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GENERALIZED COMPOSITION OPERATORS ON BESOV SPACES

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ABSTRACT. In this paper, we characterize boundedness, compactness and find the essential norm estimates for generalized composition operators between Besov spaces and S_p spaces.

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1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of holomorphic functions on the unit disc $\mathbb{D}$. Suppose φ and ψ are holomorphic functions defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The generalized composition operator $C_\varphi M_\psi$ is defined as

$$C_\varphi M_\psi(f)(z) = \psi(\varphi(z)) f(\varphi(z)) \text{ for all } f \in H(\mathbb{D}).$$

For $1 < p < \infty$, the analytic Besov space B_p is the conformally invariant space of all $f \in H(\mathbb{D})$ whose derivative f' belongs to the standard weighted Bergman space A^p_{p-2}, while the minimal space B_1 is the set of all analytic functions in $\mathbb{D}$, whose second derivative is integrable. The spaces B_p form a nested scale of conformally invariant spaces which are contained in the Bloch space B and represent a natural generalization of the classical Dirichlet space $D = B_2$ of analytic functions in $\mathbb{D}$. Besov spaces and their operators were studied extensively in the 80's and 90's in [1, 8, 14]. The work of this paper is motivated by the work of Choa and Ohno [4]. Our main objective in this article is to investigate boundedness, compactness and essential norm estimate between Besov spaces and S_p spaces.

1.1. M$\ddot{o}$bius invariant spaces.

For any $a \in \mathbb{D}$, let σ_a denote the M$\ddot{o}$bious transformation $\sigma_a : \mathbb{D} \to \mathbb{D}$ defined by

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \ z \in \mathbb{D}.$$

We denote the set of all M$\ddot{o}$bius transformations on $\mathbb{D}$ by G. Moreover, the inverse of σ_a, for any $z \in \mathbb{D}$, under function composition is σ_a itself. Also, we have

$$|\sigma'_a(z)| = \frac{1 - |a|^2}{|1 - \bar{a}z|^2}$$

and by simple calculation $1 - |\sigma_a(z)|^2 = (1 - |z|^2)|\sigma'_a(z)|$ for all $a, z \in \mathbb{D}$.

Let $1 < p < \infty, q > -1$. Then f is in the Besov type space $B_{p,q}$ if

$$\|f\|_{B_{p,q}} = \left(\int_{\mathbb{D}} | f^{'}(z) |^p \ (1 - | z |^2)^q \, dA(z) \right)^{\frac{1}{p}} < \infty, \tag{1.1}$$

where $dA(z)$ denotes the Lebesgue area measure on $\mathbb{D}$.
Also, if we take $1 < p < \infty$ and $q = p - 2$ in (1.1), then we get the analytic Besov space B_p. That is, an analytic function f is in the analytic Besov space B_p if

$$\|f\|_{B_p} = \left(\int_{\mathbb{D}} | f^{'}(z) |^p \ (1 - | z |^2)^{p-2} \, dA(z) \right)^{\frac{1}{p}} < \infty. \tag{1.2}$$

Again, if $p = 2$ and $-1 < q < \infty$ in (1.2), then we get the weighted Dirichlet spaces $\mathbf{D}_q$, and for $1 \le p \le 2$ and $q = 0$, we get the Dirichlet type spaces $\mathbf{D}^p$. Also, for $1 \le p < \infty, B_{p,p}$ is the Bergman space A^p. We can see that $| \ f(0) \ | + \|f\|_{p,q}$ is a norm on $B_{p,q}$, that makes it a Banach space. Moreover, we can observe that, for f to be in $B_{p,q}$ or B_p, it is necessary that the derivative of f belong to the weighted Bergman spaces A^p_q or A^p_{p-2}. Also, for $1 < p < q < \infty$, we have the relation $B_p \subset B_q$. The Besov space B_p is invariant under M$\ddot{o}$bius transformations, i.e., if $f \in B_p$, then $f \circ \varphi \in B_p$, for all $\varphi \in G$.

2. BOUNDEDNESS AND COMPACTNESS

In this section, we characterize boundedness and compactness of $C_\varphi M_\psi$ by using the Carleson measure technique.

2.1. **Carleson measures.** Let $I \subset \partial\mathbb{D}$ is an interval and $|I|$ denote the length of I. The Carleson square based on I is defined as $S(I) = \{z \in \mathbb{D} : 1- \mid I \mid \leq z < 1, \frac{z}{|z|} \in I\}$. If $p > 0$ and μ is a positive Borel measure on $\mathbb{D}$. Then μ is an $p-$Carleson measure if there exists a positive constant C such that

$$\mu(S(I)) \leq C|I|^p,$$

for any interval $I \subset \partial\mathbb{D}$. An 1-Carleson measure will be simply called a (classical) Carleson measure. If X is a subspace of $H(\mathbb{D}), q > 0$ and μ is a positive Borel measure in $\mathbb{D}$, then μ is said to be a $q-$Carleson measure for the space X or an $(X, q)-$Carleson measure if $X \subset L^q(d\mu)$. The $(X, q)-$Carleson measures have been characterized for many important spaces X of analytic functions in $\mathbb{D}$ and they arise in many questions involving analytic function spaces. In particular, they play a very important role in studying boundedness and compactness of operators acting between them.

Let φ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Let $\psi \in B_q$ be such that $\psi'(z)\varphi'(\varphi^{-1}(z))(1 - |z|^2) \in L^q(\mathbb{D}, d\lambda)$, where $d\lambda(z)$ is a M$\ddot{o}$bius invariant measure defined by $d\lambda(z) = (1 - |z|^2)^{-2}dA(z)$. Then we define the following measures $\mu_{\psi',\varphi',q}$ and $\mu_{\psi,\varphi',q}$ on $\mathbb{D}$ as

$$\mu_{\psi',\varphi',q}(E) = \int_{\varphi^{-1}(E)} |\psi'(z)|^q|\varphi'(\varphi^{-1}(z))|^q(1 - |z|^2)^{q-2}dA(z)$$

and

$$\mu_{\psi,\varphi',q}(E) = \int_{\varphi^{-1}(E)} |\psi(z)|^q|\varphi'(\varphi^{-1}(z))|^q(1 - |z|^2)^{q-2}dA(z),$$

where E is a measureable subset of the unit disk $\mathbb{D}$.
If $\psi \in A^q_{q-2}$, then we define the measure υ_q on $\mathbb{D}$ as

$$\upsilon_q(E) = \int_{\varphi^{-1}(E)} |\psi(z)|^q(1 - |z|^2)^{q-2}dA(z).$$

The following lemma can be prove by using [9, Page 163] and [3, Lemma 2.1].

Lemma 2.1. *Suppose $\varphi \in H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Take $\psi \in B_q$ such that $\psi'(z)\varphi'(\varphi^{-1}(z))(1-|z|^2) \in L^q(\mathbb{D}, d\lambda)$. Then*

$$\int_{\mathbb{D}} hd\mu_{\psi,\varphi',q} = \int_{\mathbb{D}} |\psi(z)|^q|\varphi'(\varphi^{-1}(z))|^q|h(\varphi(z))|^q(1 - |z|^2)^{q-2}dA(z)$$

and

$$\int_{\mathbb{D}} hd\mu_{\psi',\varphi',q} = \int_{\mathbb{D}} |\psi'(z)|^q|\varphi'(\varphi^{-1}(z))|^q|h(\varphi(z))|^q(1 - |z|^2)^{q-2}dA(z),$$

where h is any arbitrary measurable positive function in $\mathbb{D}$.

The following lemma, whose proof is omitted, will be used to prove next theorems .

Lemma 2.2. *Take $1 < p, q < \infty$ and let $\varphi \in H(\mathbb{D})$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Suppose $\psi \in B_q$ such that $C_\varphi M_\psi : B_p \to B_q$ is bounded. Then $C_\varphi M_\psi : B_p \to B_q$ is compact(weakly compact) if and only if whenever a bounded sequence say $\{f_n\}$ is in B_p and converges to zero uniformly on compact subsets of $\mathbb{D}$, then $||C_\varphi M_\psi(f_n)||_{B_q} \to 0$ (respectively, $\{C_\varphi M_\psi(f_n)\}$ is a weak null sequence in B_q).*

Now, we can prove the following theorem.

Theorem 2.3. *Fix $1 < p \leq q < \infty$. Suppose $\psi \in A^q_{q-2}$ and $\varphi \in B_p$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If υ_q is a vanishing q-Carleson measure for B_q, then $C_\varphi M_\psi : B_p \to A^q_{q-2}$ is bounded and also compact.*

Proof. Let $\{f_n\}$ be a bounded sequence in B_p such that $\{f_n\} \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since υ_q is a vanishing $q-$Carleson measure for B_q, the inclusion operator $i : B_q \to L^q(\mathbb{D}, \upsilon_q)$ is compact. Also, $B_p \subset B_q$, we have $||f_n||_{L^q(\mathbb{D}, \upsilon_q)} \to 0 \; as \; n \to \infty$. So, by Lemma 2.1, we have

$$\begin{aligned} ||C_\varphi M_\psi(f_n)||^q_{A^q_{q-2}} &= \int_{\mathbb{D}} |\psi(\varphi(z))|^q |f_n(\varphi(z))|^q (1-|z|^2)^{q-2} dA(z) \\ &= \int_{\mathbb{D}} |f_n|^q d\upsilon_q \to 0 \; as \; n \to \infty. \end{aligned}$$

Thus, $C_\varphi M_\psi : B_p \longrightarrow A^q_{q-2}$ is compact. ∎

Theorem 2.4. *Take $1 < p \le q < \infty$ and let $\varphi, \psi \in B_p$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If $\mu_{\psi', \varphi', q}$ is a vanishing q-Carleson measure for B_q, then $C_\varphi M_\psi : B_p \to B_q$ is bounded if and only if $M_{\varphi'} C_\varphi M_\psi : A^p_{p-2} \to A^q_{q-2}$ is bounded.*

Proof. Suppose $C_\varphi M_\psi : B_p \to B_q$ is bounded. Then, there exists a constant $C > 0$ such that

$$||C_\varphi M_\psi(f)||_{B_q} \le C||f||_{B_p} \text{ for all } f \in B_p.$$

Also, by Theorem 2.3, we can find a constant $M > 0$ such that

$$||C_\varphi M_{\psi'}(f)||_{A^q_{q-2}} \le M||f||_{B_p} \text{ for all } f \in B_p.$$

Let $g \in B_p$ and $f \in A^p_{p-2}$ be such that $g' = f$ and $g(0) = 0$.
Then

$$\begin{aligned} ||M_{\varphi'} C_\varphi M_\psi(f)||_{A^q_{q-2}} &= ||\varphi'(\psi o \varphi)(f o \varphi)||_{A^q_{q-2}} \\ &= ||\varphi'(\psi o \varphi)(g' o \varphi) + \varphi'(\psi' o \varphi)(g o \varphi) - \varphi'(\psi' o \varphi)(g o \varphi)||_{A^q_{q-2}} \\ &\le ||((\psi o \varphi)(g o \varphi))'||_{A^q_{q-2}} + ||\varphi'(\psi' o \varphi)(g o \varphi)||_{A^q_{q-2}} \\ &= ||C_\varphi M_\psi(g)||_{B_q} + ||C_\varphi M_{\psi'}(g)||_{A^q_{q-2}} \\ &\le (C+M)||g||_{B_p} \\ &= (C+M)||f||_{A^p_{p-2}} \\ &< \infty. \end{aligned}$$

Hence $M_{\varphi'} C_\varphi M_\psi : A^p_{p-2} \to A^q_{q-2}$ is bounded.
Conversely, suppose $M_{\varphi'} C_\varphi M_\psi : A^p_{p-2} \to A^q_{q-2}$ is bounded. Again, by Theorem 2.3, $C_\varphi M_{\psi'} : B_p \to A^q_{q-2}$ is bounded. Let $f \in B_p$ be such that $f(0) = 0$.
Then

$$\begin{aligned} ||C_\varphi M_\psi(f)||_{B_q} &= ||((\psi o \varphi)(f o \varphi))'||_{A^q_{q-2}} \\ &= ||\varphi'(\psi' o \varphi)(f o \varphi) + \varphi'(\psi o \varphi)(f' o \varphi)||_{A^q_{q-2}} \\ &\le ||C_\varphi M_{\psi'}(f)||_{A^q_{q-2}} + ||M_{\varphi'} C_\varphi M_\psi(f')||_{A^q_{q-2}} \\ &< \infty. \end{aligned}$$

∎

The following theorem can be proved by using Theorem 2.4 and Theorem 1 of [7] so we omit the proof.

Theorem 2.5. *Take $1 < p \le q < \infty$ and let $\varphi, \psi \in B_p$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If $\mu_{\psi', \varphi', q}$ is a vanishing q-Carleson measure for B_q, then $C_\varphi M_\psi : B_p \to B_q$ is bounded if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2} \right)^q d\mu_{\psi, \varphi', q}(z) < \infty.$$

Theorem 2.6. *Fix* $1 < p \leq q < \infty$. *Let* $\varphi, \psi \in B_p$ *be such that* $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. *Suppose* $\mu_{\psi',\varphi',q}$ *is a vanishing q-Carleson measure for* B_q. *Then* $C_\varphi M_\psi : B_p \to B_q$ *is compact if and only if* $M_{\varphi'} C_\varphi M_\psi : A^p_{p-2} \to A^q_{q-2}$ *is compact.*

Proof. Suppose $C_\varphi M_\psi : B_p \to B_q$ is compact and let $\{f_n\}$ be a bounded sequence in A^p_{p-2} such that $\{f_n\} \to 0$ uniformly on compact subsets of $\mathbb{D}$. Consider the function $g_n \in B_p, n \in \mathbb{N}$ such that $g_n' = f_n$ and $g_n(0) = 0$. The sequence $\{g_n\}$ also converges on compact subsets of $\mathbb{D}$ as $n \to \infty$. Since, $C_\varphi M_\psi : B_p \to B_q$ is compact, so $||C_\varphi M_\psi(g_n)||_{B_q} \to 0$ as $n \to \infty$. By Theorem 2.3, $C_\varphi M_{\psi'} : B_p \to A^q_{q-2}$ is compact, so $||C_\varphi M_\psi(g_n)||_{A^q_{q-2}}$ also converges to zero as $n \to \infty$. Now

$$\begin{aligned}
||M_{\varphi'} C_\varphi M_\psi(f_n)||_{A^q_{q-2}} &= ||\varphi'(\psi o \varphi)(f_n o \varphi)||_{A^q_{q-2}} \\
&= ||\varphi'(\psi o \varphi)(g_n' o \varphi) + \varphi'(\psi' o \varphi)(g_n o \varphi) - \varphi'(\psi' o \varphi)(g_n o \varphi)||_{A^q_{q-2}} \\
&\leq ||((\psi o \varphi)(g_n o \varphi))'||_{A^q_{q-2}} + ||\varphi'(\psi' o \varphi)(g_n o \varphi)||_{A^q_{q-2}} \\
&= ||C_\varphi M_\psi(g_n)||_{B_q} + ||C_\varphi M_{\psi'}(g_n)||_{A^q_{q-2}} \\
&\leq (C + M)||g_n||_{B_p} \\
&= (C + M)||f_n||_{A^p_{p-2}}
\end{aligned}$$

Thus, $M_{\varphi'} C_\varphi M_\psi : A^p_{p-2} \to A^q_{q-2}$ is compact.

Conversely, suppose that $M_{\varphi'} C_\varphi M_\psi : A^p_{p-2} \to A^q_{q-2}$ is compact. Again, by Theorem 2.3, $C_\varphi M_{\psi'} : B_p \to A^q_{q-2}$ is compact. Let g_n be the same sequence as in the direct part. Then, we have

$$\begin{aligned}
||C_\varphi M_\psi(g_n)||_{B_q} &= ||((\psi o \varphi)(g_n o \varphi))'||_{A^q_{q-2}} \\
&= ||\varphi'(\psi' o \varphi)(g_n o \varphi) + \varphi'(\psi o \varphi)(g_n' o \varphi)||_{A^q_{q-2}} \\
&\leq ||C_\varphi M_{\psi'}(g_n)||_{A^q_{q-2}} + ||M_{\varphi'} C_\varphi M_\psi(f_n)||_{A^q_{q-2}} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $C_\varphi M_\psi : B_p \to B_q$ is compact. ∎

The following theorem can be proved using Theorem 2.6 and Corollary 1 of [7].

Theorem 2.7. *Take* $1 < p \leq q < \infty$. *Let* $\varphi, \psi \in B_p$ *be such that* $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ *and* $C_\varphi M_\psi : B_p \to B_q$ *is bounded. Also, suppose that the measure* $\mu_{\psi',\varphi',q}$ *is a vanishing q-Carleson measure for* B_q. *Then* $C_\varphi M_\psi : B_p \to B_q$ *is compact if and only if*

$$\lim_{|a| \to 1} \sup \int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^q d\mu_{\psi,\varphi',q}(z) = 0.$$

Theorem 2.8. *Let* $1 < p \leq q < \infty$. *Let* $\varphi, \psi \in B_p$ *be such that* $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. *Also, suppose that the measure* $\mu_{\psi',\varphi',p}$ *is a vanishing p-Carleson measure for* B_p. *Then,* $C_\varphi M_\psi : B_p \to B_q$ *is bounded(compact) if and only if the measure* $\mu_{\psi,\varphi',q}$ *is a bounded (respectively vanishing) q-Carleson measure for* B_q.

Proof. Let $\{f_n\}$ be a bounded sequence in B_p such that $\{f_n\} \to 0$ as $n \to \infty$ on compact subset of $\mathbb{D}$. Suppose that $C_\varphi M_\psi : B_p \to B_q$ is compact. Then, by using Theorem 2.6, $M_{\varphi'} C_\varphi M_\psi : A^p_{p-2} \to A^q_{q-2}$ is also compact. Also, by Theorem 2.3, $C_\varphi M_{\psi'} : B_p \to A^q_{q-2}$ is compact. Therefore

$$\begin{aligned}
||M_{\varphi'} C_\varphi M_\psi(f_n')||^q_{A^q_{q-2}} &= \int_{\mathbb{D}} |\psi(\varphi(z))|^q |\varphi'(z)|^q |f_n'(\varphi(z))|^q (1 - |z|^2)^{q-2} dA(z) \\
&= \int_{\mathbb{D}} |f_n'(\omega)|^q d\mu_{\psi,\varphi',q}(\omega) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This means that the inclusion operator $i : B_p \to D_q(\mu)$ is compact. Thus, $\mu_{\psi,\varphi',q}$ is a vanishing q-Carleson measure for B_q.
Conversely, suppose that $\mu_{\psi,\varphi',q}$ is a vanishing q-Carleson measure for B_q. We will prove that $C_\varphi M_\psi : B_p \to B_q$ is compact. Let $\{f_n\}$ be a sequence as defined in direct part. Also, we have

$$((\psi o\varphi)(fo\varphi))' = (\psi' o\varphi)\varphi'(fo\varphi) + (\psi o\varphi)\varphi'(f' o\varphi).$$

So, by using Lemma 2.1, we have

$$\int_{\mathbb{D}} |\psi'(\varphi(z))|^q |\varphi'(z)|^q |f_n(\varphi(z))|^q (1-|z|^2)^{q-2} dA(z) = \int_{\mathbb{D}} |f_n(z)|^q d\mu_{\psi',\varphi',q} \to 0 \text{ as } n \to \infty$$

and

$$\int_{\mathbb{D}} |\psi(\varphi(z))|^q |\varphi'(z)|^q |f_n'(\varphi(z))|^q (1-|z|^2)^{q-2} dA(z) = \int_{\mathbb{D}} |f_n'(z)|^q d\mu_{\psi,\varphi',q} \to 0 \text{ as } n \to \infty.$$

Thus, $C_\varphi M_\psi : B_p \to B_q$ is compact. ∎

3. THE ESSENTIAL NORM

Recall that the essential norm of a bounded linear operator T is the distance from T to the compact operators, i.e., $||T||_e = \inf\{||T-K|| : \text{where K is a compact operator}\}$. Clearly T is compact if and only if its essential norm is 0. In this section, we give estimate for the essential norm of $C_\varphi M_\psi$ on Besov spaces.

Lemma 3.1. [6] *Take $0 < r < 1$ and denote $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$. Let μ be a positive Borel measure on $\mathbb{D}$. Take*

$$||\mu||_r = \sup_{|I| \le 1-r} \frac{\mu(S(I))}{|I|^p} \text{ and } ||\mu|| = \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p},$$

where I run through arcs on the unit circle. Let μ_r denote the restriction of the measure μ to the set $\mathbb{D}/\mathbb{D}_r$. Further, if μ is a Carleson measure for some Besov space, so is μ_r and $||\mu_r|| \le M||\mu||_r$, where $M > 0$ is a constant.

Lemma 3.2. [6] *For $0 < r < 1$ and $1 < p < \infty$, let*

$$||\mu||_r^* = \sup_{|a| \ge r} \int_{\mathbb{D}} |\sigma_a'(z)|^p d\mu(z).$$

Moreover, if μ is a Carleson measure for some Besov space, then $||\mu_r|| \le K||\mu||_r^*$, where K is an absolute constant.

Take $f(z) = \sum_{s=0}^{\infty} a_s z^s$ be holomorphic on $\mathbb{D}$. For a positive integer n, define the operator $R_n f(z) = \sum_{s=n+1}^{\infty} a_s z^s$ and $K_n = I - R_n$, where I is the identity map.

By using [15, Theorem 5.3.7] and [6, Proposition 3], we get the following generalization of [5, Lemma 3.16. p–134] for the Besov space.

Lemma 3.3. *If T is a bounded linear operator on B_p, then*

$$K \limsup_{n\to\infty} \|TR_n\| \le \|T\|_e \le \liminf_{n\to\infty} \|TR_n\|,$$

for some constant $K > 0$.

We will give the upper and the lower estimates for essential norm of the operator $C_\varphi M_\psi$ in the following theorem.

Theorem 3.4. *For $1 < p < \infty$, let $\varphi, \psi \in B_p$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Suppose $\mu_{\psi',\varphi',p}$ is a vanishing $p-$Carleson measure and $C_\varphi M_\psi$ is a bounded operator on B_p. Then there exist absolute constants $C_1, C_2 > 0$ such that*

$$\lim_{|a|\to 1}\sup ||(C_\varphi M_\psi)\sigma_a||^p_{B_p} \le ||C_\varphi M_\psi||^p_e \le C_1 \lim_{|a|\to 1}\sup \Phi(a) + C_2 \lim_{|a|\to 1}\sup \Psi(a),$$

where

$$\Phi(a) = \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^p d\mu_{\psi,\varphi',p} \text{ and } \Psi(a) = \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^p d\mu_{\psi',\varphi',p}.$$

Proof. Upper estimate: By Lemma 3.3, we have

$$||C_\varphi M_\psi||^p_e \le \lim_{n\to\infty}\inf ||C_\varphi M_\psi||^p_{B_p} \le \lim_{n\to\infty}\inf \sup_{||f||_{B_p}\le 1} ||(C_\varphi M_\psi R_n)f||^p_{B_p}.$$

Consider

$$||(C_\varphi M_\psi R_n)f||^p_{B_p} = |\psi(\varphi(0))(R_n f(\varphi(0)))| + ||((\psi o \varphi)(R_n f o \varphi))'||^p_{A^p_{p-2}}.$$

Now $|\psi(\varphi(0))(R_n f(\varphi(0)))| \to |\psi(\varphi(0))|$ as $n \to \infty$ which is bounded as $\psi \in B_p$. Therefore by Lemma 2.1, we have

$$\begin{aligned} ||(C_\varphi M_\psi R_n)f||^p_{B_p} &= \int_{\mathbb{D}} |(\psi(\varphi(z))(R_n f(\varphi(z))))'|^p (1-|z|^2)^{p-2} dA(z) \\ &\le \int_{\mathbb{D}} |\psi(\varphi(z))|^p |\varphi'(z)|^p |(R_n f)'(\varphi(z))|^p (1-|z|^2)^{p-2} dA(z) \\ &+ \int_{\mathbb{D}} |\psi'(\varphi(z))|^p |\varphi'(z)|^p |(R_n f)(\varphi(z))|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_{\mathbb{D}} |(R_n f)'(\omega)|^p d\mu_{\psi,\varphi',p}(\omega) + \int_{\mathbb{D}} |(R_n f)(\omega)|^p d\mu_{\psi',\varphi',p}(\omega) \\ &= I_1 + I_2. \end{aligned}$$

The last condition follows by using Theorem 2.3 and Theorem 2.4. Now, we take the integral I_1,

$$\begin{aligned} \int_{\mathbb{D}} |\ (R_n f)'(\omega)\ |^p\ d\mu_{\psi',\varphi',p}(\omega) \quad &= \quad \int_{\mathbb{D}\setminus\mathbb{D}_r} |\ (R_n f)'(\omega)\ |^p\ d\mu_{\psi',\varphi',p}(\omega) \\ &\quad + \int_{\mathbb{D}_r} |\ (R_n f)'(\omega)\ |^p\ \ d\mu_{\psi',\varphi',p}(\omega). \end{aligned}$$

Also, the measure $\mu_{\psi',\varphi',p}$ is a bounded p–Carleson measure, because the operator $C_\varphi M_\psi$ is bounded on B_p. Let $K_\omega = 1 + \log(\frac{1}{1-\overline{\omega}z})$ be the kernel for evaluation at ω. Using [5, page–133], we have

$$|R_n f(\omega)| \le \|f\|_{B_p} \|R_n K_\omega\|_{B_p}.$$

Take $0 < r < 1$ and $|\ \omega\ | \le r, z \in \mathbb{D}$. Also, take the Taylor expansion of $K_\omega = \sum_{k=1}^\infty \frac{\overline{\omega}^k z^k}{k}$. Using this Taylor expansion, we get that $|\ R_n K_\omega(z)\ | \le \sum_{k=1}^\infty \frac{r^k}{k}$. Thus, for any $\epsilon > 0$, we can find n large enough such that

$$\int_{\mathbb{D}} |\ R_n K_\omega(z)\ |^q\ (1-|\ z\ |^2)^{q-2} dA(z) < \epsilon^p.$$

Therefore, for a fixed r, we have

$$\sup_{\|f\|_{B_p}\leq 1}\int_{\mathbb{D}_r} \mid (R_n)^{'}(\omega) \mid^p \, d\mu_{\psi',\varphi',p} \to 0 \text{ as } n\to\infty.$$

Let $\mu_{\psi',\varphi',p,r}$ denotes the restriction of measure $\mu_{\psi',\varphi',p}$ to the set $\mathbb{D}\setminus\mathbb{D}_r$. So by using Lemma 3.2 and using [1], we have

$$\begin{aligned}\int_{\mathbb{D}\setminus\mathbb{D}_r} \mid (R_nf)'(\omega) \mid^p \, d\mu_{\psi',\varphi',p,r}(\omega) &\leq K \,\|\mu_{\psi',\varphi',p,r}\|\|(R_nf)^{'}\|^p_{B_p}\\ &\leq K\ M\|\mu_{\psi',\varphi',p}\|^*_r\|f\|^p_{B_p} \leq K\ M\ \|\mu_{\psi',\varphi',p}\|^*_r,\end{aligned}$$

where K and M are absolute constants and $\|\mu_{\psi',\varphi',p}\|^*_r$ is defined as in Lemma 3.2.

By following the similar techniques as above, we can show that the integral I_1 is also bounded by $K_1\ M_1\ \|\mu_{\psi,\varphi',p}\|^*_r$, where K_1 and M_1 are absolute constants.
Therefore,

$$\lim_{n\to\infty}\sup_{\|f\|_{B_p}\leq 1}\|(C_\varphi M_\psi R_n)f\|^p_{B^p} \leq \lim_{n\to\infty} K\ M\ \|\mu_{\psi',\varphi,p}\|^*_r + \lim_{n\to\infty} K_1\ M_1\ \|\mu_{\psi,\varphi',p}\|^*_r.$$

Thus, $\|C_\varphi M_\psi\|^p_e \leq K\ M\ \|\mu_{\psi,\varphi',p}\|^*_r + K_1\ M_1\ \|\mu_{\psi',\varphi,p}\|^*_r$.
Taking $r\to 1$, we have

$$\begin{aligned}&\|C_\varphi M_\psi\|^p_e \ \leq\ K\ M\ \lim_{r\to\infty}\|\mu_{\psi,\varphi',p}\|^*_r + K_1\ M_1\ \lim_{r\to\infty}\|\mu_{\psi',\varphi',p}\|^*_r\\ =\ & K\ M\ \limsup_{\mid a\mid\to 1}\int_{\mathbb{D}} \mid \sigma_a^{'}(\omega)\mid^2\ \ d\mu_{\psi,\varphi',p}(\omega) + K_1\ M_1\ \limsup_{\mid a\mid\to 1}\int_{\mathbb{D}}\mid \sigma_a^{'}(\omega)\mid^2\ \ d\mu_{\psi',\varphi',p}(\omega)\\ =\ & K\ M\ \limsup_{\mid a\mid\to 1}\int_{\mathbb{D}}(\frac{1-\mid a\mid^2}{\mid 1-\bar{a}\omega\mid^2})^p d\mu_{\psi,\varphi',p}(\omega)\\ &\quad +K_1\ M_1\ \limsup_{\mid a\mid\to 1}\int_{\mathbb{D}}(\frac{1-\mid a\mid^2}{\mid 1-\bar{a}\omega\mid)^2})^p d\mu_{\psi',\varphi',p}(\omega)\\ =\ & K\ M\ \limsup_{\mid a\mid\to 1}\Phi(a) + K_1\ M_1\ \limsup_{\mid a\mid\to 1}\Psi(a)\end{aligned}$$

which is the desired upper bound.

Now, we will prove the lower bound. Clearly, the set $\{\sigma_a : a\in\mathbb{D}\}$ is bounded in B_p. Moreover, $\sigma_a - a\to 0$ as $|a|\to 1$ uniformly on compact subsets of $\mathbb{D}$ as $|\sigma_a(z)-a| = |z|\frac{1-|a|^2}{1-\bar{a}z}$. Let K is a compact operator on B_p. Then $||K(\sigma_a - a)||_{B_p}\to 0$ as $|a|\to 1$. Thus, $||K\sigma_a||_{B_p}\to 0$ as $|a|\to 1$. Therefore, by Lemma 3.2, we have

$$\lim_{|a|\to 1} sup||(C_\varphi M_\psi)\sigma_a||^p_{B_p} \leq ||C_\varphi M_\psi - K||^p_{B_p} \leq ||C_\varphi M_\psi||^p_e$$

This completes the proof. ∎

Theorem 3.5. *For $1<p\leq q<\infty$, let $\varphi,\psi\in B_p$ be such that $\varphi(\mathbb{D})\subseteq\mathbb{D}$. Suppose $\mu_{\psi',\varphi',q}$ is a vanishing q-Carleson measure for B_q and $C_\varphi M_\psi$ is bounded from B_p into B_q. Then, there exists an absolute constant $C>0$ such that*

$$\lim_{|a|\to 1}\sup\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^q d\mu_{\psi,\varphi',q} \leq ||M_{\varphi'}C_\varphi M_\psi||^q_e \leq C\lim_{|a|\to 1}\sup\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^q d\mu_{\psi,\varphi',q}.$$

Proof. Since $C_\varphi M_\psi$ is bounded from B_p into B_q, Therefore by Theorem 2.4, $M_{\varphi'}C_\varphi M_\psi$ is also bounded operator from A^p_{p-2} into A^q_{q-2}. Now using Theorem 2 of [7], we have

$$\lim_{|a|\to 1}\sup\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^q|\psi(\varphi(\omega))\varphi'(\omega)|^q(1-|z|^2)^{q-2}dA(z)$$
$$\le ||M_{\varphi'}C_\varphi M_\psi||_e^q \le C\lim_{|a|\to 1}\sup\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^q|\psi(\varphi(\omega))\varphi'(\omega)|^q(1-|z|^2)^{q-2}dA(z).$$

Therefore

$$\lim_{|a|\to 1}\sup\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^q d\mu_{\psi,\varphi',q}\le ||M_{\varphi'}C_\varphi M_\psi||_e^q\le C\lim_{|a|\to 1}\sup\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^q d\mu_{\psi,\varphi',q}.$$

∎

4. GENERALIZED COMPOSITION OPERATORS BETWEEN S_p SPACES

In this section, we will find estimates for the essential norm of generalized composition operators. Let a positive measure on the disk $\mathbb{D}$ is defined as μ. Then we define the space $\mathbb{D}_p(\mu)$ as the space of all holomorphic functions $f \in H(\mathbb{D})$ such that $f' \in L^p(\mathbb{D},\mu)$. Moreover, the norm on $\mathbb{D}_p(\mu)$ is defined as

$$||f||_{\mathbb{D}_p(\mu)} = \left(\int_{\mathbb{D}}|f'(z)|^p d\mu(z)\right)^{1/p}. \tag{4.1}$$

Let $1 \le p \le \infty$. Then, $H^p(\mathbb{D})$ denotes the Hardy space of the unit disk $\mathbb{D}$, see [5]. The space of all those holomorphic functions on $\mathbb{D}$ whose first derivative is in the Hardy space $H^p(\mathbb{D})$ is denoted by S_p. We define the S_p norm of f as

$$||f||_{S_p} = |f(0)| + ||f'||_{H_p}. \tag{4.2}$$

We see that S_p is a Banach space with this norm.

Let $f \in H^p$. Then, according to Fatou's theorem, the radial limit $f^*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exists almost everywhere on $\partial\mathbb{D}$ and $f^* \in L^p(\partial\mathbb{D}, d\rho)$, where $d\rho(z)$ is the normalized measure on $\partial\mathbb{D}$. We can denote this radial limit by f also.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi \in H(\mathbb{D})$ be such that $\psi(z)\varphi'(\varphi^{-1}(z)) \in H^q$. Then, we define the measure $\mu_{\psi\varphi',q}$ on $\overline{\mathbb{D}}$ as

$$\mu_{\psi\varphi',q}(E) = \int_{\varphi^{-1}(E)\cap\partial\mathbb{D}}|\psi(z)\varphi'(\varphi^{-1}(z))|^q d\rho(z),$$

where E is a measurable subset of the closed unit disc $\overline{\mathbb{D}}$.

The proof of following theorem follows on similar lines as in Theorem 2.1 of [3].

Theorem 4.1. *Let $1 \le p, q \le \infty$, $\varphi \in H(\mathbb{D})$ be such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi \in S_q$. Then $C_\varphi M_\psi : S_p \to S_q$ is bounded if and only if $M_{\varphi'}C_\varphi M_\psi$ exists as a bounded operator from H^p into H^q.*
Moreover, if $(p, q) \neq (1, \infty)$, then the operator $C_\varphi M_\psi : S_p \to S_q$ is compact if and only if $M_{\varphi'}C_\varphi M_\psi : H^p \to H^q$ is compact.

By using Theorem 4.1 and Theorem 4 of [7], we can prove the following theorem.

Theorem 4.2. *Let $1 \le p, q < \infty$, $\varphi \in H(\mathbb{D})$ be such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi \in S_q$. Then $C_\varphi M_\psi : S_p \to S_q$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\partial\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^{q/p} d\mu_{\psi\varphi',q}(\omega) < \infty.$$

Proof. Let $C_\varphi M_\psi : S_p \to S_q$ be a bounded operator. Then, by Theorem 4.1, $M_{\varphi'} C_\varphi M_\psi : H^p \to H^q$ is also bounded. Therefore by Theorem 4 of [7], we have

$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^{q/p} |\psi(\varphi(\omega))|^q |\varphi'(\varphi^{-1}(\omega))|^q d\rho(\omega) < \infty.$$

Thus,

$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^{q/p} d\mu_{\psi\varphi',q}(\omega) < \infty.$$

∎

The proof of the next theorem follows from Theorem 4.1 and Theorem 5 of [7]. So we omitted the proof.

Theorem 4.3. *Let $1 \le p, q < \infty$, $\varphi \in H(\mathbb{D})$ be such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi \in S_q$. Suppose $C_\varphi M_\psi : S_p \to S_q$ is bounded . Then, there exists an absolute constant $C > 0$ such that*

$$\begin{aligned}\lim_{|a|\to 1} \sup \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^{q/p} d\mu_{\psi\varphi',q}(\omega) &\le ||M_{\varphi'} C_\varphi M_\psi||_e^q \\ &\le C \lim_{|a|\to 1} \sup \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\omega|^2}\right)^{q/p} d\mu_{\psi\varphi',q}(\omega).\end{aligned}$$

Similarly, the proof of next theorem can be done using Theorem 4.1 and Proposition 2 of [7].

Theorem 4.4. *Let $1 \le p, q < \infty$, $\varphi \in H(\mathbb{D})$ be such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi \in S_q$. Then $C_\varphi M_\psi : S_p \to S_q$ is bounded if and only if*

$$\int_0^{2\pi} \left(\int_{\Gamma(\theta)} \frac{d\mu_{\psi\varphi',q}(\omega)}{1-|\omega|^2}\right)^{\frac{p}{p-q}} d\theta < \infty,$$

where $\Gamma(\theta)$ is the Stolz angle at θ, which is defined for real θ as the convex hull of the set $\{e^{i\theta}\} \cup \{z : |z| < \sqrt{1/2}\}$.

5. HILBERT-SCHMIDT OPERATORS

In this section, we find the condition to formalize the relationship between generalized composition operators on Besov space B_2 and Hilbert-Schmidt operators. We will also study some examples based on this relationship which are already proved for H^2 and L^2_α in [4]. The proof of the following theorem follows on similar lines as the proof of Theorem 1 of [11]. So, we will omit the proof.

Theorem 5.1. *Let $\varphi, \psi \in B_2$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then, $C_\varphi M_\psi : B_2 \to B_2$ is Hilbert-Schmidt operator if and only if*

$$\int_{\mathbb{D}} \left[|\psi'(\varphi(z))|^2 |\varphi'(z)|^2 \log\frac{1}{1-|\varphi(z)|^2} + \frac{|\psi(\varphi(z))|^2|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\right] dA(z) < \infty.$$

Example 5.1. *Let $\psi(z) = (1-z)^\beta$ where $\beta > 2$ and let $\varphi(z) = 1 - \sqrt{1-z}$. Then $C_\varphi M_\psi$ is compact on B_2.*

Proof. We see that φ maps the unit disk $\mathbb{D}$ univalently into a non tangential region with vertex at the point 1. So, for $|z| \le 1$, we have

- $1 - |\varphi(z)|^2 \approx |1 - \varphi(z)| = |1 - z|^{1/2}$;
- $\psi(\varphi(z)) = (1 - z)^{\frac{\beta}{2}}$;
- $\varphi'(z) = 0 - \gamma(1 - z)^{\gamma-1}(-1) = \gamma(1 - z)^{\gamma-1}$

- $\psi'(z) = \beta(1-z)^{\beta-1}(-1) = -\beta(1-z)^{\beta-1}$.

Therefore, $\psi'(\varphi(z)) = -\beta(1-1+\sqrt{1-z})^{\beta-1} = -\beta(1-z)^{\frac{\beta-1}{2}}$ and $\varphi'(z) = 0 - \frac{1}{2\sqrt{1-z}}(-1) = \frac{1}{2}\frac{1}{\sqrt{1-z}}$. Thus,

$$\int_{\mathbb{D}}\left[|\psi'(\varphi(z))|^2|\varphi'(z)|^2 \log\frac{1}{1-|\varphi(z)|^2} + \frac{|\psi(\varphi(z))|^2|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\right]dA(z)$$
$$= \int_{\mathbb{D}}\left[\beta^2|1-z|^{\beta-1}\,\frac{1}{2^2}\frac{1}{|1-z|}\log\frac{1}{|1-z|^{1/2}} + \frac{|1-z|^{\beta}\frac{1}{2^2}\frac{1}{|1-z|}}{(|1-z|^{1/2})^2}\right]dA(z)$$
$$= \int_{\mathbb{D}}\left[\left(\frac{\beta}{2}\right)^2\left(|1-z|^{\beta-2}\log\frac{1}{|1-z|^{1/2}}\right) + \frac{1}{2^2}|1-z|^{\beta-2}\right]dA(z)$$

which is clearly compact as $\beta > 2$. ∎

Example 5.2. *Let $\psi(z) = (1-z)^{\beta}$, where $\beta > 1$ and let $\varphi(z) = \frac{(z+1)}{2}$. Then, prove that $C_\varphi M_\psi$ is compact on B_2.*

Proof. We see that φ maps the unit disk $\mathbb{D}$ univalently into a non tangential region with vertex at the point 1. Thus for $|z| \le 1$, we have

- $1-|\varphi(z)|^2 \approx |1-\varphi(z)| = |1-\left(\frac{z+1}{2}\right)| = |\frac{1-z}{2}|$
- $\psi(\varphi(z)) = (1-\frac{z+1}{2})^{\beta} = \left(\frac{1-z}{2}\right)^{\beta}$
- $\psi'(z) = \beta(1-z)^{\beta-1}(-1) = -\beta(1-z)^{\beta-1}$.

Therefore, $\psi'(\varphi(z)) = -\beta\left(1-\frac{z+1}{2}\right)^{\beta-1} = -\beta\left(\frac{1-z}{2}\right)^{\beta-1}$ and $\varphi'(z) = \frac{1}{2}$
and so

$$\int_{\mathbb{D}}\left[|\psi'(\varphi(z))|^2|\varphi'(z)|^2 \log\frac{1}{1-|\varphi(z)|^2} + \frac{|\psi(\varphi(z))|^2|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\right]dA(z)$$
$$= \int_{\mathbb{D}}\left[\frac{\beta^2}{2^{2(\beta-1)}}|1-z|^{2(\beta-1)}\frac{1}{2^2}\log\frac{1}{|\frac{1-z}{2}|} + \frac{|\frac{1-z}{2}|^{2\beta}\frac{1}{2^2}}{|\frac{1-z}{2}|^2}\right]dA(z)$$
$$= \int_{\mathbb{D}}\left[\frac{\beta^2}{2^{2\beta-2}}\frac{1}{2^2}|1-z|^{2\beta-2}\log\frac{1}{|\frac{1-z}{2}|} + \frac{1}{2^2}|\frac{1-z}{2}|^{2\beta-2}\right]dA(z)$$

which is clearly compact as $\beta > 1$. ∎

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