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SOME NOTES ON THE SEMI-OPEN SUBSPACES OF TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we obtain some new results regarding to the nowhere dense and first category set in the semi-open subspace of a topological space. More precisely, we prove that a nowhere dense set in the semi-open subspace of a topological space is equivalent as a nowhere dense set in that topological space. This implies that a first category set in the semi-open subspace of a topological space is equivalent as a first category set in that topological space. We also give some applications of these results to give some new proofs relating to the properties of semi-open set and Baire space.

Key words and phrases: Semi-open subspace; Nowhere dense set; First category set; Second category set.

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1. INTRODUCTION

Let (X,τ) be a topological space, $Y \subseteq X$, and $\tau_Y = \{Y \cap U : U \in \tau\}$. We call (Y,τ_Y) is a **subspace** of (X,τ). Note that, (Y,τ_Y) is also a topological space. A set $Z \in \tau$ (resp. $Z \in \tau_Y$) is called **open** set in (X,τ) (resp. open set in (Y,τ_Y)). Meanwhile, a set $F \subseteq X$ (resp. $F \subseteq Y$) is called **closed** set in (X,τ) (resp. closed set in (Y,τ_Y)) if $X \backslash F \in \tau$ (resp. $Y \backslash F \in \tau_Y$).

Given $Z \subseteq Y \subseteq X$. The **interior** of Z in (X,τ) (resp. in (Y,τ_Y)), denoted by $int_\tau(Z)$ (resp. by $int_{\tau_Y}(Z)$), is defined as the union of all open sets in (X,τ) (resp. in (Y,τ_Y)), contained in Z. The **closure** of Z in (X,τ) (resp. in (Y,τ_Y)), denoted by $cl_\tau(Z)$ (resp. by $cl_{\tau_Y}(Z)$), is defined as the intersection of all closed sets in (X,τ) (resp. in (Y,τ_Y)) containing Z. Using de Morgan's law, we can prove that for every $Z \subseteq X$, $X \backslash int_\tau(Z) = cl_\tau(X \backslash Z)$ and $X \backslash cl_\tau(Z) = int_\tau(X \backslash Z)$. It also can be proved that $x \in cl_\tau(Z)$ if and only if for every $U \in \tau \backslash \{\emptyset\}$ such that $x \in U$, then $U \cap Z \neq \emptyset$.

The following is the defintion of semi-open set, introduced by [1], in a topological space.

Definition 1.1. Let (Y,τ_Y) be a subspace of (X,τ) and $Z \subseteq X$ (resp. $Z \subseteq Y$). The set Z is called **semi-open** in (X,τ) (resp. in (Y,τ_Y)) if $Z \subseteq cl_\tau(int_\tau(Z))$ (resp. $Z \subseteq cl_{\tau_Y}(int_{\tau_Y}(Z))$). The set of all semi-open set in (X,τ) (resp. in (Y,τ_Y)) is denoted by $SO(X,\tau)$ (resp. $SO(Y,\tau_Y)$). If $Y \in SO(X,\tau)$, we call (Y,τ_Y) is a **semi-open subspace** of (X,τ).

For every topological spaces (X,τ), it is easy to show that $\tau \subseteq SO(X,\tau)$. This inclusion can be a proper inclusion. Furthermore, $Z \in SO(X,\tau) \backslash \{\emptyset\}$ if and only if there exists $O \in \tau \backslash \{\emptyset\}$ such that $O \subseteq Z \subseteq cl_\tau(O)$ (the proof similar to [1, Theorem 1]). The following definition is well known and can be seen, for example, in [2].

Definition 1.2. Let (Y,τ_Y) be a subspace of (X,τ) and $Z \subseteq X$ (resp. $Z \subseteq Y$).

(1) Z is called **dense** in (X,τ) (reps. in (Y,τ_Y)) if $cl_\tau(Z) = X$ (resp. $cl_{\tau_Y}(Z) = Y$).
(2) Z is called **nowhere dense** in (X,τ) (resp. in (Y,τ_Y)) if $int_\tau(cl_\tau(Z)) = \emptyset$ (resp. $int_{\tau_Y}(cl_{\tau_Y}(Z)) = \emptyset$).
(3) Z is called **first category** in (X,τ) (resp. in (Y,τ_Y)) if $Z = \cup_{n\in\mathbb{N}} Z_n$, where Z_n is nowhere dense in (X,τ) (resp. in (Y,τ_Y)) for every $n \in \mathbb{N}$.
(4) Z is called **second category** in (X,τ) (resp. in (Y,τ_Y)) if Z is not first category in (X,τ) (resp. in (Y,τ_Y)).

Given a topological space (X,τ), $Y \in SO(X,\tau)$, and $Z \subseteq Y$. In this paper, we will prove that Z is nowhere dense in (Y,τ_Y) if and only if Z is nowhere dense in (X,τ). As a result of this property, we have that Z is first category (resp. second category set) in (Y,τ_Y) if and only if Z is first category (resp. second category set) in (X,τ). To the best of the authors knowledge, these results are new. Furthermore, these results we use to provide simple proofs of some Baire space properties obtained in [5].

2. RESULTS AND DISCUSSION

In this section, we will discuss the main results of this paper. We start with the simple lemma below, which we believe is classical, but we can not find a reference that proved it formally.

Lemma 2.1. *Let (X,τ) be a topological space and $Z \subseteq X$. The set Z is nowhere dense in (X,τ) if and only if there exists $W \in \tau$ and W is dense in (X,τ) such that $W \subseteq X \backslash Z$.*

Proof. Let Z be a nowhere dense in (X,τ). Then $int_\tau(cl_\tau(Z)) = \emptyset$. So,

$$X = X \backslash \emptyset = X \backslash int_\tau(cl_\tau(Z)) = cl_\tau(X \backslash cl_\tau(Z)) = cl_\tau(int_\tau(X \backslash Z)).$$

Setting $W = int_\tau(X\backslash Z)$. Observe that $W \in \tau$ and W is dense in (X, τ) such that $W \subseteq X\backslash Z$.

Let W satisfies the assumption above. We have $X = cl_\tau(W) = cl_\tau(int_\tau(W)) = cl_\tau(int_\tau(X\backslash Z))$. Therefore

$$\emptyset = X\backslash X = X\backslash cl_\tau(int_\tau(X\backslash Z)) = int_\tau(X\backslash int_\tau(X\backslash Z)) = int_\tau(cl_\tau(X\backslash(X\backslash Z))) = int_\tau(cl_\tau(Z)).$$

This means Z is nowhere dense in (X, τ). ∎

Lemma 2.2. *Let (X, τ) be a topological space, $Y \in SO(X, \tau)$, and $Z \subseteq Y$. The set $Z \in SO(X, \tau)$ if and only if $Z \in SO(Y, \tau_Y)$.*

It is stated in [5, 6] that Lemma 2.2 belongs to [7]. However, we will prove this lemma since we do not have access to the paper [7]. Before we prove it, we need the following property.

Lemma 2.3. *Let (X, τ) be a topological space, $Y \in SO(X, \tau)$, and $O \subseteq Y$. If $O \in \tau_Y$, then $O \in SO(X, \tau)$.*

Proof. Since $O \in \tau_Y$, then $O = E \cap Y$, for some $E \in \tau$. So,

$$int_\tau(O) = int_\tau(E \cap Y) = int_\tau(E) \cap int_\tau(Y) = E \cap int_\tau(Y),$$

which gives us

$$cl_\tau(E \cap int_\tau(Y)) = cl_\tau(int_\tau(O)). \tag{2.1}$$

By the assumption $Y \in SO(X, \tau)$, we have

$$O = E \cap Y \subseteq E \cap cl_\tau(int_\tau(Y)). \tag{2.2}$$

We claim that

$$E \cap cl_\tau(int_\tau(Y)) \subseteq cl_\tau(E \cap int_\tau(Y)). \tag{2.3}$$

Let $x \in E \cap cl_\tau(int_\tau(Y))$. For every $U \in \tau$ such that $x \in U$, we have $x \in E \cap U \in \tau$. Since $x \in cl_\tau(int_\tau(Y))$, then $E \cap U \cap int_\tau(Y) \neq \emptyset$. We conclude that $x \in cl_\tau(E \cap int_\tau(Y))$ (see [3, p.13]) and the claim is proved. Combining (2.2), (2.3), and (2.1), we obtain

$$O \subseteq E \cap cl_\tau(int_\tau(Y)) \subseteq cl_\tau(E \cap int_\tau(Y)) = cl_\tau(int_\tau(O)).$$

The lemma is proved. ∎

Now, we ready to prove Lemma 2.2.

Proof of Lemma 2.2. Let $Z \in SO(X, \tau)$. Then $Z \subseteq cl_\tau(int_\tau(Z))$. Since $int_\tau(Z) \subseteq Z \subseteq Y$, we have $int_\tau(Z) \cap Y = int_\tau(Z)$. Using the fact that $int_\tau(Z) \cap Y \in \tau_Y$, we also have $int_\tau(Z) \cap Y \subseteq int_{\tau_Y}(Z)$. Whence $int_\tau(Z) \subseteq int_{\tau_Y}(Z)$ which gives us $cl_\tau(int_\tau(Z)) \subseteq cl_\tau(int_{\tau_Y}(Z))$. Thus,

$$Z = Z \cap Y \subseteq cl_\tau(int_\tau(Z)) \cap Y \subseteq cl_\tau(int_{\tau_Y}(Z)) \cap Y = cl_{\tau_Y}(int_{\tau_Y}(Z))$$

which we obtained by combining all previous informations and the property in [2, p.77] or [3, p.66]. Therefore $Z \in SO(Y, \tau_Y)$. Observe that we do not use the assumption $Y \in SO(X, \tau)$ for the proof of this necessary condition.

Let $Z \in SO(Y, \tau_Y)$. We may assume $Z \neq \emptyset$. Then there exists $O \in \tau_Y\backslash\{\emptyset\}$ such that $O \subseteq Z \subseteq cl_{\tau_Y}(O)$. According to Lemma 2.3, we have $O \in SO(X, \tau)$, which means $cl_\tau(O) \subseteq cl_\tau(int_\tau(O))$. Since $O \subseteq Z$, we also have $cl_\tau(int_\tau(O)) \subseteq cl_\tau(int_\tau(Z))$. By using the property in [2, p.77] or [3, p.66], we conclude that

$$Z \subseteq cl_{\tau_Y}(O) = cl_\tau(O) \cap Y \subseteq cl_\tau(int_\tau(O)) \subseteq cl_\tau(int_\tau(Z)).$$

Thus, $Z \in SO(X, \tau)$. ∎

Now we will investigate some properties which hold in semi-open subspace and also hold in topological space contains that semi-open subspace.

Theorem 2.4. *Let (X,τ) be a topological space, $Y \in SO(X,\tau)$, and $Z \subseteq Y$. The set Z is nowhere dense in (Y,τ_Y) if and only if Z is nowhere dense in (X,τ).*

Proof. Let Z be a nowhere dense in (Y,τ_Y). According to Lemma 2.1, there exists $W_0 \in \tau_Y$ and W_0 is dense in (Y,τ_Y), such that $W_0 \subseteq Y\backslash Z$. Let $W = int_\tau(W_0 \cup (X\backslash Y))$. Then $W \in \tau$.

We will show that W is dense in (X,τ). Suppose that $O \in \tau\backslash\{\emptyset\}$. If $O \cap Y = \emptyset$, then $O \subseteq X\backslash Y \subseteq W_0 \cup (X\backslash Y)$. This means $O \subseteq W$ and also $O \cap W \neq \emptyset$. On the other hand, if $O \cap Y \neq \emptyset$, then $O \cap Y \in \tau_Y\backslash\{\emptyset\}$. Since W_0 is dense in (Y,τ_Y), then $O \cap Y \cap W_0 \neq \emptyset$. Since $O \cap Y \cap W_0 \in \tau_Y$, then $O \cap Y \cap W_0 \in SO(Y,\tau_Y)$. By virtue of the assumption $Y \in SO(X,\tau)$ and Lemma 2.2, we obtain $O \cap Y \cap W_0 \in SO(X,\tau)$. Hence, there exists $O' \in \tau\backslash\{\emptyset\}$ such that $O' \subseteq O \cap Y \cap W_0 \subseteq O$. We also have $O' \subseteq W$ by the definition of W. Therefore, $O' \subseteq O \cap W$ and also $O \cap W \neq \emptyset$. We conclude from these two cases, W is dense in (X,τ).

Notice that $W_0 \subseteq Y\backslash Z \subseteq X\backslash Z$ and $X\backslash Y \subseteq X\backslash Z$. Then $W_0 \cup (X\backslash Y) \subseteq X\backslash Z$. Thus $W \subseteq X\backslash Z$. From this and the facts that $W \in \tau$ and W is dense in (X,τ), by Lemma 2.1, we conclude that Z is nowhere dense in (X,τ).

For the proof of the converse, we let Z be a nowhere dense in (X,τ). Then $int_t(cl_\tau(Z)) = \emptyset$. Since $cl_{\tau_Y}(Z) = cl_\tau(Z) \cap Y \subseteq cl_\tau(Z)$, then $int_{\tau_Y}(cl_{\tau_Y}(Z)) \subseteq cl_\tau(Z)$. Clearly, $int_{\tau_Y}(cl_{\tau_Y}(Z)) \in SO(Y,\tau_Y)$. Since $Y \in SO(X,\tau)$, by Lemma 2.2, we have $int_{\tau_Y}(cl_{\tau_Y}(Z)) \in SO(X,\tau)$. Hence,

$$int_{\tau_Y}(cl_{\tau_Y}(Z)) \subseteq cl_\tau(int_\tau(int_{\tau_Y}(cl_{\tau_Y}(Z)))) \subseteq cl_\tau(int_\tau(cl_\tau(Z))) = int_\tau(\emptyset) = \emptyset.$$

This tells us that Z is nowhere dense in (Y,τ_Y). ∎

As a result of Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let (X,τ) be a topological space, $Y \in SO(X,\tau)$, and $Z \subseteq Y$. The set Z is first category in (Y,τ_Y) if and only if Z is first category in (X,τ).*

Proof. Let Z be a first category in (Y,τ_Y). Then $Z = \cup_{n\in\mathbb{N}} Z_n$, where Z_n is nowhere dense in (Y,τ_Y) for every $n \in \mathbb{N}$. We have, Z_n is nowhere dense in (X,τ) for every $n \in \mathbb{N}$, according to Theorem 2.4. Thus, we have proved that Z is first category in (X,τ).

Let Z be a first category in (X,τ). We have $Z = \cup_{n\in\mathbb{N}} Z_n$, where Z_n is nowhere dense in (X,τ) for every $n \in \mathbb{N}$. Therefore, Z_n is nowhere dense in (Y,τ_Y) for every $n \in \mathbb{N}$, by using Theorem 2.4. Thus, Z is first category in (Y,τ_Y). ∎

One of immediate consequences of Corollary 2.5 is the following property.

Corollary 2.6. *Let (X,τ) be a topological space, $Y \in SO(X,\tau)$, and $Z \subseteq Y$. The set Z is second category in (Y,τ_Y) if and only if Z is second category in (X,τ).*

3. SEMI-OPEN SET AND BAIRE SPACES

In this section, we give some applications of Corollary 2.5 to study the properties of Baire space. All theorems in this section are stated in [5]. However, we give a different and simple proof in that theorems. We first recall the definition of Baire space.

Definition 3.1. Let (Y,τ_Y) be a subspace of (X,τ). The space (X,τ) (resp. (Y,τ_Y)) is called **Baire space** if for every $\{E_n : n \in \mathbb{N}\} \subseteq \tau$ (resp. $\{E_n : n \in \mathbb{N}\} \subseteq \tau_Y$) such that $cl_\tau(E_n) = X$ (resp. $cl_{\tau_Y}(E_n) = Y$) for every $n \in \mathbb{N}$, then $cl_\tau(\cap_{n\in\mathbb{N}} E_n) = X$ (resp. $cl_{\tau_Y}(\cap_{n\in\mathbb{N}} E_n) = Y$). We also call (Y,τ_Y) is a **Baire subspace** of (X,τ).

The Baire space (X, τ) can be characterized as follows: (X, τ) is a Baire space if and only if for every $Z \in \tau \backslash \{\emptyset\}$, then Z is second category in (X, τ). This classical property stated in [4, p.295]. The following theorem generalized the previous characterization.

Theorem 3.1. [5, Theorem 2.8] *Let (X, τ) be a topological space, (X, τ) is a Baire space if and only if for every $Z \in SO(X, \tau) \backslash \{\emptyset\}$, then Z is second category in (X, τ).*

Proof. Let $Z \in SO(X, \tau) \backslash \{\emptyset\}$. Suppose that Z is first category in (X, τ). Then $Z = \cup_{n \in \mathbb{N}} Z_n$, where $int_\tau(cl_\tau(Z_n)) = \emptyset$ for every $n \in \mathbb{N}$. Therefore, $cl_\tau(int_\tau(X \backslash Z_n)) = X$ for every $n \in \mathbb{N}$. Since (X, τ) is a Baire space, then $cl_\tau \left(\cap_{n \in \mathbb{N}} int_\tau(X \backslash Z_n) \right) = X$. Observe that,

$$\begin{aligned} \emptyset &= X \backslash cl_\tau \left(\cap_{n \in \mathbb{N}} int_\tau(X \backslash Z_n) \right) = int_\tau \left(X \backslash \left(\cap_{n \in \mathbb{N}} int_\tau(X \backslash Z_n) \right) \right) = int_\tau \left(\cup_{n \in \mathbb{N}} \left(X \backslash int_\tau(X \backslash Z_n) \right) \right) \\ &= int_\tau \left(\cup_{n \in \mathbb{N}} cl_\tau(X \backslash (X \backslash Z_n)) \right) = int_\tau \left(\cup_{n \in \mathbb{N}} cl_\tau(Z_n) \right) \supseteq int_\tau \left(\cup_{n \in \mathbb{N}} Z_n \right) = int_\tau(Z). \end{aligned}$$

By assumption $Z \in SO(X, \tau) \backslash \{\emptyset\}$, we have $\emptyset \neq Z \subseteq cl_\tau(int_\tau(Z)) \subseteq \emptyset$. This is a contradiction. Therefore, we must have Z is second category in (X, τ).

The converse of this theorem can be proved by similar method in [4, pp.295-296]. ∎

The remaining theorems in this section proved by application of Corrollary 2.5.

Theorem 3.2. [5, Theorem 2.9] *If (X, τ) is a Baire space, then for every $Y \in SO(X, \tau)$, (Y, τ_Y) is also a Baire space.*

Proof. Let $Z \in SO(Y, \tau_Y) \backslash \{\emptyset\}$. Suppose that Z is first category in (Y, τ_Y). According to Lemma 2.2 and Corollary 2.5, we obtain $Z \in SO(X, \tau) \backslash \{\emptyset\}$ and Z is first category in (X, τ). This contradicts to (X, τ) is Baire space, by virtue of Theorem 3.1. ∎

Definition 3.2. Let $\{U_\alpha : \alpha \in I\}$ be a collection of subsets of X. The collection $\{U_\alpha : \alpha \in I\}$ is called **pseudo-cover** for (X, τ) if $cl_\tau(\cup_{\alpha \in I} U_\alpha) = X$. If $\{U_\alpha : \alpha \in I\} \subseteq SO(X, \tau)$, we call $\{U_\alpha : \alpha \in I\}$ a **semi-open pseudo-cover** for (X, τ).

Theorem 3.3. [5, Theorem 2.10] *If there exist a semi-open pseudo-cover for (X, τ) such that its member is a Baire subspace of (X, τ), then (X, τ) is a Baire space.*

Proof. Let $Z \in SO(X, \tau) \backslash \{\emptyset\}$. According to the assumption, there exists $\{U_\alpha : \alpha \in I\}$, such that it is a semi-open pseudo-cover for (X, τ) and $(U_\alpha, \tau_{U_\alpha})$ is a Baire subspace of (X, τ), for every $\alpha \in I$. Suppose that Z is first category in (X, τ). Then, there exists $\beta \in I$ such that $Z \cap U_\beta \neq \emptyset$. It is clear that $Z \cap U_\beta$ is first category in (X, τ). From Lemma 2.2, we obtain $Z \in SO(U_\beta, \tau_{U_\beta})$. Using Corollary 2.5, we conclude that $Z \cap U_\beta$ is first category in $(U_\beta, \tau_{U_\beta})$. Since $(U_\beta, \tau_{U_\beta})$ is a Baire space and by virtue of Theorem 3.1, we have a contradiction. ∎

4. CONCLUSIONS

Let (X, τ) be a topological space, $Y \in SO(X, \tau)$, and $Z \subseteq Y$. In this paper, we have proved that Z is nowhere dense in (Y, τ_Y) if and only if Z is nowhere dense in (X, τ). This implies that Z is first category (resp. second category) in (Y, τ_Y) if and only if Z is first category (resp. second category) in (X, τ). As a consequences, we obtain some new, but simple, proof regarding to the properties of Baire spaces.

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