{PB}$, by Theorem 2.2, we have $\mathcal{N}(T - \lambda I) \subseteq \mathcal{N}(T - \lambda I)^*$. Therefore, $\mathcal{N}(T - \lambda I)^\perp$ is invariant under T. Hence, $\mathcal{N}(T - \lambda I)$ reduces T. Thus,

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix},$$

where $B = T|_{\mathcal{N}(T - \lambda I)^\perp}$. Let $x \in \mathcal{N}(B - \lambda I)$. Then

$$(T - \lambda I)\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ (B - \lambda I)x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $x \in \mathcal{N}(T - \lambda I)$. Since $B = T|_{\mathcal{N}(T - \lambda I)^\perp}$, we have $x \in \mathcal{N}(T - \lambda I)^\perp$. Thus, $x = 0$. Hence, $\mathcal{N}(B - \lambda I) = 0$. i.e, $\lambda \notin \sigma_p(B)$. Since $T = \lambda I \oplus B$, we have $\sigma(T) = \{\lambda\} \cup \sigma(B)$. ∎

Let $\mathcal{L}$ and $\mathcal{S}$ denotes the set of all compact and bounded subsets of $\mathbb{C}$ respectively. Let (X, d) be a metric space and the function $f : X \to \mathcal{S}$ is upper continuous (lower continuous) at x_0 if for each $\epsilon > 0$, there is a $\delta > 0$ such that $f(x) \subseteq (f(x_0))_\epsilon$ (respectively, $f(x_0) \subseteq (f(x))_\epsilon$) for all x with $d(x, x_0) < \delta$, where $(f(x_0))_\epsilon = \{z \in \mathbb{C} : dist(z, f(x_0)\} < \epsilon\}$. Define spectral map $\sigma : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{L}$, which maps $T \in \mathcal{B}(\mathcal{H})$ to spectrum of T ([3]). Spectral properties are studied for many class of operators for instant ([4, 5, 7, 13, 15]). Now we discuss the continuity of spectral map on the set of all totally P-posinormal operators.

Theorem 2.9. *The spectral map σ is continuous on all class of $\mathcal{PB}$ operators.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{PB}$. Then from Theorem 2.8, if $\sigma(T) = \{0\}$, then T is nilpotent. Also from the proof of Theorem 2.6, $\phi(T)$ is totally P-posinormal. From Theorem 2.8 and [5, Theorem 1.1], we have the spectral map σ is continuous on the set of all totally P-posinormal operators. ∎

3. FINITE OPERATOR

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a *finite operator* if

$$\|I - (TX - XT)\| \geq 1$$

for all $X \in \mathcal{B}(\mathcal{H})$ ([14]). Finite operator is a starting point of commutator approximation, which has many applications in quantum theory. In [14], J P Williams proved that all normal and hyponormal operators are finite. Properties of finite operators is studied in [10].

Next we show that bounded $T \in \mathcal{PB}$ is a finite operator.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$, then T is a finite operator.*

Proof. First we show that $\sigma_{ja}(T) \neq \emptyset$. Let $z \in \sigma_a(T)$. Then there exist a sequence (x_n) in $\mathcal{H}$ with $\|x_n\| = 1$ and $(T - zI)x_n \to 0$ as $n \to \infty$. Since T is totally P-posinormal, we have

$$\|(P(T - zI))^* x_n\| \leq M(z)\|(T - zI)x_n\|.$$

Hence, $\|(P(T - zI))^* x_n\| \to 0$ as $n \to \infty$. We have,

$$(P(T - zI))^* x_n = (T - zI)^{*n} x_n + \sum_{j=1}^{n-1} \overline{c_j}(T - zI)^{*j} x_n$$

Hence,

$\overline{c_1}(T - zI)^* x_n = (P(T - zI))^* x_n - (T - zI)^{*n} x_n + \sum_{j=2}^{n-1} -\overline{c_j}(T - zI)^{*j} x_n$. Since $c_1 > 0$, we have

$$\begin{aligned} \|\overline{c_1}(T - zI)^* x_n\| &\leq \| \left(P(T - zI)\right)^* x_n\| + \|(T - zI)^{*n} x_n + \sum_{j=2}^{n-1} \overline{c_j}(T - zI)^{*j} x_n\| \\ &\leq 2\| \left(P(T - zI)\right)^* x_n\|. \end{aligned}$$

Since $\|(P(T - zI))^* x_n\| \to 0$ as $n \to \infty$, we have $\|\overline{c_1}(T - zI)^* x_n\| \to 0$ as $n \to \infty$. As $c_1 > 0$, $\|(T - zI)^* x_n\| \to 0$ as $n \to \infty$. Hence, $\overline{z} \in \sigma_a(T^*)$. Thus, $z \in \sigma_{ja}(T)$. Hence $\sigma_a(T) = \sigma_{ja}(T)$. Since $\partial\sigma(T) \subset \sigma_a(T)$, we have $\sigma_{ja}(T) \neq \emptyset$. Hence, from [14, Theorem 6] T is a finite operator. ∎

Let $\mathcal{F}$ be a complex Banach space. Let $a, b \in \mathcal{F}$. If $\|a\| \leq \|a + zb\|$ for all $z \in \mathbb{C}$ then we say that a is orthogonal to b in the sense of Birkhoff. Geometrically it means that the line $\{a + zb : z \in \mathbb{C}\}$ is tangent to the open ball centered at zero and having radius $\|a\|$ ([10]).

If

$$\|A\| \leq \|A - (TX - XT)\|$$

for all $X \in \mathcal{B}(\mathcal{H})$ and for all $A \in \mathcal{N}(\delta_T)$, where $\delta_T(X) = TX - XT$, then we say that $R(\delta_T)$ is orthogonal to $\mathcal{N}(\delta_T)$.

Next we show that $R(\delta_T)$ is orthogonal to $\mathcal{N}(\delta_T)$ for a totally P-posinormal operator. For proving the result we use the following lemma.

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$ and $A \in \mathcal{B}(\mathcal{H})$ is a normal operator with $AT = TA$. Then*

$$|\lambda| \leq \|A - (TX - XT)\|$$

for all $\lambda \in \sigma_p(A)$ and for all $X \in \mathcal{B}(\mathcal{H})$.

Proof. Let $\lambda \in \sigma_p(A)$. If $\lambda = 0$, the result trivially holds. If $\lambda \neq 0$. Let $D_\lambda = \mathcal{N}(A - \lambda I)$. Since A is a normal operator with $AT = TA$ and by Fuglede-Putnam theorem, we have $A^*T = TA^*$. Hence D_λ reduces T and A. Thus the matrix representation of T and A on $D_\lambda \oplus D_\lambda^\perp$ is

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda I & 0 \\ 0 & A_2 \end{pmatrix}$$

Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

Hence $A - (TX - XT) = \begin{pmatrix} \lambda I - (T_1X_1 - X_1T_1) & B \\ R & S \end{pmatrix}$,

where $B, R, S \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} \|A - (TX - XT)\| &\geq \|\lambda I - (T_1X_1 - X_1T_1)\| \\ &= |\lambda| \, \|I - \left(T_1\frac{X_1}{\lambda} - \frac{X_1}{\lambda}T_1\right)\| \end{aligned}$$

Since D_λ is invariant under T and $T_1 = T|_{D_\lambda}$, we have T_1 is a totally P-posinormal operator from Theorem 2.1. Also from Theorem 3.1, T_1 is a finite operator. Therefore, $\|A - (TX - XT)\| \geq |\lambda|$. ∎

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$ and $A \in \mathcal{B}(\mathcal{H})$ is a normal operator with $AT = TA$. Then $R(\delta_T)$ is orthogonal to $\mathcal{N}(\delta_T)$.*

Proof. Let ϕ be the function as mentioned in Theorem 2.6. Since A is normal, $\phi(A)$ is normal. Since T is totally P-posinormal and from the proof of Theorem 2.7, we have $\phi(T)$ is totally P-posinormal. Also from Theorem 3.1, $\phi(T)$ is a finite operator. Since $AT = TA$, $\phi(A)\phi(T) = \phi(T)\phi(A)$. Let $\lambda \in \sigma_p(\phi(A))$. From Theorem 3.2, we have

$$|\lambda| \leq \|\phi(A) - (\phi(T)\phi(X) - \phi(X)\phi(T))\| = \|A - (TX - XT)\|, \tag{3.1}$$

for all $X \in \mathcal{B}(\mathcal{H})$. Since $\phi(A)$ and A are normal, we have

$$\|\phi(A)\| = \sup_{\mu \in \sigma(\phi(A))} |\mu| \text{ and } \|\mathrm{A}\| = \sup_{\mu \in \sigma(\mathrm{A})} |\mu| \tag{3.2}$$

Since A is normal and from Theorem 2.6, we have $\sigma(A) = \sigma_a(A) = \sigma_p(\phi(A))$. Hence from equation (3.1) and equation (3.2), we have

$$\|\phi(A)\| = \|A\| \leq \|A - (TX - XT)\|,$$

for all $X \in \mathcal{B}(\mathcal{H})$. ∎

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