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A UNIFYING VIEW OF SOME BANACH ALGEBRAS

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ABSTRACT. The purpose of this article is to shed light on a unifying framework for some normed algebras and, in particular, for some Banach algebras. The focus is on linear operators T between normed algebras X and Y and specified subalgebras A of Y. When the action of T on products in X satisfies a certain operative equation, the subspace $T^{-1}(A)$ is stable under the multiplication of X and is readily equipped with a family of canonical submultiplicative norms. It turns out that many familiar and important spaces are encompassed under this versatile perspective, and we offer a sampling of several such. In this sense, the article presents an alternative lens through which to view a host of normed algebras. Moreover, recognition that a normed linear space conforms to this general structure provides another avenue to confirming that it is at once stable under multiplication and also outfitted with an abundance of equivalent submultiplicative norms.

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1. INTRODUCTION

This article centers around linear operators $T : X \longrightarrow Y$ between normed algebras X and Y whose actions on products of elements of X are governed by a certain operative equation. In each such case, the pre-image $T^{-1}(A)$ of a specified subalgebra A of Y turns out to be a subalgebra of X that is automatically endowed with a family of equivalent submultiplicative norms. Under mild additional conditions, these norms on $T^{-1}(A)$ are complete. This general setting is remarkably flexible, and we devote a great deal of attention to highlighting an assortment of Banach algebras that may be construed in this way. We thereby offer a retro-fitting of several Banach algebras to a unifying framework, exposing a certain structural consistency among them. The current work may be viewed as a modest companion to Section 4 of the article [6] and the attendant articles [3], [4].

In the final section, we modify the setting slightly and introduce a second operative equation which, when satisfied, ensures that $T^{-1}(A)$ is again a subalgebra of X, this time equipped with a family of submultiplicative seminorms. We conclude with two examples of such; as a lagniappe, in each of these, a seminorm under consideration turns out to be a norm.

2. NORMED ALGEBRAS

Throughout this section and the next, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ represent normed algebras, and $(A, \|\cdot\|_A)$ is a normed subalgebra of Y that is also a two-sided ideal of Y; specifically,

(i) $\|a\|_Y \le \|a\|_A$ for all $a \in A$;
(ii) $Ay \subseteq A$ for all $y \in Y$ and $\|ay\|_A \le \|a\|_A\|y\|_Y$ for all $a \in A$ and all $y \in Y$;
(iii) $yA \subseteq A$ for all $y \in Y$ and $\|ya\|_A \le \|y\|_Y\|a\|_A$ for all $y \in Y$ and all $a \in A$.

We will be interested in functions $f, g : X \longrightarrow Y$ and $h : X \times X \longrightarrow Y$ that satisfy the following conditions:

$$\left\{\begin{array}{l} f \text{ and } g \text{ are norm-decreasing;} \\ h(u,v) \in A \text{ whenever } u, v \in T^{-1}(A)\text{; and} \\ \text{there is an } m > 0 \text{ such that } \|h(u,v)\|_A \le m\|Tu\|_A\|Tv\|_A \text{ for all } u, v \in T^{-1}(A). \end{array}\right. \tag{2.1}$$

When such functions f, g, and h implement the *operative equation*

$$T(uv) = f(u)(Tv) + (Tu)g(v) + h(u,v) \quad \text{for all } u, v \in X, \tag{2.2}$$

we define

$$\|u\|_{T,m} = \|u\|_X + m\|Tu\|_A \quad \text{for all } u \in T^{-1}(A).$$

Against this backdrop, we are prepared for the following theorem.

Theorem 2.1. *Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed algebras, that $(A, \|\cdot\|_A)$ is a normed subalgebra of Y that is also a two-sided ideal of Y, and that $T : X \longrightarrow Y$ is a linear operator. If f, g, and h are functions that satisfy conditions* (2.1) *and implement the operative equation* (2.2)*, then $(T^{-1}(A), \|\cdot\|_{T,m})$ is a normed algebra. Moreover, if $m^\star > m$, then $\|\cdot\|_{T,m^\star}$ provides an algebra norm for $T^{-1}(A)$ that is equivalent to $\|\cdot\|_{T,m}$.*

Proof. It is straightforward to confirm that the function $\|\cdot\|_{T,m}$ provides a norm for the subspace $T^{-1}(A)$ of X. We thus proceed to establish that $T^{-1}(A)$ is stable under multiplication and that $\|\cdot\|_{T,m}$ is submultiplicative on $T^{-1}(A)$. Indeed, for $u, v \in T^{-1}(A)$, we have from equation (2.2) that $T(uv) = f(u)Tv + (Tu)g(v) + h(u,v)$ which, together with conditions (ii), (iii), and

(2.1), implies that $uv \in T^{-1}(A)$. Moreover,

$$\begin{aligned}\|uv\|_{T,m} &= \|uv\|_X + m\|T(uv)\|_A \\ &= \|uv\|_X + m\|f(u)(Tv) + (Tu)g(v) + h(u,v)\|_A \\ &\le \|u\|_X\|v\|_X + m\|f(u)\|_Y\|Tv\|_A + m\|Tu\|_A\|g(v)\|_Y + m\|h(u,v)\|_A \\ &\le \|u\|_X\|v\|_X + m\|u\|_X\|Tv\|_A + m\|Tu\|_A\|v\|_X + m^2\|Tu\|_A\|Tv\|_A \\ &= (\|u\|_X + m\|Tu\|_A)(\|v\|_X + m\|Tv\|_A) \\ &= \|u\|_{T,m}\|v\|_{T,m}\end{aligned}$$

to establish the submultiplicativity of the norm $\|\cdot\|_{T,m}$.

For the final assertion, it is now routine to confirm that $\|\cdot\|_{T,m^\star}$ is also an algebra norm for $T^{-1}(A)$, and we infer the equivalence of the norms $\|\cdot\|_{T,m}$ and $\|\cdot\|_{T,m^\star}$ from the inequalities $\|u\|_{T,m} \le \|u\|_{T,m^\star} \le (m^\star/m)\|u\|_{T,m}$ for all $u \in T^{-1}(A)$. ∎

In the context of Theorem 2.1, there is a family of infinitely many equivalent algebra norms for $T^{-1}(A)$, namely, one for each sufficiently large $m > 0$. In fact, as is the case in several of the upcoming examples, when the function $h \equiv 0$, every $m > 0$ provides such a norm.

If, instead of being norm-decreasing as stipulated in (2.1), there are positive numbers c_1 and c_2, at least one of which is greater than 1, for which the functions f and g satisfy $\|f(u)\|_Y \le c_1\|u\|_X$ and $\|g(u)\|_Y \le c_2\|u\|_X$ for all $u \in X$, then the number $c = \max\{c_1, c_2\}$ satisfies $\|uv\|_{T,m} \le c\|u\|_{T,m}\|v\|_{T,m}$ for all $u, v \in T^{-1}(A)$. In this case, there is a submultiplicative norm for $T^{-1}(A)$ that is equivalent to $\|\cdot\|_{T,m}$ ([1], Lemma 4.8).

The following corollary shows that many familiar operators automatically satisfy conditions (2.1) and equation (2.2). In advance of the corollary, we recall that, by Definition 1.4.5 of [5], an algebra X is *left faithful* if the equality $\{w \in X : wX = 0\} = \{0\}$ holds and *right faithful* if the equality $\{w \in X : Xw = 0\} = \{0\}$ holds.

Corollary 2.2. *In the setting of Theorem 2.1, if any of the following conditions is satisfied, then $(T^{-1}(A), \|\cdot\|_{T,m})$ is a normed algebra:*

(a) *$T : X \longrightarrow Y$ is an algebra homomorphism and $m \ge 1$;*
(b) *$T : X \longrightarrow X$ is a derivation on X and $m > 0$;*
(c) *the algebra X is left or right faithful, $T : X \longrightarrow X$ is a multiplier on X, and $m > 0$.*

Proof. (a) If T is an algebra homomorphism, then the equality $T(uv) = TuTv$ holds for all $u, v \in X$. Equation (2.2) thus holds with the choices $f \equiv g \equiv 0$ on X and the function $h : X \times X \longrightarrow Y$ given by $h(u,v) = TuTv$ for all $u, v \in X$. The conditions (2.1) are also satisfied since, in particular, if $u, v \in T^{-1}(A)$, then $h(u,v) = TuTv \in A$ and $\|h(u,v)\|_A = \|TuTv\|_A \le \|Tu\|_A\|Tv\|_A$. The result is therefore an immediate consequence of Theorem 2.1.

(b) If T is a derivation on X, then, by definition, the equality $T(uv) = uTv + (Tu)v$ holds for all $u, v \in X$. Conditions (2.1) and equation (2.2) are thus satisfied with the choices $f(u) = g(u) = u$ for all $u \in X$, the function $h \equiv 0$ on $X \times X$, and any $m > 0$. The result follows from Theorem 2.1.

(c) If $T : X \longrightarrow X$ is a multiplier, then, by definition, $(Tu)v = uTv$ for all $u, v \in X$. It follows that, for any $w \in X$,

$$w(T(uv) - uTv) = wT(uv) - w(Tu)v = (Tw)(uv - uv) = 0.$$

Thus, if X is left faithful, then $T(uv) = uTv$ for all $u, v \in X$, so that conditions (2.1) and equation (2.2) are satisfied with the choices $f(u) = u$ for all $u \in X$, $g \equiv 0$ on X, $h \equiv 0$ on $X \times X$, and any $m > 0$. The result follows by Theorem 2.1. The argument in the case that X is right faithful is similar. ∎

3. BANACH ALGEBRAS

Theorem 3.1. *Suppose that $(X, \|\cdot\|_X)$ is a Banach algebra, that $(A, \|\cdot\|_A)$ is a Banach subalgebra of the normed algebra $(Y, \|\cdot\|_Y)$ that is also a two-sided ideal of Y, and that $T : X \longrightarrow Y$ is a bounded linear operator. If $(T^{-1}(A), \|\cdot\|_{T,m})$ is a normed algebra, then $(T^{-1}(A), \|\cdot\|_{T,m})$ is a Banach algebra.*

Proof. If $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the normed algebra $(T^{-1}(A), \|\cdot\|_{T,m})$, then it follows from the definition of $\|\cdot\|_{T,m}$ that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(Tu_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $(X, \|\cdot\|_X)$ and $(A, \|\cdot\|_A)$, respectively. Thus, there exist elements $x \in X$ and $a \in A$ such that $u_n \xrightarrow{\|\cdot\|_X} x$ and $T(u_n) \xrightarrow{\|\cdot\|_A} a$ as $n \to \infty$. Condition (i) on the subalgebra A implies that $T(u_n) \xrightarrow{\|\cdot\|_Y} a$ as $n \to \infty$. On the other hand, since T is bounded, $T(u_n) \xrightarrow{\|\cdot\|_Y} Tx$ as $n \to \infty$. Consequently, $Tx = a$. Finally, since

$$\|u_n - x\|_{T,m} = \|u_n - x\|_X + m\|T(u_n) - T(x)\|_A = \|u_n - x\|_X + m\|T(u_n) - a\|_A,$$

we conclude that $u_n \xrightarrow{\|\cdot\|_{T,m}} x$ as $n \to \infty$ which completes the proof. ∎

Example 3.1. SEQUENCES OF GENERALIZED BOUNDED VARIATION. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ both denote the Banach algebra $(\ell^\infty, \|\cdot\|_\infty)$ and, for $p \geq 1$, let $(A, \|\cdot\|_A)$ denote the Banach subalgebra $(\ell^p, \|\cdot\|_p)$ of $(\ell^\infty, \|\cdot\|_\infty)$. Next, let $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ represent a sequence of positive real numbers that is bounded away from zero, and define $T : \ell^\infty \longrightarrow \ell^\infty$ by

$$T\mathbf{x} = \Big(\frac{x_2 - x_1}{\lambda_1^{1/p}}, \frac{x_3 - x_2}{\lambda_2^{1/p}}, \frac{x_4 - x_3}{\lambda_3^{1/p}}, \dots\Big) \quad \text{for all } \mathbf{x} = (x_1, x_2, \dots) \in \ell^\infty.$$

Then T is a bounded linear operator on ℓ^∞, and if the sequences $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ are elements of ℓ^∞, then the n-th term of the sequence $T(\mathbf{x}\mathbf{y})$ is

$$x_{n+1}\Big(\frac{y_{n+1} - y_n}{\lambda_n^{1/p}}\Big) + \Big(\frac{x_{n+1} - x_n}{\lambda_n^{1/p}}\Big)y_n.$$

It follows that $T(\mathbf{x}\mathbf{y}) = L\mathbf{x}T\mathbf{y} + (T\mathbf{x})\mathbf{y}$, where L is the left shift operator on ℓ^∞, that is,

$$L\mathbf{x} = L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

The operative equation (2.2) thus holds with the norm-decreasing function L in the role of f, the identity function on X in the role of g, and the function $h \equiv 0$ on $X \times X$. Since these clearly satisfy conditions (2.1), Theorem 2.1 implies that, for any $m > 0$, the function defined by

$$\|\mathbf{x}\|_{T,m} = \|\mathbf{x}\|_X + m\|T\mathbf{x}\|_A = \|\mathbf{x}\|_\infty + m\Big(\sum_{n=1}^{\infty} \frac{|x_{n+1} - x_n|^p}{\lambda_n}\Big)^{1/p} \quad \text{for all } \mathbf{x} \in T^{-1}(\ell^p)$$

provides a norm for the algebra $T^{-1}(\ell^p)$; in fact, as a consequence of Theorem 3.1, the pair $(T^{-1}(\ell^p), \|\cdot\|_{T,m})$ is a Banach algebra. The classical Banach algebra $(bv, \|\cdot\|_{bv})$ of *sequences of bounded variation* ([5], Example 4.1.44) arises, for example, when $p = 1$, $m = 1$, and Λ is the constant sequence whose terms are all ones. Moreover, the condition that the sequence Λ is bounded away from zero ensures that bv is a subalgebra of any of the Banach algebras $(T^{-1}(\ell^p), \|\mathbf{x}\|_{T,m})$ of *sequences of generalized bounded variation.* □

3.1. Suppose that $(X, \|\cdot\|_X)$ is a Banach algebra, that $(A, \|\cdot\|_A)$ is a Banach subalgebra of the normed algebra $(Y, \|\cdot\|_Y)$ that is also a two-sided ideal of Y, and that $T_\alpha : X \longrightarrow Y$ is a bounded linear operator for each α in the index set J. Suppose further that, for each $\alpha \in J$, the pair $(T_\alpha^{-1}(A), \|\cdot\|_{T_\alpha, m_\alpha})$ is a normed subalgebra of $(X, \|\cdot\|_X)$. By Theorem 3.1, $(T_\alpha^{-1}(A), \|\cdot\|_{T_\alpha, m_\alpha})$ is, therefore, a Banach algebra for each $\alpha \in J$. Let

$$W = \{u \in \bigcap_{\alpha \in J} T_\alpha^{-1}(A) : \sup_{\alpha \in J} \|u\|_{T_\alpha, m_\alpha} < \infty\},$$

and define

$$\|u\|_W = \sup_{\alpha \in J} \|u\|_{T_\alpha, m_\alpha} = \|u\|_X + \sup_{\alpha \in J} m_\alpha \|T_\alpha(u)\|_A \quad \text{for all } u \in W.$$

Theorem 3.2. *In the setting detailed above, $(W, \|\cdot\|_W)$ is a Banach algebra.*

Proof. It is routine to verify that $(W, \|\cdot\|_W)$ is a normed linear space, so we confirm that the norm $\|\cdot\|_W$ is submultiplicative and complete. For $u, v \in W$ and $\alpha \in J$,

$$\|uv\|_{T_\alpha, m_\alpha} \leq \|u\|_{T_\alpha, m_\alpha} \|v\|_{T_\alpha, m_\alpha} \leq \|u\|_W \|v\|_W,$$

which implies that $uv \in W$ and that $\|uv\|_W \leq \|u\|_W \|v\|_W$.

Next, suppose that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the normed algebra $(W, \|\cdot\|_W)$. By the definition of $\|\cdot\|_W$, it follows that $(u_n)_{n \in \mathbb{N}}$ is $\|\cdot\|_X$-Cauchy and, for each $\alpha \in J$, the sequence $m_\alpha T_\alpha(u_n)$ is $\|\cdot\|_A$-Cauchy. Thus, there are elements $x \in X$ and $a_\alpha \in A$ such that $u_n \xrightarrow{\|\cdot\|_X} x$ and $m_\alpha T_\alpha(u_n) \xrightarrow{\|\cdot\|_A} a_\alpha$ as $n \to \infty$. The continuity of T_α ensures that $m_\alpha T_\alpha(u_n) \xrightarrow{\|\cdot\|_Y} m_\alpha T_\alpha(x) \in Y$ as $n \to \infty$. It follows that, for each $\alpha \in J$, and $\varepsilon > 0$ given, there is an integer $N \in \mathbb{N}$ so large that whenever $n \geq N$, we have

$$\begin{aligned}
\|m_\alpha T_\alpha(x) - a_\alpha\|_Y &\leq \|m_\alpha T_\alpha(x) - m_\alpha T_\alpha(u_n)\|_Y + \|m_\alpha T_\alpha(u_n) - a_\alpha\|_Y \\
&< \frac{\varepsilon}{2} + \|m_\alpha T_\alpha(u_n) - a_\alpha\|_A \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore $m_\alpha T_\alpha(x) = a_\alpha$ for each $\alpha \in J$, which implies that $x \in T_\alpha^{-1}(A)$ for all $\alpha \in J$, that is, $x \in \bigcap_{\alpha \in J} T_\alpha^{-1}(A)$.

Now, for each $\alpha \in J$, there is an integer $N_\alpha \in \mathbb{N}$ for which $\|m_\alpha T_\alpha(u_n) - m_\alpha T_\alpha(x)\|_A < 1$ whenever $n \geq N_\alpha$. Thus,

$$\begin{aligned}
\|m_\alpha T_\alpha(x)\|_A &\leq \|m_\alpha T_\alpha(x) - m_\alpha T_\alpha(u_{N_\alpha})\|_A + \|m_\alpha T_\alpha(u_{N_\alpha})\|_A \\
&< 1 + \|u_{N_\alpha}\|_W \\
&\leq 1 + M,
\end{aligned}$$

where the number $M > 0$ is an upper bound for the $\|\cdot\|_W$-Cauchy sequence $(u_n)_{n \in \mathbb{N}}$. Consequently, $\sup_{\alpha \in J} \|m_\alpha T_\alpha(x)\|_A$ is finite, so $x \in W$. It remains to show that $u_n \xrightarrow{\|\cdot\|_W} x$ as $n \to \infty$. To this end, for any $\alpha \in J$ and any integers $i, j \in \mathbb{N}$:

$$\begin{aligned}
&\big| \|u_i - u_j\|_{T_\alpha, m_\alpha} - \|x - u_j\|_{T_\alpha, m_\alpha} \big| \\
&\qquad = \big| (\|u_i - u_j\|_X + \|m_\alpha T_\alpha(u_i - u_j)\|_A) - (\|x - u_j\|_X + \|m_\alpha T_\alpha(x - u_j)\|_A) \big| \\
&\qquad \leq \big| \|u_i - u_j\|_X - \|x - u_j\|_X \big| + \big| \|m_\alpha T_\alpha(u_i - u_j)\|_A - \|m_\alpha T_\alpha(x - u_j)\|_A \big| \\
&\qquad \leq \|u_i - x\|_X + \|m_\alpha T_\alpha(u_i - x)\|_A,
\end{aligned}$$

which tends to zero as $i \to \infty$. Thus,

$$\lim_{i\to\infty} \|u_i - u_j\|_{T_\alpha,m_\alpha} = \|x - u_j\|_{T_\alpha,m_\alpha} \quad \text{for all } j \in \mathbb{N}. \tag{3.1}$$

Now, because $(u_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_W$-Cauchy sequence, let $N \in \mathbb{N}$ be so large that $\|u_i - u_j\|_W < \varepsilon/2$ for all $i, j \geq N$. By definition of $\|\cdot\|_W$, for any $\alpha \in J$, we have $\|u_i - u_j\|_{T_\alpha,m_\alpha} < \varepsilon/2$ for all $i, j \geq N$. As a consequence of equation (3.1), $\|x - u_j\|_{T_\alpha,m_\alpha} \leq \varepsilon/2$ for all $j \geq N$. Finally, therefore, for all $j \geq N$,

$$\|x - u_j\|_W = \sup_{\alpha\in J} \|x - u_j\|_{T_\alpha,m_\alpha} \leq \varepsilon/2 < \varepsilon,$$

to complete the proof. ∎

Example 3.2. LIPSCHITZ FUNCTIONS. Let $(X, \|\cdot\|_X) = (\mathrm{Bd}[a,b], \|\cdot\|_\infty)$ and $(Y, \|\cdot\|_Y) = (A, \|\cdot\|_A) = (\mathbb{C}, |\cdot|)$, and fix $\delta \in (0,1]$. For a given pair $\alpha = (x,y)$ of distinct numbers in the interval $[a,b]$, define $T_\alpha : \mathrm{Bd}[a,b] \longrightarrow \mathbb{C}$ by

$$T_\alpha(f) = \frac{f(x) - f(y)}{|x-y|^\delta} \quad \text{for all } f \in \mathrm{Bd}[a,b].$$

It follows that T_α is a bounded linear operator and, for $f, g \in \mathrm{Bd}[a,b]$,

$$T_\alpha(fg) = \frac{fg(x) - fg(y)}{|x-y|^\delta} = f(x)\frac{g(x) - g(y)}{|x-y|^\delta} + \frac{f(x) - f(y)}{|x-y|^\delta} g(y) = \varepsilon_x(f)T_\alpha(g) + T_\alpha(f)\varepsilon_y(g),$$

where ε_x and ε_y denote the norm-decreasing evaluation functionals on $\mathrm{Bd}[a,b]$ at x and y, respectively. Conditions (2.1) and the operative equation (2.2) are thus satisfied with the choices ε_x, ε_y, and $h_\alpha \equiv 0$ so that, by Theorems 2.1 and 3.1, for any choice of $m_\alpha > 0$, the function

$$\|f\|_{T_\alpha,m_\alpha} = \|f\|_\infty + m_\alpha|T_\alpha(f)| = \|f\|_\infty + m_\alpha\frac{|f(x) - f(y)|}{|x-y|^\delta}$$

provides a complete submultiplicative norm for the algebra $T_\alpha^{-1}(\mathbb{C}) = \mathrm{Bd}[a,b]$.

Now, let J denote the collection of all pairs $\alpha = (x,y)$ of distinct numbers $a \leq x < y \leq b$ from the interval $[a,b]$ and fix a number $m_\alpha > 0$ for each $\alpha \in J$. Theorem 3.2 thus ensures that the pair $(W, \|\cdot\|_W)$ is a Banach algebra, where

$$W = \{f \in \bigcap_{\alpha\in J} T_\alpha^{-1}(\mathbb{C}) : \sup_{\alpha\in J} \|f\|_{T_\alpha,m_\alpha} < \infty\}$$

and

$$\|f\|_W = \sup_{\alpha\in J} \|f\|_{T_\alpha,m_\alpha} = \|f\|_\infty + \sup_{\alpha\in J} m_\alpha|T_\alpha(f)| \quad \text{for all } f \in W.$$

In particular, with the choice $m_\alpha = 1$ for all $\alpha \in J$, the function that assigns the number

$$\|f\|_\infty + \sup\left\{\frac{|f(x) - f(y)|}{|x-y|^\delta} : x \neq y\right\} \tag{3.2}$$

to each f is a complete algebra norm for the space $\mathrm{Lip}_\delta[\mathrm{a,b}]$ of *Lipschitz functions*, which addresses a part of Exercise 4.13 of [1] in the case of the compact metric space $[a,b]$. □

Example 3.3. FUNCTIONS OF BOUNDED SECOND VARIATION. Let X denote the algebra $\mathrm{Lip}_1[a,b]$ of Lipschitz functions that is the W of the preceding Example 3.2, with the complete algebra norm $\|\cdot\|_X$ prescribed by expression (3.2) for the choice $\delta = 1$. Further, let $(Y, \|\cdot\|_Y) = (\ell^\infty, \|\cdot\|_\infty)$, with subalgebra $(A, \|\cdot\|_A) = (\ell^1, \|\cdot\|_1)$. For a partition $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of the interval $[a,b]$, define the bounded linear operator $T_\pi : \mathrm{Lip}_1[a,b] \longrightarrow \ell^\infty$ by

$$T_\pi(f) = (\Delta_1(f) - \Delta_0(f),\, \Delta_2(f) - \Delta_1(f),\, \ldots,\, \Delta_n(f) - \Delta_{n-1}(f),\, 0,\, 0,\, \ldots)$$

for all $f \in \mathrm{Lip}_1[a,b]$, where $\Delta_0(f) = 0$ and $\Delta_k(f)$ denotes the difference quotient

$$\Delta_k(f) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} \quad \text{for } k = 1, 2, \ldots n.$$

Computation of $\Delta_k(fg)$ for functions $f, g \in \mathrm{Lip}_1[a,b]$ and $k = 1, 2, \ldots n$ yields

$$\Delta_k(fg) = f(t_k)\Delta_k(g) + \Delta_k(f)g(t_{k-1})$$

so that for $k = 1, 2, \ldots, n$, the k-th term of the sequence $T_\pi(fg)$ is

$$\begin{aligned}\Delta_k(fg) - \Delta_{k-1}(fg) &= f(t_k)\Delta_k(g) + \Delta_k(f)g(t_{k-1}) - f(t_{k-1})\Delta_{k-1}(g) - \Delta_{k-1}(f)g(t_{k-2}) \\ &= f(t_k)(\Delta_k(g) - \Delta_{k-1}(g)) + (\Delta_k(f) - \Delta_{k-1}(f))g(t_{k-1}) \\ &\quad + \Delta_{k-1}(g)\Delta_k(f)(t_k - t_{k-1}) + \Delta_{k-1}(f)\Delta_{k-1}(g)(t_{k-1} - t_{k-2}).\end{aligned}$$

Equation (2.2) is thus satisfied, specifically,

$$T_\pi(fg) = \sigma_\pi(f)T_\pi g + (T_\pi f)\tau_\pi(g) + h_\pi(f,g) \quad \text{for all } f, g \in \mathrm{Lip}_1[a,b],$$

where the norm-decreasing functions $\sigma_\pi, \tau_\pi : \mathrm{Lip}_1[a,b] \longrightarrow \ell^\infty$ are defined by

$$\sigma_\pi(f) = (f(t_1), f(t_2), \ldots, f(t_n), 0, 0, \ldots) \quad \text{and} \quad \tau_\pi(g) = (g(t_0), g(t_1), \ldots, g(t_{n-1}), 0, 0, \ldots)$$

for all $f, g \in \mathrm{Lip}_1[a,b]$, and the function $h_\pi : \mathrm{Lip}_1[a,b] \times \mathrm{Lip}_1[a,b] \longrightarrow \ell^\infty$ assigns to the pair (f,g) the bounded sequence having k-th term

$$\Delta_{k-1}(g)\Delta_k(f)(t_k - t_{k-1}) + \Delta_{k-1}(f)\Delta_{k-1}(g)(t_{k-1} - t_{k-2})$$

for $k = 2, 3, \ldots, n$ and zeros otherwise. The range of h_π thus clearly lies in $A = \ell^1$, and estimation of $\|h_\pi(f,g)\|_1$ yields

$$\begin{aligned}\|h_\pi(f,g)\|_1 &= \sum_{k=2}^{n} |\Delta_{k-1}(g)\Delta_k(f)(t_k - t_{k-1}) + \Delta_{k-1}(f)\Delta_{k-1}(g)(t_{k-1} - t_{k-2})| \\ &\le \sum_{k=2}^{n} \big(|\Delta_{k-1}(g)|\,|\Delta_k(f)|\,(t_k - t_{k-1}) + |\Delta_{k-1}(f)|\,|\Delta_{k-1}(g)|\,(t_{k-1} - t_{k-2}) \big) \\ &\le 2(b-a) \sup_{1\le k\le n} |\Delta_k(f)| \sup_{1\le k\le n-1} |\Delta_k(g)| \\ &\le 2(b-a)\|T_\pi(f)\|_1 \|T_\pi(g)\|_1,\end{aligned}$$

where the last inequality is a result of the following simple telescoping computation for any $k = 1, 2, \ldots, n$:

$$\begin{aligned}|\Delta_k(f)| &= |\Delta_k(f) - \Delta_{k-1}(f) + \Delta_{k-1}(f) - \cdots + \Delta_2(f) - \Delta_1(f) + \Delta_1(f) - \Delta_0(f)| \\ &\le |\Delta_1(f) - \Delta_0(f)| + |\Delta_2(f) - \Delta_1(f)| + \cdots + |\Delta_k(f) - \Delta_{k-1}(f)| \\ &\le \|T_\pi(f)\|_1.\end{aligned}$$

The conditions (2.1) are thus all satisfied, so, by Theorems 2.1 and 3.1, for any partition π of the interval $[a,b]$, the preimage $T_\pi^{-1}(\ell^1) = \mathrm{Lip}_1[a,b]$ is a Banach algebra when equipped with the norm

$$\begin{aligned}\|f\|_{T_\pi, m_\pi} &= \|f\|_X + m_\pi \|T_\pi(f)\|_1 \\ &= \|f\|_\infty + \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|} : x \ne y \right\} + m_\pi \sum_{k=1}^{n} |\Delta_k(f) - \Delta_{k-1}(f)| \\ &= \|f\|_\infty + \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|} : x \ne y \right\} + m_\pi \Big(|\Delta_1(f)| + \sum_{k=2}^{n} |\Delta_k(f) - \Delta_{k-1}(f)| \Big)\end{aligned}$$

for any number $m_\pi \geq 2(b-a)$. An application of Theorem 3.2 leads to the conclusion that

$$W = \{f \in \bigcap_{\pi \in \mathcal{P}} T_\pi^{-1}(\ell^1) : \sup_{\pi \in \mathcal{P}} \|f\|_{T_\pi, m_\pi} < \infty\} \tag{3.3}$$

is a Banach algebra when equipped with the complete algebra norm $\|f\|_W = \sup_{\pi \in \mathcal{P}} \|f\|_{T_\pi, m_\pi}$ for all $f \in W$.

The value $\sup_{\pi \in \mathcal{P}} \sum_{k=2}^n |\Delta_k(f) - \Delta_{k-1}(f)|$, where the supremum is taken over the collection $\mathcal{P}$ of all partitions of $[a,b]$, is called the *total second variation* of the function f on the interval $[a,b]$. Functions whose total second variation are finite play a role in an extension of the Riemann-Stieltjes integral, as is comprehensively detailed in [7]. Moreover, with $m_\pi = 2(b-a)$, say, for all partitions $\pi \in \mathcal{P}$, Lemma 1.2 of [7] implies that the equality $W = BV_2[a,b]$ holds, where W is our algebra (3.3) and $BV_2[a,b]$ denotes the space of *functions of bounded second variation*. The present example thus parallels and re-contextualizes the similar result Theorem 5.1 of [4] in which a different Banach algebra norm is provided for $BV_2[a,b]$. Evidently the algebra $BV_2[a,b]$ is semi-simple so, by *Johnson's uniqueness-of-norm theorem* ([1], Corollary 5.29), these Banach algebra norms on $BV_2[a,b]$ are equivalent. □

Example 3.4. FUNCTIONS OF GENERALIZED BOUNDED VARIATION. Let $(X, \|\cdot\|_X) = (\mathrm{Bd}[a,b], \|\cdot\|_\infty)$, $(Y, \|\cdot\|_Y) = (\ell^\infty, \|\cdot\|_\infty)$, and $(A, \|\cdot\|_A) = (\ell^p, \|\cdot\|_p)$, where $p \geq 1$ and, as in Example 3.1, let $\Lambda = (\lambda_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers that is bounded away from zero. In addition, fix a partition $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of the compact interval $[a,b]$, and define $T_\pi : \mathrm{Bd}[a,b] \longrightarrow \ell^\infty$ to be the bounded linear operator that assigns to a function $f \in \mathrm{Bd}[a,b]$ the sequence

$$T_\pi(f) = (\Delta_1(f), \Delta_2(f), \ldots, \Delta_n(f), 0, 0, \ldots),$$

where here

$$\Delta_k(f) = \frac{f(x_k) - f(x_{k-1})}{\lambda_k^{1/p}} \quad \text{for } k = 1, 2, \ldots, n.$$

Thus, for $f, g \in \mathrm{Bd}[a,b]$ and $k = 1, 2, \ldots, n$, the k-th entry of the sequence $T_\pi(fg)$ is

$$\Delta_k(fg) = f(x_k)\Delta_k(g) + \Delta_k(f)g(x_{k-1}),$$

and all the remaining entries are zeros. It follows that for functions $f, g \in \mathrm{Bd}[a,b]$,

$$T_\pi(fg) = \sigma_\pi(f)T_\pi(g) + T_\pi(f)\tau_\pi(g),$$

where $\sigma_\pi, \tau_\pi : \mathrm{Bd}[a,b] \longrightarrow \ell^\infty$ are defined by

$$\sigma_\pi(f) = (f(x_1), \ldots, f(x_n), 0, 0, \ldots) \quad \text{and} \quad \tau_\pi(g) = (g(x_0), \ldots, g(x_{n-1}), 0, 0, \ldots).$$

Conditions (2.1) and the operative equation (2.2) are thus satisfied with the choices σ_π, τ_π, and $h_\pi \equiv 0$. By Theorems 2.1 and 3.1, it follows that, for each partition π of the interval $[a,b]$, and any $m_\pi > 0$, the pair $(T_\pi^{-1}(\ell^p), \|\cdot\|_{T_\pi, m_\pi})$ is a Banach algebra. The algebra $T_\pi^{-1}(\ell^p) = \mathrm{Bd}[a,b]$ so that

$$\|f\|_{T_\pi, m_\pi} = \|f\|_\infty + m_\pi \left(\sum_{k=1}^n \frac{|f(x_k) - f(x_{k-1})|^p}{\lambda_k} \right)^{1/p}$$

provides a submultiplicative norm for $\mathrm{Bd}[a,b]$.

Now, let $\mathcal{P}$ denote the collection of all partitions of the interval $[a,b]$ and fix a number $m_\pi > 0$ for each $\pi \in \mathcal{P}$. Theorem 3.2 ensures that the space

$$W_p = \{f \in \bigcap_{\pi \in \mathcal{P}} T_\pi^{-1}(\ell^p) : \sup_{\pi \in \mathcal{P}} \|f\|_{T_\pi, m_\pi} < \infty\}$$

is rendered a Banach algebra when equipped with the complete submultiplicative norm

$$\|f\|_{W_p} = \sup_{\pi\in\mathcal{P}} \|f\|_{T_\pi,m_\pi} = \|f\|_\infty + \sup_{\pi\in\mathcal{P}} m_\pi \|T_\pi(f)\|_p \quad \text{for all } f \in W_p.$$

With the canonical choice $m_\pi = 1$ for each partition $\pi \in \mathcal{P}$, the spaces W_p are the spaces of *functions of generalized bounded variation* in the sense of Waterman-Shiba that were introduced in 1980 by Shiba [9]. For $p = 1$, it turns out that W_1 is the space ΛBV studied by Waterman in [10], and the classical space BV of *functions of bounded variation* arises, for example, when $m_\pi = 1$ for all $\pi \in \mathcal{P}$, $p = 1$, and Λ is the constant sequence of all ones. □

Example 3.5. ALGEBRAS OF JAMES TYPE. In this example, we allow the algebras $X = Y$ to represent any of ℓ^∞, c, or c_0 equipped with the supremum norm $\|\cdot\|_\infty$, while $(A, \|\cdot\|_A) = (\ell^p, \|\cdot\|_p)$ for some $p \geq 1$. Denote by $\mathcal{Q}$ the set of integer k-tuples $q = (q_1, \dots, q_k)$ such that $k \geq 2$ and $1 \leq q_1 < q_2 < \cdots < q_k$. For $q \in \mathcal{Q}$, define the bounded linear operator $T_q : X \longrightarrow Y$ by

$$T_q(\mathbf{x}) = \big(x_{q_2} - x_{q_1}, x_{q_3} - x_{q_2}, \dots, x_{q_k} - x_{q_{k-1}}, x_{q_k} - x_{q_1}, 0, 0, \dots\big)$$

for all $\mathbf{x} = (x_1, x_2, \dots) \in X$. For sequences $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ in X, and distinct indices $i, j \in \{1, 2, \dots, k\}$, the equality

$$(x_{q_i} y_{q_i} - x_{q_j} y_{q_j}) = x_{q_i}(y_{q_i} - y_{q_j}) + (x_{q_i} - x_{q_j}) y_{q_j}$$

implies that

$$T_q(\mathbf{xy}) = \sigma_q(\mathbf{x}) T_q(\mathbf{y}) + T_q(\mathbf{x}) \tau_q(\mathbf{y}),$$

where $\sigma_q, \tau_q : X \longrightarrow Y$ are defined by

$$\sigma_q(\mathbf{x}) = (x_{q_2}, x_{q_3}, \dots, x_{q_{k-1}}, x_{q_k}, x_{q_k}, 0, 0, \dots)$$

and

$$\tau_q(\mathbf{y}) = (y_{q_1}, y_{q_2}, \dots, y_{q_{k-2}}, y_{q_{k-1}}, y_{q_1}, 0, 0, \dots).$$

Clearly, $\|\sigma_q(\mathbf{x})\|_\infty \leq \|\mathbf{x}\|_\infty$ and $\|\tau_q(\mathbf{y})\|_\infty \leq \|\mathbf{y}\|_\infty$ so that conditions (2.1) and the operative equation (2.2) are satisfied with the choices σ_q, τ_q, and $h_q \equiv 0$. Thus, by Theorems 2.1 and 3.1, for each $q \in \mathcal{Q}$ and arbitrary $m_q > 0$, a Banach algebra norm for $T_q^{-1}(\ell^p)$ is prescribed by the function

$$\|\mathbf{x}\|_{T_q,m_q} = \|\mathbf{x}\|_\infty + m_q \|T_q(\mathbf{x})\|_p = \|\mathbf{x}\|_\infty + m_q \Big(\sum_{j=1}^{k-1} |x_{q_{j+1}} - x_{q_j}|^p + |x_{q_k} - x_{q_1}|^p \Big)^{1/p}$$

for all $\mathbf{x} \in T_q^{-1}(\ell^p)$. In fact, $T_q^{-1}(\ell^p) = X$ so each function $\|\mathbf{x}\|_{T_q,m_q}$ provides a complete submultiplicative norm for the algebra X. Furthermore, as a consequence of Theorem 3.2, the space

$$W = \{\mathbf{x} \in \bigcap_{q\in\mathcal{Q}} T_q^{-1}(\ell^p) : \sup_{q\in\mathcal{Q}} m_q \|T_q(\mathbf{x})\|_p < \infty\}$$

is rendered a Banach algebra when equipped with the complete submultiplicative norm

$$\|\mathbf{x}\|_W = \sup_{q\in\mathcal{Q}} \|\mathbf{x}\|_{T_q,m_q} = \|\mathbf{x}\|_\infty + \sup_{q\in\mathcal{Q}} m_q \|T_q(\mathbf{x})\|_p \quad \text{for all } \mathbf{x} \in W.$$

In the case that $X = Y = c_0$, $A = \ell^2$, and $m_q = 1$ for all $q \in \mathcal{Q}$, the resulting algebra W is the famous *James' space*, a non-reflexive Banach space that is isometrically isomorphic to its second dual. When $X = Y = \ell^\infty$, it turns out that W is the unitization of James' space. A detailed and comprehensive analysis of James' space equipped with a Banach algebra norm different from those presented here is undertaken in the article [2]; Example 4.1.45 of [5] is also based on [2]. Among its many interesting properties, James' algebra is semi-simple so, by *Johnson's uniqueness-of-norm theorem* ([1], Corollary 5.29), all Banach algebra norms on it are equivalent. □

Example 3.6. ALGEBRAS OF FEINSTEIN TYPE. As in Example 3.5, here, too, we allow the algebras $X = Y$ to represent any of ℓ^∞, c, or c_0 equipped with the supremum norm $\|\cdot\|_\infty$, while $(A, \|\cdot\|_A) = (\ell^p, \|\cdot\|_p)$ for some $p \geq 1$. For $n \in \mathbb{N}$, define the bounded linear operator $T_n : X \longrightarrow Y$ by

$$T_n(\mathbf{x}) = \frac{1}{n}\,(x_2 - x_1, 2(x_3 - x_2), 3(x_4 - x_3), \ldots, n(x_{n+1} - x_n), 0, 0, \ldots)$$

for all $\mathbf{x} = (x_1, x_2, \ldots) \in X$. Given sequences $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ in X, and an integer $1 \leq k \leq n$, the k-th term of the sequence $T_n(\mathbf{xy})$ is

$$\frac{k}{n}(x_{k+1}y_{k+1} - x_k y_k) = x_{k+1} \cdot \frac{k}{n}(y_{k+1} - y_k) + \frac{k}{n}(x_{k+1} - x_k) \cdot y_k.$$

Consequently, the formula

$$T_n(\mathbf{xy}) = L(\mathbf{x})T_n(\mathbf{y}) + T_n(\mathbf{x})\mathbf{y}$$

holds, where, as in Example 3.1, L is the (norm-decreasing) left shift operator on X. Conditions (2.1) and the operative equation (2.2) are thus clearly satisfied with L, the identity function on X, and the function $h \equiv 0$ for each $n \in \mathbb{N}$. By Theorem 3.1, for each $n \in \mathbb{N}$ and arbitrary $m_n > 0$, the pair $(T_n^{-1}(\ell^p), \|\cdot\|_{T_n,m_n})$ is a Banach algebra, where

$$\|\mathbf{x}\|_{T_n,m_n} = \|\mathbf{x}\|_\infty + m_n\|T_n(\mathbf{x})\|_p = \|\mathbf{x}\|_\infty + \frac{m_n}{n}\Big(\sum_{k=1}^{n} k^p|x_{n+1} - x_n|^p\Big)^{1/p}$$

for all $\mathbf{x} \in T_n^{-1}(\ell^p)$. Similar to Example 3.5, for each $n \in \mathbb{N}$ the equality $T_n^{-1}(\ell^p) = X$ holds, so the function $\|\mathbf{x}\|_{T_n,m_n}$ provides a submultiplicative norm for the algebra X.

Next, define

$$W = \{\mathbf{x} \in \bigcap_{n\in\mathbb{N}} T_n^{-1}(\ell^p) : \sup_{n\in\mathbb{N}} m_n\|T_n(\mathbf{x})\|_p < \infty\},$$

and apply Theorem 3.2 to conclude that W a Banach algebra when equipped with the complete submultiplicative norm

$$\|\mathbf{x}\|_W = \sup_{n\in\mathbb{N}} \|\mathbf{x}\|_{T_n,m_n} = \|\mathbf{x}\|_\infty + \sup_{n\in\mathbb{N}} m_n\|T_n(\mathbf{x})\|_p \quad \text{for all } \mathbf{x} \in W.$$

The choices $X = Y = c_0$, $A = \ell^1$, and $m_q = 1$ for all $q \in \mathcal{Q}$ (as defined in Example 3.5) result in a Banach algebra W that coincides with one constructed by J. F. Feinstein; its remarkable properties are highlighted and detailed in Example 4.1.46 of [5]. Here, too, Feinstein's algebra is semi-simple, so *Johnson's uniqueness-of-norm theorem* ([1], Corollary 5.29) guarantees that it supports a unique complete norm. □

4. SEMINORMED ALGEBRAS

In this section, we begin with an algebra X and again suppose that $(Y, \|\cdot\|_Y)$ is a normed algebra with a normed subalgebra $(A, \|\cdot\|_A)$ that is also a two-sided ideal of Y satisfying the conditions (i), (ii), and (iii) of Section 2. This time, however, we are interested in a multiplicative linear functional φ on X and a function $h : X \times X \longrightarrow Y$ that satisfies the relevant parts of conditions (2.1), specifically,

$$\text{(4.1)} \qquad \begin{cases} h(u, v) \in A \text{ whenever } u, v \in T^{-1}(A); \text{ and} \\ \text{there is an } m > 0 \text{ such that } \|h(u, v)\|_A \leq m\|Tu\|_A\|Tv\|_A \text{ for all } u, v \in T^{-1}(A). \end{cases}$$

When φ and h implement the second *operative equation*

$$T(uv) = \varphi(u)(Tv) + \varphi(v)(Tu) + h(u,v) \quad \text{for all } u, v \in X, \tag{4.2}$$

we define

$$\rho_{T,m}(u) = |\varphi(u)| + m\|Tu\|_A \quad \text{for all } u \in T^{-1}(A).$$

In this setting, we have the following analogue of Theorem 2.1.

Theorem 4.1. *Suppose that X is an algebra, that $(A, \|\cdot\|_A)$ is a normed subalgebra of the normed algebra $(Y, \|\cdot\|_Y)$ that is also a two-sided ideal of Y, and that $T : X \longrightarrow Y$ is a linear operator. Suppose further that φ is a multiplicative linear functional on X and that h is a function that satisfies conditions* (4.1). *If the operative equation* (4.2) *holds, then $T^{-1}(A)$ is an algebra for which $\rho_{T,m}$ provides a submultiplicative seminorm. Moreover, if $m^\star > m$, then $\rho_{T,m^\star}$ is a submultiplicative seminorm for $T^{-1}(A)$ that is equivalent to $\rho_{T,m}$.*

Proof. It is straightforward to check that $\rho_{T,m}$ is a seminorm for the subspace $T^{-1}(A)$ of X. The proofs of the stability of $T^{-1}(A)$ under multiplication, the submultiplicativity of $\rho_{T,m}$ on $T^{-1}(A)$, and the equivalence of the seminorms all proceed as in the proof of Theorem 2.1 *mutatis mutandis*. ∎

Example 4.1. SEQUENCES OF BOUNDED VARIATION. Let X denote the algebra ℓ^∞, and let $(Y, \|\cdot\|_Y)$ denote the Banach algebra $(\ell^\infty, \|\cdot\|_\infty)$ with Banach subalgebra $(A, \|\cdot\|_A) = (\ell^1, \|\cdot\|_1)$. Define the operator $T : X \longrightarrow Y$ by

$$T\mathbf{x} = (\Delta_1(\mathbf{x}), \Delta_2(\mathbf{x}), \Delta_3(\mathbf{x}), \dots) \quad \text{for all } \mathbf{x} = (x_1, x_2, x_3, \dots) \in X,$$

where, in this example, $\Delta_k(\mathbf{x})$ simply represents the difference $x_{k+1} - x_k$ for all $k \in \mathbb{N}$. Thus, for sequences $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $\mathbf{y} = (y_1, y_2, y_3, \dots)$ in ℓ^∞, the k-th entry of $T(\mathbf{xy})$ is

$$\begin{aligned}
\Delta_k(\mathbf{xy}) &= x_{k+1}y_{k+1} - x_k y_k \\
&= x_{k+1}\Delta_k(\mathbf{y}) + y_k\Delta_k(\mathbf{x}) \\
&= x_1\Delta_k(\mathbf{y}) + y_1\Delta_k(\mathbf{x}) + (x_{k+1} - x_1)\Delta_k(\mathbf{y}) + (y_k - y_1)\Delta_k(\mathbf{x}) \\
&= \varphi(\mathbf{x})T\mathbf{y} + \varphi(\mathbf{y})T\mathbf{x} + h(\mathbf{x}, \mathbf{y}),
\end{aligned}$$

where φ denotes the projection functional of ℓ^∞ onto the first entry, that is, $\varphi(\mathbf{x}) = x_1$ for all $\mathbf{x} = (x_1, x_2, x_3, \dots) \in \ell^\infty$, and the function $h : \ell^\infty \times \ell^\infty \longrightarrow \ell^\infty$ assigns to each pair $(\mathbf{x}, \mathbf{y})$ of bounded sequences the sequence $h(\mathbf{x}, \mathbf{y}) \in \ell^\infty$ whose k-th entry is

$$(x_{k+1} - x_1)\Delta_k(\mathbf{y}) + (y_k - y_1)\Delta_k(\mathbf{x})$$

for all $k \in \mathbb{N}$. Since the operative equation (4.2) holds with these choices of φ and h, it remains to establish that the conditions (4.1) are satisfied. To this end, let $\mathbf{x}, \mathbf{y} \in T^{-1}(\ell^1)$, and consider the following computations which are motivated by those in [8].

For any integer $n \geq 2$,

$$\begin{aligned}
&\sum_{k=1}^{n-1} |(x_{k+1}-x_1)\Delta_k(\mathbf{y}) + (y_k - y_1)\Delta_k(\mathbf{x})| \\
&\leq \sum_{k=1}^{n-1} |(x_{k+1}-x_1)\,\Delta_k(\mathbf{y})| + \sum_{k=1}^{n-1} |(y_k - y_1)\,\Delta_k(\mathbf{x})| \\
&= \sum_{k=1}^{n-1} \Big|\Big(\sum_{j=1}^{k} \Delta_j(\mathbf{x})\Big)\Delta_k(\mathbf{y})\Big| + \sum_{k=1}^{n-1} \Big|\Big(\sum_{j=1}^{k-1} \Delta_j(\mathbf{y})\Big)\Delta_k(\mathbf{x})\Big| \\
&\leq \sum_{k=1}^{n-1}\sum_{j=1}^{k} |\Delta_j(\mathbf{x})|\,|\Delta_k(\mathbf{y})| + \sum_{k=1}^{n-1}\sum_{j=1}^{k-1} |\Delta_j(\mathbf{y})|\,|\Delta_k(\mathbf{x})| \\
&= \sum_{j=1}^{n-1}\sum_{k=j}^{n-1} |\Delta_j(\mathbf{x})|\,|\Delta_k(\mathbf{y})| + \sum_{k=1}^{n-1}\sum_{j=1}^{k-1} |\Delta_j(\mathbf{y})|\,|\Delta_k(\mathbf{x})| \\
&= \sum_{k=1}^{n-1}\sum_{j=k}^{n-1} |\Delta_k(\mathbf{x})|\,|\Delta_j(\mathbf{y})| + \sum_{k=1}^{n-1}\sum_{j=1}^{k-1} |\Delta_j(\mathbf{y})|\,|\Delta_k(\mathbf{x})| \\
&= \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} |\Delta_k(\mathbf{x})|\,|\Delta_j(\mathbf{y})| \\
&= \sum_{k=1}^{n-1} |\Delta_k(\mathbf{x})| \sum_{j=1}^{n-1} |\Delta_j(\mathbf{y})| \\
&\leq \|T(\mathbf{x})\|_1 \|T(\mathbf{y})\|_1.
\end{aligned}$$

Consequently, the image $h(\mathbf{x},\mathbf{y})$ of any pair of sequences $(\mathbf{x},\mathbf{y}) \in T^{-1}(\ell^1) \times T^{-1}(\ell^1)$ lies in ℓ^1 as required by (4.1) and, moreover, the estimate $\|h(\mathbf{x},\mathbf{y})\|_1 \leq \|T(\mathbf{x})\|_1\|T(\mathbf{y})\|_1$ holds. Theorem 4.1 thus implies that, for any choice of $m \geq 1$, the definition

$$\rho_{T,m}(\mathbf{x}) = |\varphi(\mathbf{x})| + m\|T\mathbf{x}\|_A = |x_1| + m\sum_{n=1}^{\infty} |x_{n+1}-x_n| \quad \text{for all } \mathbf{x} \in T^{-1}(\ell^1)$$

provides a seminorm for the algebra $bv = T^{-1}(\ell^1)$ of *sequences of bounded variation*. Actually, if $\mathbf{x} \in T^{-1}(\ell^1)$ satisfies $\rho_{T,m}(\mathbf{x}) = 0$, then $\mathbf{x} \in \ker(\varphi) \cap \ker(T) = \{\mathbf{0}\}$ so that, in fact, the function $\rho_{T,m}(\mathbf{x})$ is a norm for the algebra $bv = T^{-1}(\ell^1)$.

Note that, without an appropriate choice for the factor m, the norm $\rho_{T,m}$ is not, in general, submultiplicative. Consider, for example, the sequence $\mathbf{x} = (0, 1, \frac{1}{2}, 0, 0, \dots) \in \ell^\infty$ and let $m = \frac{1}{4}$. Then $\mathbf{x}^2 = (0, 1, \frac{1}{4}, 0, 0, \dots)$ so that

$$\rho_{T,m}(\mathbf{x}^2) = 0 + \frac{1}{4}\left(1 + \frac{3}{4} + \frac{1}{4}\right) = \frac{1}{2} \quad \text{and} \quad \rho_{T,m}(\mathbf{x}) = 0 + \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Thus, $\rho_{T,m}(\mathbf{x}^2) = \frac{1}{2} > \frac{1}{4} = \big(\rho_{T,m}(\mathbf{x})\big)^2$ which confirms that $\rho_{T,m}$ is not submultiplicative. □

4.1. Paralleling the situation in subsection 3.1, suppose that $T_\alpha : X \longrightarrow Y$ is a linear operator for each α in the index set J and that $(T_\alpha^{-1}(A), \rho_{T_\alpha,m_\alpha})$ is a seminormed subalgebra of X. Let

$$W = \{u \in \bigcap_{\alpha\in J} T_\alpha^{-1}(A) : \sup_{\alpha\in J} m_\alpha \|T_\alpha(u)\|_A < \infty\}, \tag{4.3}$$

and define

$$\rho_W(u) = \sup_{\alpha\in J} \rho_{T_\alpha,m_\alpha}(u) = |\varphi(u)| + \sup_{\alpha\in J} m_\alpha \|T_\alpha(u)\|_A \quad \text{for all } u \in W. \tag{4.4}$$

It is straightforward to confirm that (W, ρ_W) is a seminormed algebra.

Example 4.2. LIPSCHITZ FUNCTIONS - REDUX. Let X denote the algebra $\mathrm{Bd}[a, b]$, and let $(Y, \|\cdot\|_Y)$ and $(A, \|\cdot\|_A)$ both denote the Banach algebra $(\mathbb{C}^2, \|\cdot\|_\infty)$. For a fixed $\delta \in (0, 1]$ and a pair $\alpha = (x, y)$ of numbers that satisfy $a < x < y \le b$, define $T_\alpha : \mathrm{Bd}[a, b] \longrightarrow (\mathbb{C}^2, \|\cdot\|_\infty)$ by

$$T_\alpha(f) = \left(\frac{f(x) - f(a)}{(x - a)^\delta}, \frac{f(y) - f(x)}{(y - x)^\delta}\right) \quad \text{for all } f \in \mathrm{Bd}[a, b].$$

It follows that, for $f, g \in \mathrm{Bd}[a, b]$,

$$\begin{aligned}
T_\alpha(fg) &= \left(\frac{fg(x) - fg(a)}{(x - a)^\delta}, \frac{fg(y) - fg(x)}{(y - x)^\delta}\right)\\
&= \left(f(x)\frac{g(x) - g(a)}{(x - a)^\delta} + g(a)\frac{f(x) - f(a)}{(x - a)^\delta}, f(y)\frac{g(y) - g(x)}{(y - x)^\delta} + g(x)\frac{f(y) - f(x)}{(y - x)^\delta}\right)\\
&= (f(x), f(y))T_\alpha(g) + (g(a), g(x))T_\alpha(f)\\
&= (f(x) - f(a), f(y) - f(a))T_\alpha(g) + (f(a), f(a))T_\alpha(g)\\
&\qquad + (g(a), g(a))T_\alpha(f) + (0, g(x) - g(a))T_\alpha(f)\\
&= \varphi(f)T_\alpha(g) + \varphi(g)T_\alpha(f) + h_\alpha(f, g),
\end{aligned}$$

where the functional φ on $\mathrm{Bd}[a, b]$ represents point evaluation at a and

$$h_\alpha(f, g) = (f(x) - f(a), f(y) - f(a))T_\alpha(g) + (0, g(x) - g(a))T_\alpha(f)$$

for all $(f, g) \in \mathrm{Bd}[a, b] \times \mathrm{Bd}[a, b]$. The functions φ and h thus implement the operative equation (4.2), and we can estimate $\|h_\alpha(f, g)\|_\infty$ by:

$$\begin{aligned}
&\|(f(x) - f(a), f(y) - f(a))T_\alpha(g) + (0, g(x) - g(a))T_\alpha(f)\|_\infty\\
&= \|(f(x) - f(a), f(y) - f(x))T_\alpha(g) + (0, f(x) - f(a))T_\alpha(g) + (0, g(x) - g(a))T_\alpha(f)\|_\infty\\
&\le \|\big((x - a)^\delta, (y - x)^\delta\big)\|_\infty \|T_\alpha(f)\|_\infty \|T_\alpha(g)\|_\infty + 2(x - a)^\delta \|T_\alpha(f)\|_\infty \|T_\alpha(g)\|_\infty\\
&\le 3(b - a)^\delta \|T_\alpha(f)\|_\infty \|T_\alpha(g)\|_\infty
\end{aligned}$$

Conditions (4.1) are thus satisfied so that by Theorem 4.1, for any number $m_\alpha \ge 3(b - a)^\delta$, the function ρ_{T_α,m_α} given by

$$\rho_{T_\alpha,m_\alpha}(f) = |\varphi(f)| + m_\alpha\|T_\alpha(f)\|_\infty = |f(a)| + m_\alpha \max\left\{\frac{|f(x) - f(a)|}{(x - a)^\delta}, \frac{|f(y) - f(x)|}{(y - x)^\delta}\right\}$$

for all $f \in T_\alpha^{-1}(\mathbb{C}^2)$ provides a submultiplicative seminorm for the algebra $T_\alpha^{-1}(\mathbb{C}^2) = \mathrm{Bd}[a, b]$. Note that without an appropriate factor m_α, the norm ρ_{T_α,m_α} is not, in general, submultiplicative. Consider, for example, the pair of numbers $\alpha = (\frac{1}{2}, 1)$ from the interval $[0, 1]$, the choice $m_\alpha = 1$, and the resulting function

$$\rho_\alpha(f) = |f(0)| + \max\left\{\frac{|f(\frac{1}{2}) - f(0)|}{\frac{1}{2} - 0}, \frac{|f(1) - f(\frac{1}{2})|}{1 - \frac{1}{2}}\right\} \quad \text{for all } f \in \mathrm{Bd}[0, 1].$$

Then, for $f : [0, 1] \longrightarrow \mathbb{R}$ defined by $f(t) = t$ for all $t \in [0, 1]$, we find $\rho_\alpha(f^2) = 3/2 > 1 = (\rho(f))^2$ which confirms that $\rho_\alpha(f)$ is not submultiplicative on $\mathrm{Bd}[0, 1]$.

Now, if $m_\alpha = 3(b-a)^\delta$, say, for all pairs α from the set $J = \{(x,y) : a < x < y \le b\}$, then, apropos of equations (4.3) and (4.4), the function

$$\rho_W(f) = \sup_{\alpha\in J} \rho_{T_\alpha, m_\alpha}(f) = |f(a)| + 3(b-a)^\delta \sup_{\alpha\in J}\left\{\frac{|f(x)-f(a)|}{(x-a)^\delta}, \frac{|f(y)-f(x)|}{(y-x)^\delta}\right\} \tag{4.5}$$

provides a submultiplicative seminorm for the subalgebra W of $\mathrm{Bd}[a,b]$ consisting of those $f \in \mathrm{Bd}[a,b]$ for which

$$\sup_{\alpha\in J}\left\{\frac{|f(x)-f(a)|}{(x-a)^\delta}, \frac{|f(y)-f(x)|}{(y-x)^\delta}\right\} < \infty.$$

Evidently, W is thus the subalgebra of $\mathrm{Bd}[a,b]$ consisting of those $f \in \mathrm{Bd}[a,b]$ for which

$$\sup\left\{\frac{|f(y)-f(x)|}{|y-x|^\delta} : x \neq y\right\} < \infty,$$

that is, $W = \mathrm{Lip}_\delta[a,b]$ and

$$\rho_W(f) = |f(a)| + 3(b-a)^\delta \sup\left\{\frac{|f(y)-f(x)|}{|y-x|^\delta} : x \neq y\right\}.$$

Moreover, one may easily check that if $\rho_W(f) = 0$, then $f \equiv 0$ on $[a,b]$, so that ρ_W actually provides an algebra norm for $\mathrm{Lip}_\delta[a,b]$. $\square$

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