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ON SOME NONLINEAR RETARDED INTEGRODIFFERENTIAL INEQUALITIES IN TWO AND *N* INDEPENDENT VARIABLES AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we establish some new nonlinear retarded integrodifferential inequalities in two and n independent variables. Some applications are given as illustration.

Key words and phrases: Nonlinear inequalities; Retarded integrodifferential inequalities; Functions of two or n variables; Partial integrodifferential equations.

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1. INTRODUCTION

The integral inequalities with a term of delay are utilized a lot in the study and modeling of partial differential equations with a term of delay. A number of researchers [4, 6, 13] have already established their basic properties, such as generalizations in the bidimensional and multidimensional cases, applications to retarded partial differential equations, and existence as well as uniqueness of solutions .

The integrodifferential inequalities for functions of two or n variables are very significant for assuming the existence and uniqueness of solutions of the Wendroff-type integrodifferential inequalities and equations [2, 3, 7, 11, 12]; they are also useful for studying the boundedness of solutions of nonlinear partial integrodifferential equations with delay for functions of two or n variables [1, 5, 8, 10].

Pachpatte [9] presented one of the Wendroff-type nonlinear integrodifferential inequalities for two-variable functions as follows:

Lemma 1.1. *(see Theorem 1 [9]) Let $\phi(x,y)$ and $c(x,y)$ be nonnegative continuous functions defined for $x \geq 0,\ y \geq 0$, and $\phi(x,0) = \phi(0,y) = 0$ for which the inequality*

$$\phi_{xy}(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y c(s,t)(\phi(s,t) + \phi_{xy}(s,t))dsdt,$$

holds for $x \geq 0,\ y \geq 0$, where $a(x),\ b(y) > 0$; $a'(x)$ and $b'(y) \geq 0$ are continuous functions defined for $x \geq 0,\ y \geq 0$. Then

$$\begin{aligned} \phi_{xy}(x,y) \quad &\leq \quad a(x) + b(y) + \int_0^x \int_0^y c(s,t) \left[\frac{[a(0)+b(t)][a(s)+b(0)]}{[a(0)+b(0)]} \right. \\ &\left. \times exp\left(\int_0^s \int_0^t [1 + c(m,n)]dmdn \right) \right] dsdt. \end{aligned}$$

2. MAIN RESULTS

In this section, some results of nonlinear retarded integrodifferential inequalities in two independent variables are presented.
In what follows, $x_0,\ y_0 \in \mathbb{R}_+$, with $x_0 \leq x,\ y_0 \leq y$.

Theorem 2.1. *Let $u(x,y)$, $c(x,y)$, $a(x,y)$, $Du(x,y)$ and $D_i u(x,y)$ be nonnegative continuous functions for all $i = 1,2$ defined for $x, y \in \mathbb{R}_+$, and $\alpha,\ \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions for each variable, with $\alpha(x) \leq x$ on $\mathbb{R}_+$ and $\beta(y) \leq y$ on $\mathbb{R}_+$. Let $c(x,y)$ be a nondecreasing function for each variable $x, y \in \mathbb{R}_+$, and $u(x_0,y) = u(x,y_0) = 0$. If*

$$Du(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)[u(s,t) + Du(s,t)]dsdt, \tag{2.1}$$

for $x, y \in \mathbb{R}_+$, *then*

$$\begin{aligned} Du(x,y) &\leq c(x,y)\Bigg[1 + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(s,t) \\ &\quad \times \exp\left(\int\limits_{\alpha(x_0)}^{s} \int\limits_{\beta(y_0)}^{t} (1 + a(\tau,\sigma))d\tau d\sigma\right) dsdt\Bigg], \end{aligned} \tag{2.2}$$

for $x, y \in \mathbb{R}_+$.

Proof: Fix any $X,\ Y \in \mathbb{R}_+$. Then, for $x_0 \leq x \leq X$ and $y_0 \leq y \leq Y$, we have

$$Du(x,y) \leq z(x,y), \tag{2.3}$$

where $z(x,y)$ is a function defined by

$$z(x,y) = c(X,Y) + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(s,t)[u(s,t) + Du(s,t)]dsdt, \tag{2.4}$$

then $z(x_0, y) = z(x, y_0) = c(X,Y)$. By integrating both sides of (2.3),

$$u(x,y) \leq \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} z(s,t)dsdt. \tag{2.5}$$

By differentiating (2.4),

$$Dz(x,y) \leq a(x,y)[u(x,y) + Du(x,y)]\alpha'(x)\beta'(y). \tag{2.6}$$

Now, using (2.3) and (2.5) in (2.6) we get

$$Dz(x,y) \leq a(x,y)\left[z(x,y) + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} z(s,t)dsdt\right]\alpha'(x)\beta'(y). \tag{2.7}$$

If we put

$$v(x,y) = z(x,y) + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} z(s,t)dsdt, \tag{2.8}$$

then $v(x_0, y) = v(x, y_0) = c(X,Y)$, and

$$Dv(x,y) \leq Dz(x,y) + z(x,y)\alpha'(x)\beta'(y).$$

By taking $Dz(x,y) \leq a(x,y)v(x,y)\alpha'(x)\beta'(y)$ from (2.7) and $z(x,y) \leq v(x,y)$ from (2.8), we have

$$Dv(x,y) \leq [1 + a(x,y)]v(x,y)\alpha'(x)\beta'(y).$$

Now, it is possible to estimate $v(x,y)$ by

$$v(x,y) \leq c(X,Y)\exp\left[\int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} (1 + a(s,t))dsdt\right]. \tag{2.9}$$

By substituting (2.9) in (2.7), integrating both sides, and using $z(x_0, y) = z(x, y_0) = c(X, Y)$, it yields

$$z(x,y) \leq c(X,Y) + c(X,Y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) \exp\left[\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} (1 + a(\tau,\sigma)) d\tau d\sigma\right] dsdt.$$

We obtain the inequality (2.2) by substituting the value of $z(x, y)$ in (2.3) because X and Y are arbitraries.

Remark 2.1. It is enough to put $\alpha(x_0) = \beta(y_0) = 0,\ \alpha(x) = x,\ \beta(y) = y$, and $c(x, y) = c_1(x) + c_2(y)$ in Theorem 2.1 so as to obtain Theorem 1 in [9].

Theorem 2.2. *Let $u(x, y), c(x, y), a(x, y), \alpha$, and β be defined as in Theorem 2.1, and assuming that $b(x, y)$ is nonnegative continuous function. If*

$$\begin{aligned} u(x,y) &\leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) u(s,t) dsdt \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) \left(\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(\tau,\sigma) u(\tau,\sigma) d\tau d\sigma \right) dsdt, \end{aligned} \tag{2.10}$$

for $x, y \in \mathbb{R}_+$, then

$$\begin{aligned} u(x,y) &\leq c(x,y) \exp\left[\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) dsdt \right. \\ &\quad \left. + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) \left(\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(\tau,\sigma) d\tau d\sigma \right) dsdt \right], \end{aligned} \tag{2.11}$$

for $x, y \in \mathbb{R}_+$.

Proof: Since $c(x, y)$ is nonnegative and nondecreasing, from (2.10) we have

$$\begin{aligned} \frac{u(x,y)}{c(x,y)} &\leq 1 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) \frac{u(s,t)}{c(s,t)} dsdt \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) \left(\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(\tau,\sigma) \frac{u(\tau,\sigma)}{c(\tau,\sigma)} d\tau d\sigma \right) dsdt. \end{aligned} \tag{2.12}$$

Define a function $z(x, y)$ by the right side of the last inequality. Then $z(x, y) > 0$, $z(x_0, y) = z(x, y_0) = 1$, $\frac{u(x,y)}{c(x,y)} \leq z(x, y)$, and

$$Dz(x,y) \leq z(x,y) \left[a(x,y) + a(x,y) \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) dsdt \right) \right] \alpha'(x) \beta'(y).$$

i.e.

$$\frac{Dz(x,y)z(x,y)}{z^2(x,y)}-\frac{D_1z(x,y)D_2z(x,y)}{z^2(x,y)} \le \Bigg[a(x,y)$$

$$+a(x,y)\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)dsdt\right)\Bigg]\alpha'(x)\beta'(y). \tag{2.13}$$

Thus

$$D_2\left[\frac{D_1z(x,y)}{z(x,y)}\right] \le \left[a(x,y)+a(x,y)\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)dsdt\right)\right]\alpha'(x)\beta'(y). \tag{2.14}$$

By keeping y fixed, setting $x=s$, and integrating from x_0 to x in (2.14), and again by keeping x fixed, setting $y=t$, and integrating from y_0 to y in the resulting inequality, we have

$$z(x,y) \le \exp\left[\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)dsdt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)\left(\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} b(\tau,\sigma)d\tau d\sigma\right)dsdt\right].$$

Finally, since $\frac{u(x,y)}{c(x,y)} \le z(x,y)$ we obtain the inequality (2.11).

Remark 2.2. (i) It is enough to put $\alpha(x_0)=\beta(y_0)=0,\ \alpha(x)=x,\ \beta(y)=y$, and $c(x,y)=c_1(x)+c_2(y)$ in Theorem 2.2 so as to obtain Theorem 3 in [9].
(ii) If $b(x,y)=0$, the bound obtained in (2.11) reduces to

$$u(x,y) \le c(x,y)\exp\left[\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)dsdt\right]. \tag{2.15}$$

Theorem 2.3. *Under the same hypotheses of Theorem* 2.2, *and assuming that* $f(x,y)$ *is nonnegative continuous and nondecreasing function, let* $K(u(x,y))$ *be a real-valued, positive, continuous, strictly nondecreasing, sub-additive, and sub-multiplicative function for* $u(x,y)\ge 0$, *and* $H(u(x,y))$ *be a real-valued, continuous, positive, and nondecreasing function defined for* $x,y\in\mathbb{R}_+$. *If*

$$Du(x,y) \le c(x,y)+f(x,y)H\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)K(u(s,t))dsdt\right)$$

$$+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)Du(s,t)dsdt, \tag{2.16}$$

for $x,y\in\mathbb{R}_+$, *then*

$$Du(x,y) \le \Bigg\{c(x,y)+f(x,y)H\Bigg(G^{-1}\Bigg[G(\xi)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)$$

$$\times K(f(s,t)p(s,t))dsdt\Bigg]\Bigg)\Bigg\}\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)dsdt\right), \tag{2.17}$$

for all $x, y \in \mathbb{R}_+$, *where*

$$p(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \exp\left(\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(\tau,\sigma) d\tau d\sigma \right) dsdt. \tag{2.18}$$

$$\xi = \int_{\alpha(x_0)}^{\infty} \int_{\beta(y_0)}^{\infty} a(s,t) K(c(s,t)p(s,t)) dsdt. \tag{2.19}$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{K(H(s))},\ r \geq r_0 \geq 0, \tag{2.20}$$

where G^{-1} *is the inverse function of* G, *and* $G(\xi) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) K(f(s,t)\ p(s,t)) dsdt \in dom(G^{-1})$ *for* $x, y \in \mathbb{R}_+$.

Proof: From (2.16), we have

$$Du(x,y) \leq z(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) Du(s,t) dsdt, \tag{2.21}$$

where $z(x,y)$ is a function defined by

$$z(x,y) = c(x,y) + f(x,y) H\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) K(u(s,t)) dsdt \right). \tag{2.22}$$

We note that $z(x,y)$ is a positive, continuous, and nondecreasing function for $x, y \in \mathbb{R}_+$. Using (2.15) from Theorem 2.2 in (2.21), we get

$$Du(x,y) \leq z(x,y) \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) dsdt \right). \tag{2.23}$$

By integration, first with respect to x from x_0 to x, and then with respect to y from y_0 to y in the last inequality, we obtain

$$u(x,y) \leq z(x,y) p(x,y), \tag{2.24}$$

where $p(x,y)$ is defined in (2.18). From (2.22) we have

$$z(x,y) = c(x,y) + f(x,y) H(v(x,y)), \tag{2.25}$$

where

$$v(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) K(u(s,t)) dsdt. \tag{2.26}$$

Now, using (2.25) in (2.24) we get

$$u(x,y) \leq [c(x,y) + f(x,y) H(v(x,y))] p(x,y). \tag{2.27}$$

From (2.26) and (2.27) and since K is a sub-additive and sub-multiplicative function, we obtain

$$\begin{aligned} v(x,y) &\leq \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)K([c(s,t)+f(s,t)H(v(s,t))]p(s,t))dsdt \\ &\leq \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)K(c(s,t)p(s,t))dsdt \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)K(f(s,t)p(s,t))K(H(v(s,t)))dsdt. \end{aligned}$$

Therefore

$$v(x,y) \leq \Phi(x,y), \tag{2.28}$$

where $\Phi(x,y)$ is a function defined by

$$\begin{aligned} \Phi(x,y) &= \int_{\alpha(x_0)}^{\infty}\int_{\beta(y_0)}^{\infty} a(s,t)K(c(s,t)p(s,t))dsdt \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)K(f(s,t)p(s,t))K(H(v(s,t)))dsdt, \end{aligned} \tag{2.29}$$

then

$$\Phi(x_0,y) = \Phi(x,y_0) = \int_{\alpha(x_0)}^{\infty}\int_{\beta(y_0)}^{\infty} a(s,t)K(c(s,t)p(s,t))dsdt = \xi. \tag{2.30}$$

Clearly, $\Phi(x,y)$ is a positive and nondecreasing function for y. So

$$D_1\Phi(x,y) \leq K(H(\Phi(x,y))) \int_{\beta(y_0)}^{\beta(y)} a(x,t)K(f(x,t)p(x,t))dt\ \alpha'(x).$$

From (2.20) we have

$$D_1G(\Phi(x,y)) = \frac{D_1\Phi(x,y)}{K(H(\Phi(x,y)))} \leq \int_{\beta(y_0)}^{\beta(y)} a(x,t)K(f(x,t)p(x,t))dt\ \alpha'(x). \tag{2.31}$$

Now, by setting $x = s$ and integrating from x_0 to x in (2.31), and using (2.30) we get

$$\Phi(x,y) \leq G^{-1}\left[G(\xi) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)K(f(s,t)p(s,t))dsdt\right]. \tag{2.32}$$

Finally, by substituting (2.25), (2.28), and (2.32) in (2.23) we obtain the inequality (2.17).

Remark 2.3. (i) From the inequalities $(2.27), (2.28)$, and (2.32) in the proof of Theorem 2.3 we get the following inequality

$$\begin{aligned} u(x,y) \;\; \leq \;\; & \Bigg\{ c(x,y) + f(x,y) H \Bigg(G^{-1} \Bigg[G(\xi) \\ & + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) K(f(s,t) p(s,t)) dsdt \Bigg] \Bigg) \Bigg\} p(x,y). \end{aligned}$$

(ii) It is enough to put $\alpha(x_0) = \beta(y_0) = 0,\ \alpha(x) = x,\ \beta(y) = y$, $c(x,y) = c_1(x) + c_2(y)$, $f(x,y) = 1,\ H(x) = K(x) = x$, and $a(x,y) = b(x,y)$ so that Theorem 3.3 reduces to Theorem 1 in [9].

Corollary 2.4. *Under the same hypotheses of Theorem* 2.3, *and if*

$$\begin{aligned} Du(x,y) \;\; \leq \;\; & c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) K(u(s,t)) dsdt \\ & + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) Du(s,t) dsdt, \end{aligned} \tag{2.33}$$

for $x, y \in \mathbb{R}_+$, *then*

$$\begin{aligned} Du(x,y) \;\; \leq \;\; & \Bigg\{ c(x,y) + T^{-1} \Bigg[T(\xi) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) K(p(s,t)) dsdt \Bigg] \Bigg\} \\ & \times \exp \Bigg(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) dsdt \Bigg), \end{aligned} \tag{2.34}$$

for all $x, y \in \mathbb{R}_+$, *where* $p(x,y)$ *and* ξ *are defined in Theorem* 2.3.

$$T(r) = \int_{r_0}^{r} \frac{ds}{K(s)},\ r \geq r_0 \geq 0,$$

where T^{-1} *is the inverse function of* T, *and* $T(\xi) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) K(p(s,t)) dsdt \in dom(T^{-1})$ *for* $x, y \in \mathbb{R}_+$.

Proof: The proof of this Corollary follows the same steps as in Theorem 2.3.

Remark 2.4. (i) It is enough to put $H(x) = x$ and $f(x,y) = 1$ in Theorem 2.3 so as to obtain the result in Corollary 2.4.
(ii) It is enough to put $\alpha(x_0) = \beta(y_0) = 0,\ \alpha(x) = x,\ \beta(y) = y$, $c(x,y) = c_1(x) + c_2(y)$, $K(x) = x$, and $a(x,y) = b(x,y)$ so as Corollary 2.4 reduces to Theorem 1 in [9].

Corollary 2.5. *Under the same hypotheses of Theorem* 2.2, *and if*

$$Du(x,y) \leq M + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) u(s,t) dsdt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) Du(s,t) dsdt, \tag{2.35}$$

for $x, y \in \mathbb{R}_+$, where $M > 0$ is constant, then we obtain the following results:

$$\begin{aligned}(1)\ Du(x,y) &\le M\left\{1+\left(\int_{\alpha(x_0)}^{\infty}\int_{\beta(y_0)}^{\infty} a(s,t)p(s,t)dsdt\right)\right.\\ &\quad\left.\times\ \exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)p(s,t)dsdt\right)\right\}\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)dsdt\right).\end{aligned}$$

$$\begin{aligned}(2)\ u(x,y) \le M\Bigg\{1 \ &+\ \left(\int_{\alpha(x_0)}^{\infty}\int_{\beta(y_0)}^{\infty} a(s,t)p(s,t)dsdt\right)\\ &\times\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)p(s,t)dsdt\right)\Bigg\}p(x,y),\end{aligned}$$

for all $x, y \in \mathbb{R}_+$, where $p(x, y)$ is defined in Theorem 2.3.

Proof: The results of this Corollary can be obtained by setting $K(x) = x$ and $c(x, y) = M$ in Corollary 2.4.

3. RETARDED NONLINEAR INTEGRODIFFERENTIAL INEQUALITIES IN *n* INDEPENDENT VARIABLES

This section is devoted to presenting some results of nonlinear retarded integrodifferential inequalities in n independent variables.

In what follows, $D = D_1D_2...D_n$, where $D_i = \frac{\partial}{\partial x_i}$, for $i = 1, 2, ..., n$. For $x = (x_1, x_2, ..., x_n)$, $t = (t_1, t_2, ..., t_n),\ x^0 = (x_1^0, x_2^0, ..., x_n^0) \in \mathbb{R}_+^n$(where $\mathbb{R}_+^n = [0, \infty)$ is a subset of $\mathbb{R}^n, n \ge 1$), we assume:
For $x, t \in \mathbb{R}_+^n$, we write $t \le x$ whenever $t_i \le x_i$, $i = 1, 2, .., n$, and $x \ge x_0 \ge 0 \in \mathbb{R}_+^n$. For any $X = (X_1, X_2, ..., X_n) \in \mathbb{R}_+^n$, we write $x^0 \le x \le X$ whenever $x_i^0 \le x_i \le X_i, i = 1, 2, ..., n$.

$$\tilde{\alpha}(x) = (\alpha_1(x_1), \alpha_2(x_2), ..., \alpha_n(x_n)) \in \mathbb{R}_+^n,\ and\ \tilde{\beta}(x) = (\beta_1(x_1), \beta_2(x_2), ..., \beta_n(x_n)) \in \mathbb{R}_+^n.$$

We assume $\tilde{\alpha}(x) \le x$ and $\tilde{\beta}(x) \le x$ whenever $\alpha_i(x_i) \le x_i$ and $\beta_i(x_i) \le x_i$ respectively for $i = 1, 2, ..., n$, and

$$\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} dt = \int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)}\int_{\alpha_2(x_2^0)}^{\alpha_2(x_2)} ... \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} ...dt_n...dt_1,\ and\ \int_{\tilde{\beta}(x^0)}^{\tilde{\beta}(x)} dt = \int_{\beta_1(x_1^0)}^{\beta_1(x_1)}\int_{\beta_2(x_2^0)}^{\beta_2(x_2)} ... \int_{\beta_n(x_n^0)}^{\beta_n(x_n)} ...dt_n...dt_1.$$

The main results are established in the following theorems.

Theorem 3.1. *Let $u(x), c(x)$ and $a(x)$ be nonnegative continuous functions defined for $x \in \mathbb{R}_+^n$, and $\tilde{\alpha} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n)$ be a nondecreasing function for each variable, with $\tilde{\alpha}(x) \le x$ on $\mathbb{R}_+^n$. We consider that $c(x)$ is nondecreasing for each variable $x \in \mathbb{R}_+^n$. If*

$$u(x) \le c(x) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)u(t)dt, \tag{3.1}$$

for $x \in \mathbb{R}_+^n$, *then*

$$u(x) \le c(x) \exp\left(\int\limits_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)dt \right), \tag{3.2}$$

for $x \in \mathbb{R}_+^n$.

Proof: As $c(x)$ is nonnegative and nondecreasing, then from (3.1) we have

$$\frac{u(x)}{c(x)} \le 1 + \int\limits_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)\frac{u(t)}{c(t)}dt. \tag{3.3}$$

where $z(x)$ is a function defined by the right side of (3.3). Then $z(x) > 0$, $z(x_1^0, x_2, ..., x_n) = 1$, $z(..., x_i^0, x_{i+1}, ..., x_n) = 1,\ \forall i = 2, ..., n$, $\frac{u(x)}{c(x)} \le z(x)$, and

$$Dz(x) \le a(x)z(x)\tilde{\alpha}'(x).$$

Therefore

$$D_n\left(\frac{D_1...D_{n-1}z(x)}{z(x)}\right) \le a(x)\tilde{\alpha}'(x). \tag{3.4}$$

By integrating (3.4) with respect to x_n from x_n^0 to x_n, we have

$$\frac{D_1...D_{n-1}z(x)}{z(x)} \le \int\limits_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} a(x_1, ...x_{n-1}, t_n)dt_n\alpha_1'(x_1)....\alpha_{n-1}'(x_{n-1}),$$

thus

$$\frac{z(x)D_1...D_{n-1}z(x)}{z^2(x)} - \frac{D_{n-1}z(x)(D_1...D_{n-2}Dz(x))}{z^2(x)} \le \int\limits_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} a(x_1, ...x_{n-1}, t_n)dt_n\alpha_1'(x_1)....\alpha_{n-1}'(x_{n-1}),$$

hence

$$D_{n-1}\left(\frac{D_1...D_{n-2}z(x)}{z(x)}\right) \le \int\limits_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} a(x_1, ...x_{n-1}, t_n)dt_n\alpha_1'(x_1)....\alpha_{n-1}'(x_{n-1}).$$

The integration of this inequality with respect to x_{n-1} from x_{n-1}^0 to x_{n-1} yields

$$\frac{D_1...D_{n-2}z(x)}{z(x)} \le \int\limits_{\alpha_{n-1}(x_{n-1}^0)}^{\alpha_{n-1}(x_{n-1})} \int\limits_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} a(x_1, ...x_{n-2}, t_{n-1}, t_n)dt_n dt_{n-1}\alpha_1'(x_1)....\alpha_{n-2}'(x_{n-2}).$$

By continuing this process, we arrive at

$$\frac{D_1z(x)}{z(x)} \le \int\limits_{\alpha_2(x_2^0)}^{\alpha_2(x_2)} ... \int\limits_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} a(x_1, t_2, t_3, ..., t_{n-1}, t_n)dt_n...dt_2\alpha_1'(x_1). \tag{3.5}$$

By integrating (3.5) with respect to x_1 from x_1^0 to x_1, we get

$$z(x) \leq \exp\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)dt\right).$$

Finally, because $\frac{u(x)}{c(x)} \leq z(x)$ we obtain the inequality (3.2).

Remark 3.1. When $n = 2$, $x \in \mathbb{R}_+^2$, $(x_1^0, x_2^0) = (0,0)$, $\alpha_1(x_1) = x$, $\alpha_2(x_2) = y$, and $c(x) = c_1(x) + c_2(y)$ then Theorem 3.1 reduces to Lemma 1 in [9].

Theorem 3.2. *Under the same hypotheses of Theorem* 2.3, *and if*

$$Du(x) \leq c(x) + f(x)H\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(u(t))dt\right) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)Du(t)dt, \tag{3.6}$$

for $x \in \mathbb{R}_+^n$, *then*

$$\begin{aligned} Du(x) \;\; \leq \;\; & \Bigg\{c(x) + f(x)H\Bigg(G^{-1}\Bigg[G(\xi) \\ & + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dt\Bigg]\Bigg)\Bigg\} \exp\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)dt\right), \end{aligned} \tag{3.7}$$

for all $x \in \mathbb{R}_+^n$, *where*

$$p(x) = \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} \exp\left(\int_{\tilde{\alpha}(x^0)}^{s} b(\tau)d\tau\right)dt. \tag{3.8}$$

$$\xi = \int_{\tilde{\alpha}(x^0)}^{\infty} a(t)K(c(t)p(t))dt. \tag{3.9}$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{K(H(s))}, \; r \geq r_0 \geq 0, \tag{3.10}$$

where G^{-1} *is the inverse function of* G, *and* $G(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dsdt \in dom(G^{-1})$ *for* $x \in \mathbb{R}_+^n$.

Proof: It is possible to get the above result by following the same steps as in Theorem 2.3 and making simple modifications.

Remark 3.2. (i) Based on the inequalitie (3.6) and the equation (3.8), we can obtain the following result:

$$u(x) \;\; \leq \;\; \Bigg\{c(x) + f(x)H\Bigg(G^{-1}\Bigg[G(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dt\Bigg]\Bigg)\Bigg\}p(x). \tag{3.11}$$

(ii) It is enough to put $n = 2,\ x \in \mathbb{R}_+^2,\ (x_1^0, x_2^0) = (0,0),\ \alpha_1(x_1) = x,\ \alpha_2(x_2) = y,\ c(x) = c_1(x) + c_2(y),\ f(x) = 1,\ H(x) = K(x) = 1$, and $a(x) = b(x)$ so as Theorem 3.2 reduces to Theorem 1 in [9].

Corollary 3.3. *Under the same hypotheses of Theorem* 3.2, *and if*

$$Du(x) \le c(x) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(u(t))dt + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)Du(t)dt, \tag{3.12}$$

for $x \in \mathbb{R}_+^n$, *then*

$$Du(x) \le \left\{ c(x) + T^{-1}\left[T(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(p(t))dt \right] \right\} \exp\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)dt \right), \tag{3.13}$$

for all $x \in \mathbb{R}_+^n$, *where* $p(x)$ *and* ξ *are defined in Theorem* 3.2, *and*

$$T(r) = \int_{r_0}^{r} \frac{ds}{K(s)},\ r \ge r_0 \ge 0,$$

where T^{-1} *is the inverse function of* T, *and* $T(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(p(t))dt \in dom(T^{-1})$ *for* $x \in \mathbb{R}_+^n$.

Proof: We note that the proof of this Corollary follows the same steps as in Theorem 3.2.

Remark 3.3. It is enough to put $H(x) = x$ and $f(x) = 1$ in Theorem 3.2 so as to obtain the result in Corollary 3.3.

Theorem 3.4. *Under the same hypotheses of Theorem* 3.2, *and if*

$$Du(x) \le c(x) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(u(t))dt + \int_{\tilde{\beta}(x^0)}^{\tilde{\beta}(x)} b(t)Du(t)dt, \tag{3.14}$$

for $x \in \mathbb{R}_+^n$, *then*

$$Du(x) \le \left\{ c(x) + T^{-1}\left[T(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(p(t))dt \right] \right\} \exp\left(\int_{\tilde{\beta}(x^0)}^{\tilde{\beta}(x)} b(t)dt \right), \tag{3.15}$$

and

$$u(x) \le \left\{ c(x) + T^{-1}\left[T(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(p(t))dt \right] \right\} p(x), \tag{3.16}$$

for all $x \in \mathbb{R}_+^n$, *where* T *and* ξ *are defined in Corollary* 3.3, *and*

$$p(x) = \int_{\tilde{\beta}(x^0)}^{\tilde{\beta}(x)} \exp\left(\int_{\tilde{\beta}(x^0)}^{s} b(\tau)d\tau \right) dt.$$

Proof: Again, the proof of this theorem follows the same steps as in Theorem 3.2.

Remark 3.4. If $\tilde{\alpha}(x) = \tilde{\beta}(x)$, then Theorem 3.4 reduces to Corollary 3.3.

4. APPLICATIONS

This section suggests some applications of our results in order to study the boundedness and continuity of solutions of some nonlinear partial integrodifferential equations with delay.

APPLICATION 1: Suppose the following equation for functions of two independent variables

$$Du(x,y) = f(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(x,y,s,t,u(s,t),Du(s,t))dsdt, \tag{4.1}$$

with the boundary conditions $u(x_0, y) = u(x, y_0) = 0$, for $x, y \in \mathbb{R}_+$, where $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $h : \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions so that

$$|f(x,y)| \leq M,$$

and

$$|h(x,y,s,t,u(s,t),Du(s,t))| \leq a(s,t)|u(s,t)| + b(s,t)|Du(s,t)|,$$

for $x, y \in \mathbb{R}_+$, where $M > 0$ is constant and $a(x,y)$ and $b(x,y)$ are nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$. If $u(x,y)$ is any solution of Problem (4.1), then

$$|Du(x,y)| \leq M + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)|u(s,t)|dsdt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t)|Du(s,t)|dsdt.$$

Now, it is possible to obtain the bound on the solution $u(x,y)$ of (4.1) by applying Corollary 2.5 (inequality 2)

$$|u(x,y)| \leq M\left\{1 + \left(\int_{\alpha(x_0)}^{\infty} \int_{\beta(y_0)}^{\infty} a(s,t)p(s,t)dsdt\right) \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)p(s,t)dsdt\right)\right\} p(x,y),$$

for all $x, y \in \mathbb{R}_+$, where $p(x,y)$ is defined in Corollary 2.5.

APPLICATION 2: Suppose the following equation for functions of *n* independent variables

$$\begin{aligned} Du(x) &= q(x) + f(x)H\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} Q(x,t,u(t),K(u(t)))dt\right) \\ &\quad + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} W(x,t,u(t),Du(t))dt, \end{aligned} \tag{4.2}$$

with the conditions $u(x_1^0, x_2, ..., x_n) = 0$, $u(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ..., x_n) = 0$ for any $i = 2, ..., n$, where f, K, and H are defined in Theorem 3.2. $q : \mathbb{R}_+^n \to \mathbb{R}$ and $Q, W : \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions so that

$$|q(x)| \leq M,$$

and

$$|Q(x,t,u(t),K(u(t)))| \leq a(t)K(|u(t)|),$$

$$|W(x,t,u(t),Du(t))| \leq b(t)|Du(t)|,$$

for $x \in \mathbb{R}_+^n$, where $M > 0$ is constant, $a(x)$ and $b(x)$ are nonnegative continuous functions defined for $x \in \mathbb{R}_+^n$. If $u(x)$ is any solution of Problem (4.2), then

$$|Du(x)| \leq M + f(x)H\left(\int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(|u(t)|)dt\right) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} b(t)|Du(t)|dt.$$

Now, it is possible to obtain the bound on the solution $u(x)$ of (4.2) by applying Theorem 3.2 and Remark 3.2 (inequality 3.11) with $c(x) = M$

$$|u(x)| \leq \left\{M + f(x)H\left(G^{-1}\left[G(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(f(t)p(t))dt\right]\right)\right\}p(x),$$

for all $x \in \mathbb{R}_+^n$, where $p(x), G$ and ξ are defined in Theorem 3.2.

APPLICATION 3: Suppose the following equation for functions of *n* independent variables

$$Du(x) = q(x) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} Q(x,t,u(t),K(u(t)))dt + \int_{\tilde{\beta}(x^0)}^{\tilde{\beta}(x)} W(x,t,u(t),Du(t))dt, \tag{4.3}$$

with the conditions $u(x_1^0, x_2, ..., x_n) = 0$, $u(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ..., x_n) = 0$ for any $i = 2, ..., n$, where K is defined in Theorem 3.2. $q : \mathbb{R}_+^n \to \mathbb{R}$ and $Q, W : \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions so that

$$|q(x)| \leq M,$$

and

$$|Q(x,t,u(t),K(u(t)))| \leq a(t)K(|u(t)|),$$

$$|W(x,t,u(t),Du(t))| \leq b(t)|Du(t)|,$$

for $x \in \mathbb{R}_+^n$, where $M > 0$ is constant, $a(x)$ and $b(x)$ are nonnegative continuous functions defined for $x \in \mathbb{R}_+^n$. If $u(x)$ is any solution of Problem (4.3), then

$$|Du(x)| \leq M + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(|u(t)|)dt + \int_{\tilde{\beta}(x^0)}^{\tilde{\beta}(x)} b(t)|Du(t)|dt.$$

Now, it is possible to obtain the bound on the solution $u(x)$ of (4.3) by applying Theorem 3.4 with $c(x) = M$

$$|u(x)| \leq \left\{M + T^{-1}\left[T(\xi) + \int_{\tilde{\alpha}(x^0)}^{\tilde{\alpha}(x)} a(t)K(p(t))dt\right]\right\}p(x),$$

for all $x \in \mathbb{R}_+^n$, where $p(x), T$ and ξ are defined in Theorem 3.4.

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