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SOME MODULI AND INEQUALITIES RELATED TO BIRKHOFF ORTHOGONALITY IN BANACH SPACES

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ABSTRACT. In this paper, we shall consider two new constants $\delta_B(X)$ and $\rho_B(X)$, which are the modulus of convexity and the modulus of smoothness related to Birkhoff orthogonality, respectively. The connections between these two constants and other well-known constants are established by some equalities and inequalities. Meanwhile, we obtain two characterizations of Hilbert spaces in terms of these two constants, study the relationships between the constants $\delta_B(X)$, $\rho_B(X)$ and the fixed point property for nonexpansive mappings. Furthermore, we also give a characterization of the Radon plane with affine regular hexagonal unit sphere.

Key words and phrases: Birkhoff orthogonality; modulus of convexity; modulus of smoothness; Radon plane.

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1. Introduction

Let X be a Banach space with $\dim X \geq 2$, we denote the unit ball and the unit sphere by B_X and S_X, respectively.

Recall that the space X is called uniformly convex [9], if, for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $x, y \in S_X$ with $\|x - y\| \geq \epsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

The modulus of convexity of X is defined in [9] by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| = \epsilon\right\},\ \epsilon \in [0, 2].$$

The characteristic of convexity, $\epsilon_0(X) = \sup\{\epsilon \in [0, 2] : \delta_X(\epsilon) = 0\}$, is defined in [15]. The space X is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, or equivalently $\epsilon_0(X) = 0$. The meaning of the modulus of convexity is: if we take $x, y \in S_X$ far apart, then this modulus measures "how far" the middle point of the segment joining them must be from S_X.

Later, there have appeared different kinds of expressions of $\delta_X(\epsilon)$ in [27]:

$$\begin{aligned}\delta_X(\epsilon) &= \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon\right\}\\ &= \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \epsilon\right\}.\end{aligned}$$

Recall that the space X is called uniformly smooth [10], if, for any $\epsilon > 0$, there exists $\delta > 0$, such that if $x \in S_X$ and $\|y\| \leq \delta$, then

$$\|x + y\| + \|x - y\| < 2 + \epsilon\|y\|.$$

If instead one wants to measure "how far" the same point can be from S_X, Banaś [4] considered the following parameter:

$$\rho_X(\epsilon) = \sup\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| = \epsilon\right\}.$$

It turns out to be a "modulus of smoothness". For more results about the modulus of convexity and the modulus of smoothness, we refer the reader to [6, 11, 21].

Let x, y be two elements in a Hilbert space. Then an element $x \in X$ is said to be orthogonal to $y \in X$ (denoted by $x \perp y$) if $\langle x, y\rangle = 0$. In the general setting of Banach space, many notions of orthogonality have been introduced by means of equivalent propositions to the usual orthogonality in Hilbert spaces. For example, Roberts [24] introduced Roberts orthogonality: for any $x, y \in X$, x is said to be Roberts orthogonal to y (denoted by $x \perp_R y$), if $\|x + \lambda y\| = \|x - \lambda y\|$ for any $\lambda \in \mathbb{R}$; Birkhoff [8] introduced Birkhoff orthogonality: for any $x, y \in X$, x is said to Birkhoff orthogonal to y (denoted by $x \perp_B y$), if $\|x + \lambda y\| \geq \|x\|$ for any $\lambda \in \mathbb{R}$; James [16] introduced isosceles orthogonality: x is said to be isosceles orthogonal to y (denoted by $x \perp_I y$), if $\|x + y\| = \|x - y\|$.

Recall that the space X is called uniformly non-square (see [17]), if there exists $\delta > 0$ such that either $\frac{\|x-y\|}{2} \leq 1 - \delta$, or $\frac{\|x+y\|}{2} \leq 1 - \delta$.

The James constant $J(X)$ and the Schäffer constant $S(X)$ are defined in [13] as follows:

$$J(X) = \sup\{\min\{\|x + y\|, \|x - y\|\} : x, y \in S_X\},$$

$$S(X) = \inf\{\max\{\|x + y\|, \|x - y\|\} : x, y \in S_X\}.$$

Recently, Baronti and Papini [7] introduced the following constants:

$$J_B(X) = \sup\{\min\{\|x + y\|, \|x - y\|\} : x, y \in S_X, x \perp_B y\},$$

$$S_B(X) = \inf\{\max\{\|x+y\|, \|x-y\|\} : x, y \in S_X, x \perp_B y\}.$$

Moreover, the various properties of these constants are given in [7, 13, 19]:

(1) $1 \leq S(X) \leq S_B(X) \leq J_B(X) \leq J(X) \leq 2$.

(2) $J(X)S(X) = 2$.

(3) X is not uniformly non-square if and only if $J(X) = 2$.

(4) X is not uniformly non-square if and only if $J_B(X) = 2$.

(5) X is not uniformly non-square if and only if $S_B(X) = 1$.

Inspired by the excellent works mentioned above, we shall consider the following constants in this paper:

$$\delta_B(X) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\},$$

$$\rho_B(X) = \sup\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\}.$$

The meanings of these moduli related to Birkhoff orthogonality: if we take $x, y \in S_X$ such that $x \perp_B y$, then these moduli measure "how far" the middle point of the segment joining them must be from S_X.

The arrangement of this paper is as follows:

In Section 2, we consider the constant $\delta_B(X)$. First, we give some relationships between $\delta_B(X)$ and other well-known constants by some inequalities. Second, we obtain a characterization of Hilbert spaces in terms of $\delta_B(X)$, and establish the relationship between $\delta_B(X)$ and the fixed point property for nonexpansive mappings. Finally, we study $\delta_B(X)$ in Radon planes. The bounds of $\delta_B(X)$ in Radon planes are given. Moreover, we use the lower bound to characterize the Radon plane with affine regular hexagonal unit sphere.

In Section 3, the constant $\rho_B(X)$ is considered. First, some relations between $\rho_B(X)$ and other well-known constants are presented by some equalities and inequalities. Second, we give a characterization of Hilbert spaces in terms of $\rho_B(X)$, and consider the relation between $\rho_B(X)$ and the fixed point property for nonexpansive mappings.

In Section 4, we summarize the results obtained in this paper.

2. THE MODULUS OF CONVEXITY RELATED TO BIRKHOFF ORTHOGONALITY

2.1. The estimates for $\delta_B(X)$ in terms of other geometric constants.

First, we shall state the relation between the new geometric constant $\delta_B(X)$ and the classical modulus of convexity $\delta_X(\epsilon)$.

Proposition 2.1. *Let X be a Banach space. Then $\delta_B(X) \geq \delta_X(1)$.*

Proof. For each pair of points $x, y \in S_X$ such that $x \perp_B y$, we can obtain $\|x + \lambda y\| \geq 1$ for any real number λ. In particular, we have $\|x - y\| \geq 1$. Thus, we can obtain

$$\begin{aligned}\delta_B(X) &= \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\} \\ &\geq \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, \|x-y\| \geq 1\right\} \\ &= \delta_X(1),\end{aligned}$$

which completes the proof. ∎

Corollary 2.2. *Let X be a Banach space. If X is uniformly convex, then $\delta_B(X) > 0$.*

Proof. By Lemma 2 in [15], if the space X is uniformly convex, then we have $\delta_X(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. Hence, by Proposition 2.1, we can obtain the desired result. ∎

Next, we give the bounds of $\delta_B(X)$.

Proposition 2.3. *Let X be a Banach space. Then $0 \leq \delta_B(X) \leq 1 - \frac{\sqrt{2}}{2}$.*

Proof. It is clear that $\delta_X(1) \geq 0$ by the definition of $\delta_X(\epsilon)$, therefore we have $\delta_B(X) \geq 0$ by Proposition 2.1. Notice that there exist $x, y \in S_X$ such that $x \perp_B y$ and $\|x+y\| \geq \sqrt{2}$ (see [3], P.141). Then we have

$$\delta_B(X) \leq 1 - \frac{\|x+y\|}{2} \leq 1 - \frac{\sqrt{2}}{2}.$$

This completes the proof. ∎

We consider the condition of $\delta_B(X) = 0$.

Example 2.1. *Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Then $\delta_B(X) = 0$.*

Proof. Let $x = (1, 1)$ and $y = (1, 0)$. Then one can easily verify that $x, y \in S_X$ and $x \perp_B y$. Thus, we obtain

$$\delta_B(X) \leq 1 - \frac{1}{2}\|x+y\| = 0,$$

which implies that $\delta_B(X) = 0$ by Proposition 2.3. ∎

Recall that the space X is called strictly convex, if for any $x, y \in S_X$ with $x \neq y$, then $\|x+y\| < 2$. Now, we give the definition of strict convexity related to Birkhoff orthogonality.

Definition 2.1. Let X be a real normed linear space. If for any $x, y \in S_X$ such that $x \perp_B y$, we have $\|x+y\| < 2$, then X is strictly convex related to Birkhoff orthogonality.

Next, we discuss the relation between $\delta_B(X)$ and strict convexity related to Birkhoff orthogonality.

Proposition 2.4. *Let X be a finite-dimensional Banach space. If $\delta_B(X) = 0$, then X is not strictly convex related to Birkhoff orthogonality.*

Proof. Suppose that $\delta_B(X) = 0$, there exist $x_n, y_n \in S_X$ satisfying $x_n \perp_B y_n$ and

$$\lim_{n\to\infty}\left(1 - \frac{1}{2}\|x_n + y_n\|\right) = 0.$$

Since the unit sphere of finite-dimensional Banach space is compact, there exist $x_0, y_0 \in S_X$ such that $x_0 \perp_B y_0$ and

$$1 - \frac{1}{2}\|x_0 + y_0\| = 0,$$

then we have $\|x_0 + y_0\| = 2$, which implies that X is not strictly convex related to Birkhoff orthogonality. ∎

In a similar way, we can obtain that the following proposition holds:

Proposition 2.5. *Let X be a finite-dimensional Banach space. If $\epsilon_0(X) > 0$, then X is not strictly convex.*

Proposition 2.6. *Let X be a Banach space. If X is not strictly convex related to Birkhoff orthogonality, then $\delta_B(X) = 0$.*

Proof. Assume that X is not strictly convex related to Birkhoff orthogonality, then there exist $x, y \in S_X$ such that $x \perp_B y$ and $\|x+y\| = 2$. Thus we can obtain

$$0 \leq \delta_B(X) \leq 1 - \frac{1}{2}\|x+y\| = 0$$

by Proposition 2.3, which implies that $\delta_B(X) = 0$. ∎

In fact, there exists a two-dimensional Banach space X for which $0 < \delta_B(X) < 1 - \frac{\sqrt{2}}{2}$.

Example 2.2. *Let X be the space $\mathbb{R}^2$ endowed with the norm*

$$\|x\| = \|(x_1, x_2)\| = \max\left\{\sqrt{\frac{x_1^2}{4} + x_2^2}, \sqrt{x_1^2 + \frac{x_2^2}{4}}\right\}.$$

Then $0 < \delta_B(X) \leq 1 - \frac{2}{5}\sqrt{5} < 1 - \frac{\sqrt{2}}{2}$.

Proof. Let $x = \left(\frac{2}{5}\sqrt{5}, \frac{2}{5}\sqrt{5}\right)$ and $y = \left(\frac{2}{5}\sqrt{5}, -\frac{2}{5}\sqrt{5}\right)$. It is easy for us to obtain $x, y \in S_X$. Then for any $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
\|x + \lambda y\| &= \left\|\left(\frac{2}{5}(1+\lambda)\sqrt{5}, \frac{2}{5}(1-\lambda)\sqrt{5}\right)\right\| \\
&= \max\left\{\sqrt{\frac{1}{5}(1+\lambda)^2 + \frac{4}{5}(1-\lambda)^2}, \sqrt{\frac{4}{5}(1+\lambda)^2 + \frac{1}{5}(1-\lambda)^2}\right\} \\
&= \max\left\{\sqrt{\lambda^2 - \frac{6}{5}\lambda + 1}, \sqrt{\lambda^2 + \frac{6}{5}\lambda + 1}\right\} \\
&= \max\left\{\sqrt{\left(\lambda - \frac{3}{5}\right)^2 + \frac{16}{25}}, \sqrt{\left(\lambda + \frac{3}{5}\right)^2 + \frac{16}{25}}\right\}.
\end{aligned}$$

Thus, to obtain $x \perp_B y$, we only need to consider the following cases:

Case 1: $\lambda \geq 0$.

Then,

$$\|x + \lambda y\| = \sqrt{\left(\lambda + \frac{3}{5}\right)^2 + \frac{16}{25}} \geq 1 = \|x\|.$$

Case 2: $\lambda \leq 0$.

Then,

$$\|x + \lambda y\| = \sqrt{\left(\lambda - \frac{3}{5}\right)^2 + \frac{16}{25}} \geq 1 = \|x\|.$$

Thus, we can obtain

$$\delta_B(X) \leq 1 - \frac{\|x+y\|}{2} = 1 - \frac{2}{5}\sqrt{5} < 1 - \frac{\sqrt{2}}{2}.$$

Actually, we can obtain that X is strictly convex. In fact, let $x = (x_1, x_2) \in S_X$ and $y = (y_1, y_2) \in S_X$ such that $\|x+y\| = 2$. By a direct calculation, we have

$$\|x\| = \max\left\{\sqrt{\frac{x_1^2}{4} + x_2^2}, \sqrt{x_1^2 + \frac{x_2^2}{4}}\right\} = 1,$$

$$\|y\| = \max\left\{\sqrt{\frac{y_1^2}{4}+y_2^2}, \sqrt{y_1^2+\frac{y_2^2}{4}}\right\} = 1,$$

$$\|x+y\| = \max\left\{\sqrt{\frac{(x_1+y_1)^2}{4}+(x_2+y_2)^2}, \sqrt{(x_1+y_1)^2+\frac{(x_2+y_2)^2}{4}}\right\} = 2.$$

Hence, we have $x_1 = y_1$ and $x_2 = y_2$, which implies that $x = y$. Thus $\epsilon_0(X) = 0$ by Proposition 2.5. Then we can obtain $\delta_B(X) > 0$ by Proposition 2.1. This completes the proof. ∎

Next we discuss the relation between $\delta_B(X)$ and the constant $J_B(X)$.

Proposition 2.7. *Let X be a Banach space. Then $\delta_B(X) \leq 1 - \frac{1}{2}J_B(X)$.*

Proof. It is clear that $\|x+y\| \geq \min\{\|x+y\|, \|x-y\|\}$, then we can obtain

$$\begin{aligned}
\delta_B(X) &= \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\} \\
&= 1 - \frac{1}{2}\sup\{\|x+y\| \ : \ x, y \in S_X, x \perp_B y\} \\
&\leq 1 - \frac{1}{2}\sup\{\min\{\|x+y\|, \|x-y\|\} \ : \ x, y \in S_X, x \perp_B y\} \\
&= 1 - \frac{1}{2}J_B(X).
\end{aligned}$$

This completes the proof. ∎

The constant $BR(X)$ which estimates the distance of $x, y \in S_X$ satisfying $x \perp_B y$ from being Roberts orthogonal to each other has been performed in [23]:

$$BR(X) = \sup_{\alpha>0}\left\{\frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} : x, y \in S_X, x \perp_B y\right\}.$$

In the following, we shall discuss the relation between $\delta_B(X)$ and $BR(X)$.

Proposition 2.8. *Let X be a Banach space with $\delta_B(X) > 0$. Then $BR(X) \leq \frac{1}{1+\delta_B(X)}$ and thus $BR(X) < 1$.*

Proof. For each pair of elements $x, y \in S_X$ such that $x \perp_B y$, we can obtain $\|x - \alpha y\| \geq 1$ for any real number α. From the definition of $\delta_B(X)$, we have

$$\|x+y\| \leq 2(1-\delta_B(X)).$$

Case 1: $0 < \alpha \leq 1$.
Then

$$\|x+\alpha y\| = \|(1-\alpha)x + \alpha(x+y)\| \leq 1 - \alpha + 2\alpha(1-\delta_B(X)).$$

Thus, we obtain

$$\frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} \leq 1 - 2\delta_B(X) < 1 - \delta_B(X) \leq \frac{1}{1+\delta_B(X)}.$$

Case 2: $1 < \alpha \leq 2(1+\delta_B(X))$.
Then

$$\|x+\alpha y\| = \|(x+y) + (\alpha-1)y\| \leq \|x+y\| + \alpha - 1.$$

Therefore, we have

$$\frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} \leq \frac{2(1-\delta_B(X)) + \alpha - 2}{\alpha} \leq 1 - \frac{2\delta_B(X)}{\alpha} \leq \frac{1}{1+\delta_B(X)}.$$

Case 3: $\alpha > 2(1+\delta_B(X))$.
Then we obtain

$$\frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} \leq \frac{2}{\alpha} \leq \frac{1}{1+\delta_B(X)}.$$

This completes the proof. ∎

By Proposition 2.8, we can obtain the following corollary:

Corollary 2.9. *Let X be a Banach space. If $BR(X) = 1$, then $\delta_B(X) = 0$.*

Remark 2.1. By Proposition 2.1, Proposition 2.8 and Corollary 2.9, we can obtain Theorem 2.3 and Theorem 2.4 in [23].

The constant $A_2(X)$ is defined as follows (see [5, 26]):

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} : x, y \in S_X\right\}.$$

Now we shall study the relationship between $\delta_B(X)$ and $A_2(X)$.

Proposition 2.10. *Let X be a Banach space. Then $\delta_B(X) \geq \frac{3}{2} - A_2(X)$.*

Proof. For any $x, y \in S_X$ such that $x \perp_B y$, we have $\|x+\lambda y\| \geq 1$ for any real number λ. In particular, we can obtain $\|x-y\| \geq 1$. Then we have

$$\begin{aligned} 1 - \frac{\|x+y\|}{2} &= 1 + \frac{\|x-y\|}{2} - \frac{\|x+y\| + \|x-y\|}{2} \\ &\geq \frac{3}{2} - \frac{\|x+y\| + \|x-y\|}{2}. \end{aligned}$$

Hence we can obtain

$$\begin{aligned} \delta_B(X) &= \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\} \\ &\geq \inf\left\{\frac{3}{2} - \frac{\|x+y\| + \|x-y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\} \\ &= \frac{3}{2} - \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\} \\ &\geq \frac{3}{2} - \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} \ : \ x, y \in S_X\right\} \\ &= \frac{3}{2} - A_2(X). \end{aligned}$$

This completes the proof. ∎

To study the relationship between the constant $\delta_B(X)$ and the modified von Neumann-Jordan constant $C'_{NJ}(X)$, let us recall that the definition of the constant $C'_{NJ}(X)$ as follows (see [1, 12]):

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} \ : \ x, y \in S_X\right\}.$$

Proposition 2.11. *Let X be a Banach space. Then $\delta_B(X) \geq 1 - \frac{\sqrt{4C'_{NJ}(X)-1}}{2}$.*

Proof. According to the definition of the constant $C'_{NJ}(X)$, for all $x, y \in S_X$, we have

$$\|x+y\|^2 + \|x-y\|^2 \leq 4C'_{NJ}(X).$$

Then for any $x, y \in S_X$ with $x \perp_B y$, it is easy for us to see that $\|x-y\| \geq 1$. Hence we can obtain

$$\sup\{\|x+y\| \ : \ x, y \in S_X, x \perp_B y\} \leq \sqrt{4C'_{NJ}(X) - 1},$$

then we have

$$\begin{aligned} \delta_B(X) &= \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\} \\ &= 1 - \frac{1}{2}\sup\{\|x+y\| \ : \ x, y \in S_X, x \perp_B y\} \\ &\geq 1 - \frac{1}{2}\sqrt{4C'_{NJ}(X) - 1}, \end{aligned}$$

which completes the proof. ∎

2.2. Some geometric properties related to $\delta_B(X)$.

In this section, we will study some certain geometric properties related to this modulus. First, in order to characterize Hilbert spaces in terms of $\delta_B(X)$, we need the following lemma:

Lemma 2.12. [2, 25] *Let X be a normed linear space and $\lambda > 0$ be a fixed number. Then the following statements are equivalent:*
(1) X is an inner product space.
(2) $x, y \in S_X$, $x \perp_B y \Rightarrow \|\lambda x + y\| \leq \sqrt{1+\lambda^2}$.
(3) $x, y \in S_X$, $x \perp_B y \Rightarrow \|\lambda x + y\| \geq \sqrt{1+\lambda^2}$.
(4) $x, y \in S_X$, $x \perp_B y \Rightarrow \|\lambda x + y\| = \sqrt{1+\lambda^2}$.

Theorem 2.13. *Let X be a Banach space. Then $\delta_B(X) = 1 - \frac{\sqrt{2}}{2}$ if and only if X is a Hilbert space.*

Proof. If X is a Hilbert space, then for any $x, y \in S_X$ such that $x \perp_B y$, we have $\|x+y\| = \sqrt{2}$ by Lemma 2.12. Hence, we can obtain

$$1 - \frac{1}{2}\|x+y\| = 1 - \frac{\sqrt{2}}{2},$$

which implies that $\delta_B(X) = 1 - \frac{\sqrt{2}}{2}$.

Conversely, suppose that $\delta_B(X) = 1 - \frac{\sqrt{2}}{2}$, then for all $x, y \in S_X$ satisfying $x \perp_B y$, we have

$$1 - \frac{1}{2}\|x+y\| \geq 1 - \frac{\sqrt{2}}{2},$$

which means that $\|x+y\| \leq \sqrt{2}$. Thus X is a Hilbert space by Lemma 2.12. ∎

Next, we discuss the relationship between $\delta_B(X)$ and uniformly non-square Banach space.

Proposition 2.14. *Let X be a Banach space. If $\delta_B(X) > 0$, then X is uniformly non-square.*

Proof. Suppose conversely that X is not uniformly non-square, then $J_B(X) = 2$ from Theorem 4.3 in [7]. Thus, we can obtain $\delta_B(X) = 0$ by Proposition 2.3 and Proposition 2.7. This contradicts $\delta_B(X) > 0$, hence X is a uniformly non-square Banach space. ∎

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for any $x, y \in C$. The space X is said to have the fixed point property (for nonexpansive mappings) if for every nonempty bounded closed convex set $C \subset X$ and every nonexpansive mapping $T : C \to C$, there is an element $x \in C$ such that $Tx = x$, i.e. a fixed point of T.

It is worth nothing that the uniformly non-square Banach space always has the fixed point property (see [14]). Thus, according to the Proposition 2.14 we may conclude the following result:

Theorem 2.15. *Let X be a Banach space. If $\delta_B(X) > 0$, then X has the fixed pointed property.*

2.3. **The constant $\delta_B(X)$ in Radon planes.**

Note that an orthogonality notion "$\perp$" is called symmetric, if $x \perp y$ implies $y \perp x$. The usual orthogonality in Hilbert spaces is, of course, symmetric. However, the Birkhoff orthogonality in Banach spaces is not symmetric in general. In [18], James proved the following result:

Theorem 2.16. *(see* [18]*) A normed linear space X whose dimension is at least three is an inner product if and only if Birkhoff orthogonality is symmetric in X.*

The assumption on the dimension of the space in the above theorem cannot be omitted. A two-dimensional normed linear space in which Birkhoff orthogonality is symmetric is called Radon plane. For more results about Radon planes, we refer the reader to [2, 19].

Since Radon planes are Banach spaces, now we present the following result:

Proposition 2.17. *Let X be a Radon plane. Then $0 \leq \delta_B(X) \leq 1 - \frac{\sqrt{2}}{2}$.*

In the following, we will provide an example to illustrate that the lower bound shown in the above result is sharp and the converse of Proposition 2.14 is not true.

Example 2.3. *Let X be a Radon plane $\ell_\infty - \ell_1$, that is, the space $\mathbb{R}^2$ with the norm defined by*

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_\infty, & (x_1 x_2 \geq 0), \\ \|(x_1, x_2)\|_1, & (x_1 x_2 \leq 0). \end{cases}$$

Then $\delta_B(X) = 0$.

Proof. It is well known that X is uniformly non-square. Now let $x = (1, 0)$ and $y = (1, 1)$, it is clear that $x, y \in S_X$. Actually, we also have $x \perp_B y$. In fact, to obtain $x \perp_B y$, we only need to consider the following three cases:

Case 1: $1 + \lambda \geq 0$ and $\lambda \geq 0$.

Then, we obtain $\lambda \geq 0$ and

$$\|x + \lambda y\| = \|(1 + \lambda, \lambda)\| = \max\{|1 + \lambda|, |\lambda|\} = 1 + \lambda \geq 1 = \|x\|.$$

Case 2: $(1 + \lambda)\lambda \leq 0$.

Then, we have $-1 \leq \lambda \leq 0$ and

$$\|x + \lambda y\| = \|(1 + \lambda, \lambda)\| = |1 + \lambda| + |\lambda| = 1 + \lambda - \lambda = 1 = \|x\|.$$

Case 3: $1 + \lambda \leq 0$ and $\lambda \leq 0$.

Then we obtain $\lambda \leq -1$ and

$$\|x + \lambda y\| = \|(1 + \lambda, \lambda)\| = \max\{|1 + \lambda|, |\lambda|\} = -\lambda \geq 1 = \|x\|.$$

Hence, by Proposition 2.17, we have

$$0 \leq \delta_B(X) \leq 1 - \frac{\|x+y\|}{2} = 0.$$

This completes the proof. ∎

Notice that the unit sphere of $\ell_\infty - \ell_1$ is actually an affine regular hexagon, then $\ell_\infty - \ell_1$ is a Radon plane with $\delta_B(X) = 0$ and such that its unit sphere is an affine regular hexagon. In fact, this also holds in general, see the following result:

Theorem 2.18. *Let X be a Radon plane. Then $\delta_B(X) = 0$ if and only if its unit sphere S_X is an affine regular hexagon.*

Proof. If $\delta_B(X) = 0$, then we have $A_2(X) \geq \frac{3}{2}$ by Proposition 2.10. Thus we can obtain $A_2(X) = \frac{3}{2}$ by Theorem 3.1 in [22]. So S_X is an affine regular hexagon by Theorem 5.1 in [22].

Conversely, suppose that S_X is an affine regular hexagon. Then there exist $u, v \in S_X$ such that $\pm u$, $\pm v$, $\pm(u+v)$ are the vertices of S_X.

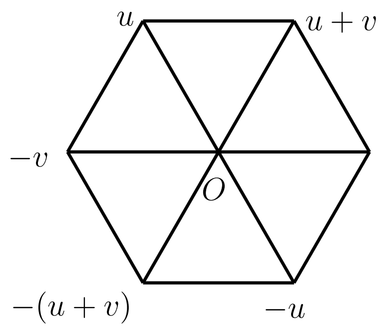

Figure 1. Affine regular hexagonal unit sphere.

Let $x = u + v \in S_X$ and $y = u \in S_X$. Then one can deduce that $x \perp_B y$ by considering the following four cases for $\lambda \in \mathbb{R}$:

Case 1: $\lambda \geq 1$.

Then,

$$\|x + \lambda y\| = \|(1+\lambda)u + v\| \geq (1+\lambda) - 1 = \lambda \geq 1 = \|x\|.$$

Case 2: $\lambda \leq -1$.

Then,

$$\|x + \lambda y\| = \|(1+\lambda)u + v\| \geq 1 - (1+\lambda) = -\lambda \geq 1 = \|x\|.$$

Case 3: $0 \leq \lambda \leq 1$.

Then,

$$\begin{aligned}\|x + \lambda y\| &= \|(1+\lambda)u + v\| = (1+\lambda)\left\|u + \frac{1}{1+\lambda}v\right\| \\ &= (1+\lambda)\left\|\left(1 - \frac{1}{1+\lambda}\right)u + \frac{1}{1+\lambda}(u+v)\right\| \\ &= 1 + \lambda \geq 1 = \|x\|.\end{aligned}$$

Case 4: $-1 \leq \lambda \leq 0$.

Then,

$$\|x + \lambda y\| = \|(1+\lambda)u + v\| = \|[1 - (-\lambda)](u+v) + (-\lambda)v\| = 1 \geq \|x\|.$$

Moreover, we can also obtain

$$\|x+y\| = \|2u+v\| = 2\left\|u+\frac{1}{2}v\right\| = 2\left\|\frac{1}{2}(u+v)+\frac{1}{2}u\right\| = 2,$$

then we have

$$0 \leq \delta_B(X) \leq 1-\frac{1}{2}\|x+y\| = 0,$$

by Proposition 2.17, which implies that $\delta_B(X)=0$. This completes the proof. ∎

3. THE MODULUS OF SMOOTHNESS RELATED TO BIRKHOFF-JAMES ORTHOGONALITY

3.1. The estimates for $\rho_B(X)$ in terms of other geometric constants.

First, we shall study the relation between $\rho_B(X)$ and the constant $\mu^{'}(X)$. Now we recall that definition of the constant $\mu^{'}(X)$ (see [3]) as follows:

$$\mu^{'}(X) = \sup\left\{\frac{2}{\|x+y\|} \ : \ x,y\in S_X, x\perp_B y\right\}.$$

Proposition 3.1. *Let X be a Banach space. Then $\rho_B(X) = 1-\frac{1}{\mu^{'}(X)}$*

Proof. According to the definition of $\mu^{'}(X)$, we have

$$\inf\{\|x+y\| \ : \ x,y\in S_X, x\perp_B y\} = \frac{2}{\mu^{'}(X)}.$$

Then we can obtain

$$\begin{aligned}\rho_B(X) &= \sup\left\{1-\frac{\|x+y\|}{2} \ : \ x,y\in S_X, x\perp_B y\right\}\\ &= 1-\frac{1}{2}\inf\{\|x+y\| \ : \ x,y\in S_X, x\perp_B y\}\\ &= 1-\frac{1}{\mu^{'}(X)}.\end{aligned}$$

This completes the proof. ∎

Corollary 3.2. *Let X be a Banach space. Then $1-\frac{\sqrt{2}}{2} \leq \rho_B(X) \leq \frac{1}{2}$.*

Proof. From [3], we have $\sqrt{2} \leq \mu^{'}(X) \leq 2$. Then we can obtain the desired result by Proposition 3.1. ∎

The following example shows that the upper bounds of $\rho_B(X)$ given in the above result is sharp.

Example 3.1. *Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Then $\rho_B(X) = \frac{1}{2}$.*

Proof. Let $x=(-1,1)$ and $y=(1,0)$. Then one can easily verify that $x,y\in S_X$ and $x\perp_B y$. Thus, we obtain

$$\rho_B(X) \geq 1-\frac{1}{2}\|x+y\| = \frac{1}{2},$$

which implies that $\rho_B(X)=\frac{1}{2}$ by Corollary 3.2. ∎

In the following, we intend to present the relationship between $\rho_B(X)$ and $S_B(X)$.

Proposition 3.3. *Let X be a Banach space. Then $\rho_B(X) \geq 1-\frac{1}{2}S_B(X)$.*

Proof. It is easy for us to obtain $\|x+y\| \leq \max\{\|x+y\|, \|x-y\|\}$, then we have

$$\begin{aligned}\rho_B(X) &= \sup\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, x \perp_B y\right\} \\ &= 1 - \frac{1}{2}\inf\{\|x+y\| \ : \ x, y \in S_X, x \perp_B y\} \\ &\geq 1 - \frac{1}{2}\inf\{\max\{\|x+y\|, \|x-y\|\} \ : \ x, y \in S_X, x \perp_B y\} \\ &= 1 - \frac{1}{2}S_B(X),\end{aligned}$$

which completes the proof. ∎

3.2. **Some geometric properties related to $\rho_B(X)$.**

In this section, we will obtain a characterization of Hilbert spaces in terms of $\rho_B(X)$ and establish the relationship between $\rho_B(X)$ and the fixed point property for nonexpansive mappings.

First, it is easy for us to see that the lower bound of $\rho_B(X)$ can be used to characterize the Hilbert space.

Theorem 3.4. *Let X be a Banach space. Then $\rho_B(X) = 1 - \frac{\sqrt{2}}{2}$ if and only if X is a Hilbert space.*

Proof. Assume that X is a Hilbert space, then for any $x, y \in S_X$ satisfying $x \perp_B y$, we can obtain $\|x+y\| = \sqrt{2}$ by Lemma 2.12. Thus, we have

$$1 - \frac{\|x+y\|}{2} = 1 - \frac{\sqrt{2}}{2},$$

which implies that $\rho_B(X) = 1 - \frac{\sqrt{2}}{2}$.

On the other hand, suppose that $\rho_B(X) = 1 - \frac{\sqrt{2}}{2}$, then for any $x, y \in S_X$ such that $x \perp_B y$, we can obtain

$$1 - \frac{\|x+y\|}{2} \leq 1 - \frac{\sqrt{2}}{2},$$

which means that $\|x+y\| \geq \sqrt{2}$. Hence X is a Hilbert space by Lemma 2.12. ∎

By Theorem 3.4 and Proposition 3.1, we can obtain Corollary 3.5, which implies Theorem 4.8.19 in [3].

Corollary 3.5. *Let X be a Banach space. Then the following statements are equivalent:*
(1) $\mu'(X) = \sqrt{2}$.
(2) $\rho_B(X) = 1 - \frac{\sqrt{2}}{2}$.
(3) X is a Hilbert space.

In the following, we will discuss the relationship between $\rho_B(X)$ and uniform non-squareness.

Proposition 3.6. *Let X be a Banach space. If $\rho_B(X) < \frac{1}{2}$, then X is uniformly non-square.*

Proof. Assume conversely that X is not a uniformly non-square Banach space, then $S_B(X) = 1$ (see Theorem 3.2 in [7]). Thus we can obtain $\rho_B(X) = \frac{1}{2}$ by Proposition 3.3 and Corollary 3.2. This contradicts $\rho_B(X) < \frac{1}{2}$, hence X is a uniformly non-square Banach space. ∎

In fact, the converse of Proposition 3.6 is not true. Now we provide the counterexample as follows:

Example 3.2. *Let X be the space $\mathbb{R}^2$ endowed with the norm*

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_1, & x_1x_2 \geq 0, \\ \|(x_1, x_2)\|_\infty, & x_1x_2 \leq 0. \end{cases}$$

Then $\rho_B(X) = \frac{1}{2}$.

Proof. It is obvious that X is uniformly non-square. Let $x = (1, 0)$ and $y = (-1, 1)$, it is clear that $x, y \in S_X$. Actually, we also have $x \perp_B y$. In fact, to obtain $x \perp_B y$, we only need to consider the following three cases:

Case 1: $(1 - \lambda)\lambda \geq 0$.
Then, we have $0 \leq \lambda \leq 1$ and

$$\|x + \lambda y\| = \|(1 - \lambda, \lambda)\| = |1 - \lambda| + |\lambda| = 1 = \|x\|.$$

Case 2: $\lambda \geq 1$.
Then, we obtain

$$\|x + \lambda y\| = \|(1 - \lambda, \lambda)\| = \max\{|1 - \lambda|, |\lambda|\} = \lambda \geq 1 = \|x\|.$$

Case 3: $\lambda \leq 0$.
Then, we have

$$\|x + \lambda y\| = \|(1 - \lambda, \lambda)\| = \max\{|1 - \lambda|, |\lambda|\} = -\lambda + 1 \geq 1 = \|x\|.$$

Thus, by Corollary 3.2, we have

$$\frac{1}{2} = 1 - \frac{1}{2}\|x + y\| \leq \rho_B(X) \leq \frac{1}{2}.$$

This completes the proof. ∎

By Proposition 3.1, Proposition 3.6 and Example 3.2, we can obtain the following corollary:

Corollary 3.7. *Let X be a Banach space. If $\mu'(X) < 2$, then X is uniformly non-square, and the converse is not true.*

By Proposition 3.6, we can obtain the following result:

Theorem 3.8. *Let X be a Banach space. If $\rho_B(X) < \frac{1}{2}$, then X has the fixed pointed property.*

4. CONCLUSIONS

In this paper, we introduce two new constants $\delta_B(X)$ and $\rho_B(X)$, which are the modulus of convexity and the modulus of smoothness related to Birkhoff orthogonality, respectively. It makes sense to investigate the relationships between the two new constants and other well-known constants by some equalities and inequalities, characterize the Hilbert space in terms of them. Meanwhile, we establish the relations between $\delta_B(X)$ and the fixed point property. Moreover, we provide a study of the constant $\delta_B(X)$ in Radon planes. The characterization of the Radon plane with affine regular hexagonal unit sphere in terms of $\delta_B(X)$ is obtained. How can the constant $\rho_B(X)$ be utilized to characterize more geometrical properties? Besides the geometric constants mentioned in the paper, what other important geometric constants are closely related to $\delta_B(X)$ and $\rho_B(X)$? Henceforth, more results about the two constants $\delta_B(X)$ and $\rho_B(X)$ will be presented in future research for the readers who are interested in the theory of geometrical constants in Banach spaces.

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