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# SEMIGROUP OF LINEAR OPERATOR IN BICOMPLEX SCALARS

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ABSTRACT. In this paper, we have studied the generators of $C_0$-semigroups of bicomplex linear operators on $\mathbb{BC}$-Banach modules. This work is based on [5].

*Key words and phrases:* $\mathbb{D}$-bounded, $\mathbb{D}$-valued norm; $\mathbb{BC}$-Banach module; $\mathbb{BC}$-linear operator; $C_0$-semigroup and generator of $C_0$-semigroup.

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## 1. INTRODUCTION

The theory of bicomplex numbers is an area of active research for quite a long time since the innovative work of Segre in search of special algebras. In 1892, Segre proposed the concept of bicomplex numbers which can be thought of as a generalization of complex numbers. Because not every non zero bicomplex number has a multiplicative inverse, the set of bicomplex numbers is a commutative ring with unity that contains the field of complex numbers but does not form a field, the study of zero divisors in bicomplex analysis is introduced. For a long time, bicomplex numbers have been explored. For a recent works on bicomplex analysis and its application we refer to [1],[11],[12],[13] and [15].

The theory of one parameter semigroups of linear operators on Banach spaces began in the early part of the twentieth century. The theory attained a certain level of knowledge in 1970s and 1980s. Semigroups have become key tools for functional differential equations in quantum physics and infinite dimensional control theory, in addition to classic areas such as partial differential equations and stochastic processes.

For details on semigroups theory, we refer to [5], [6] and [7].

## 2. PRELIMINARIES

The set $\mathbb{BC}$ of bicomplex numbers is defined as

$$\mathbb{BC} = \{Z = w_1 + \mathbf{j}w_2 \mid z_1, z_2 \in \mathbb{C}(\mathbf{i})\},$$

where $\mathbf{i}$ and $\mathbf{j}$ are imaginary units such that $\mathbf{ij} = \mathbf{ji}, \mathbf{i}^2 = \mathbf{j}^2 = -1$ and $\mathbb{C}(\mathbf{i})$ is the set of complex numbers with the imaginary unit $\mathbf{i}$. The set $\mathbb{BC}$ of bicomplex numbers form a ring under the usual addition and multiplication of bicomplex numbers. Moreover, $\mathbb{BC}$ is a module over itself. The set of positive hyperbolic number is denoted by $\mathbb{D}^+$ which is a subset of $\mathbb{D}$ is given by

$$\mathbb{D}^+ = \{\beta_1 + \mathbf{k}\beta_2 : \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\}.$$

We can discuss three conjugations for bicomplex numbers in the same way we can do for usual complex numbers, because $\mathbb{BC}$ comprises two imaginary units with squares equal to -1 and a hyperbolic units with square equal to 1.

(i) $\overline{Z} = \overline{z}_1 + \mathbf{j}\overline{z}_2 \ (the\ bar - conjugation)$;
(ii) $Z^\dagger = z_1 - \mathbf{j}z_2,\ (the\ \dagger -conjugation)$;
(iii) $Z^* = \overline{z}_1 - \mathbf{j}\overline{z}_2,\ (the\ * -conjugation)$,

where $\overline{z}_1, \overline{z}_2$ denote the usual complex conjugates to $z_1, z_2 \in \mathbb{C}(\mathbf{i})$.

If $Z = z_1 + \mathbf{j}z_2 \neq 0$ is such that $Z \cdot Z^\dagger = z_1^2 + z_2^2 = 0$, then $Z$ is a zero divisor. The set of zero divisors $\mathbb{NC}$ of $\mathbb{BC}$ is, thus, given by

$$\mathbb{NC} = \left\{Z \mid Z \neq 0,\ w_1^2 + w_2^2 = 0\right\},$$

and is called the null cone and $\mathbb{NC}_0 = \mathbb{NC} \cup \{0\}$.

The hyperbolic numbers $\mathbf{e}$ and $\mathbf{e}^\dagger$ defined as

$$\mathbf{e} = \frac{1+\mathbf{k}}{2} \text{ and } \mathbf{e}^\dagger = \frac{1-\mathbf{k}}{2},$$

are zero divisors, which are linearly independent in the $\mathbb{C}(\mathbf{i})$-vector space $\mathbb{BC}$ and satisfy the following properties:

$$\mathbf{e}^2 = \mathbf{e},\ (\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger,\ \mathbf{e}^* = \mathbf{e},\ (\mathbf{e}^\dagger)^* = \mathbf{e}^\dagger,\ \mathbf{e} + \mathbf{e}^\dagger = 1 \text{ and } \mathbf{e} \cdot \mathbf{e}^\dagger = 0.$$

Any bicomplex number $Z = z_1 + \mathbf{j}z_2$ can be uniquely written as

$$Z = \mathbf{e}w_1 + \mathbf{e}^\dagger w_2, \tag{2.1}$$

where $w_1 = z_1 - \mathbf{i}z_2$ and $w_2 = z_1 + \mathbf{i}z_2$ are elements of $\mathbb{C}(\mathbf{i})$. Formula (2.1) is called the idempotent representation of a bicomplex number $Z$. A hyperbolic number $\alpha = \beta_1 + \mathbf{k}\beta_2$ in idempotent representation can be written as

$$\alpha = \mathbf{e}\alpha_1 + \mathbf{e}^\dagger\alpha_2,$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. We say that $\alpha$ is a positive hyperbolic number if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$.
Writing these hyperbolic numbers in their idempotent form $\alpha = \mathbf{e}\alpha_1 + \mathbf{e}^\dagger\alpha_2$ and $\gamma = \gamma_1\mathbf{e} + \gamma_2\mathbf{e}^\dagger$, with real numbers $\alpha_1, \alpha_2, \gamma_1$ and $\gamma_2$, we have that

$$\alpha \preceq \gamma \text{ iff } \alpha_1 \leq \gamma_1 \text{ and } \alpha_2 \leq \gamma_2.$$

If $\gamma - \alpha \in \mathbb{D}^+ \setminus \{0\}$, we write $\gamma \succ \alpha$. This imples that $z \in \mathbb{D}^+$ is equivalent to $z \succeq 0$ and that $z \in \mathbb{D}^+ \setminus \{0\}$ is equivalent to $z \succ 0$. Now, given two hyperbolic numbers $a$ and $b$, $a \preceq b$, the set

$$[a, b]_{\mathbb{D}} = \{z \in \mathbb{D} : a \preceq z \preceq b\}$$

is called hyperbolic interval.
Consider the mappings

$$\pi_{1,\mathbf{i}}, \pi_{2,\mathbf{i}} : \mathbb{BC} \longrightarrow \mathbb{C}(\mathbf{i})$$

given by

$$\pi_{l,\mathbf{i}}(z) = \pi_{l,\mathbf{i}}(\alpha_1\mathbf{e} + \alpha_2\mathbf{e}^\dagger) := \alpha_l \in \mathbb{C}(\mathbf{i}).$$

These maps are nothing but the projections onto the $coordinate\ axis$ in $\mathbb{C}^2(\mathbf{i})$ with the basis $\{\mathbf{e}, \mathbf{e}^\dagger\}$.

**Definition 2.1.** Let $X$ be a subset of $\mathbb{BC}$. Then $X$ is said to be a product-type set if $X = X_1\mathbf{e} + X_2\mathbf{e}^\dagger$, where $X_1 := \pi_{1,\mathbf{i}}(X)$ and $X_2 := \pi_{2,\mathbf{i}}(X)$.

**Definition 2.2.** Let $X$ be a product-type set in $\mathbb{BC}$. Then a function $\Phi : X = X_1\mathbf{e} + X_2\mathbf{e}^\dagger \subset \mathbb{BC} \to \mathbb{BC}$ is said to be a product-type function if there exist $\Phi_i : X_i \to \mathbb{C}$ for $i = 1, 2$ such that $\Phi(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger) = \Phi_1(\beta_1)\mathbf{e} + \Phi_2(\beta_2)\mathbf{e}^\dagger$ for all $\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in X$.

A module defined over the ring of bicomplex numbers $\mathbb{BC}$ (or ring of hyperbolic numbers $\mathbb{D}$) is called a $\mathbb{BC}$-module (or $\mathbb{D}$-module). Consider the set $X_1 = \mathbf{e}X$ and $X_2 = \mathbf{e}^\dagger X$. Then $X_1 \cap X_2 = \{0\}$. Thus, we can write

$$X = \mathbf{e}X_1 + \mathbf{e}^\dagger X_2, \tag{2.2}$$

where $X_1 = \mathbf{e}X$ and $X_2 = \mathbf{e}^\dagger X$ are $\mathbb{C}(\mathbf{i})$-vector (or $\mathbb{R}$-vector) spaces. Equation (2.2) is called the idempotent decomposition of $X$.

**Definition 2.3.** Let $X$ be a $\mathbb{BC}$-module. A function $\|\cdot\|_{\mathbb{D}} : X \to \mathbb{D}^+$ is said to be a hyperbolic-valued norm (or $\mathbb{D}$-valued norm) on $X$ if it satisfies the following properties:

(a) $\|x\|_{\mathbb{D}} = 0$ if and only if $x = 0$.
(b) $\|\mu x\|_{\mathbb{D}} = |\mu|_{\mathbf{k}} \|x\|_{\mathbb{D}}, \ \forall\, x \in X, \ \forall\, \mu \in \mathbb{BC}$.
(c) $\|x + y\|_{\mathbb{D}} \prec \|x\|_{\mathbb{D}} + \|y\|_{\mathbb{D}}, \ \forall\, x, y \in X$.

The $\mathbb{BC}$-module can be endowed canonically with the hyperbolic, or $\mathbb{D}$-valued norm denoted by $\|\cdot\|_{\mathbb{D}}$ as follows:

$$\|x\|_{\mathbb{D}} = \left\|x_1\mathbf{e} + x_2\mathbf{e}^\dagger\right\|_{\mathbb{D}} = \|x_1\|_1 \mathbf{e} + \|x_2\|_2 \mathbf{e}^\dagger \tag{2.3}$$

**Theorem 2.1.** [10] *A $\mathbb{BC}$-module $(X, \|\cdot\|)$ is a $\mathbb{BC}$-Banach module if and only if $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are complex Banach spaces.*

**Definition 2.4.** The operator $\Phi_{\mathbf{t}} : X \to X$ is called $\mathbb{D}$-bounded if there exists $m \in \mathbb{D}^+$ such that for any $x \in X$ one has

$$\|\Phi_{\mathbf{t}}(x)\|_{\mathbb{D}} \preceq m\|x\|_{\mathbb{D}}.$$

For further details, we refer the reader to [1], [3] [10], [11] and [15].

## 3. SOME BASIC PROPERTIES OF GENERATORS

In this section, we introduce the notion of differentiability of strongly continuous semigroup and also we discuss some properties of generators of semigroup.
Let $X$ be the $\mathbb{BC}$-Banach module and $\mathfrak{L}(X)$ denote the space of all $\mathbb{D}$-bounded bicomplex linear operators on $X$. We know that $\mathfrak{L}(X)$ is a $\mathbb{D}$-normed $\mathbb{BC}$-Banach algebra with respect to operator norm

$$\|\Phi\|_{\mathbb{D}} \quad = \quad \sup\left\{\|\Phi x\|_{\mathbb{D}} : \|x\|_{\mathbb{D}} \prec 1\right\}. \tag{3.1}$$

The norm in equation (3.1) is the hyperbolic norm of $\Phi$. Hence, we can write

$$\|\Phi\|_{\mathbb{D}} = \|\Phi_1\|_1\mathbf{e} + \|\Phi_2\|_2\mathbf{e}^\dagger,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the usual norms on $\Phi_1$ and $\Phi_2$ respectively (cf. [1, page 76]).
Let $\Phi : \mathbb{D}^+ \to \mathfrak{L}(X)$ be a mapping on a set of positive hyperbolic numbers $\mathbb{D}^+$. Write $\mathbb{D}^+ = \mathbb{R}^+\mathbf{e} + \mathbb{R}^+\mathbf{e}^\dagger$, so that any element $\mathbf{t} \in \mathbb{D}^+$ is of the form $\mathbf{t} = \mathbf{t}_1\mathbf{e} + \mathbf{t}_2\mathbf{e}^\dagger$, where $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^+$.
We can write $\mathfrak{L}(X)$ as $\mathfrak{L}(X) = \mathfrak{L}(X_1)\mathbf{e} + \mathfrak{L}(X_2)\mathbf{e}^\dagger$, where $X_1$ and $X_2$ are the Banach spaces and $\Phi_{\mathbf{t}} \in \mathfrak{L}(X)$ is a mapping from $X$ to $X$.
The linearity of $\Phi_{\mathbf{t}}$ gives

$$\begin{aligned}
\Phi_{\mathbf{t}}[x] &= \Phi_{\mathbf{t}}[x\mathbf{e}\cdot\mathbf{e} + x\mathbf{e}^\dagger\cdot\mathbf{e}^\dagger] \\
&= \Phi_{\mathbf{t}}[x\mathbf{e}]\cdot\mathbf{e} + \Phi_{\mathbf{t}}[x\mathbf{e}^\dagger]\cdot\mathbf{e}^\dagger \\
&= (\Phi_{\mathbf{t}}[x\mathbf{e}]\cdot\mathbf{e})\cdot\mathbf{e} + (\Phi_{\mathbf{t}}[x\mathbf{e}^\dagger]\cdot\mathbf{e}^\dagger)\cdot\mathbf{e}^\dagger,
\end{aligned}$$

and introducing the operators $\Phi_{1,\mathbf{t}}$ and $\Phi_{2,\mathbf{t}}$ are given by

$$\Phi_{1,\mathbf{t}}[x] = \Phi_{\mathbf{t}}[x\mathbf{e}]\cdot\mathbf{e},\ \Phi_{2,\mathbf{t}}[x] = \Phi_{\mathbf{t}}[x\mathbf{e}^\dagger]\cdot\mathbf{e}^\dagger.$$

Therefore, the idempotent representation of a bicomplex linear operator $\Phi_{\mathbf{t}}$ is given by

$$\Phi_{\mathbf{t}} = \Phi_{1,\mathbf{t}}\mathbf{e} + \Phi_{2,\mathbf{t}}\mathbf{e}^\dagger = \Phi_{1,\mathbf{t}_1}\mathbf{e} + \Phi_{2,\mathbf{t}_2}\mathbf{e}^\dagger,$$

where $\Phi_{1,\mathbf{t}_1} : X_1 \to X_1$ and $\Phi_{2,\mathbf{t}_2} : X_2 \to X_2$ are the mappings on $X_1$ and $X_2$ respectively.

**Definition 3.1.** Let $X$ be a $\mathbb{BC}$-Banach module and let the mapping $\Phi : \mathbb{D}^+ \to \mathfrak{L}(X)$ have the property:
(i) For all $\mathbf{t}, \mathbf{s} \in \mathbb{D}^+,\ \Phi_{\mathbf{t}+\mathbf{s}} = \Phi_{\mathbf{t}}\Phi_{\mathbf{s}}$ and $\Phi_0 = I$, the identity operator on $X$.
(ii) $\lim\limits_{\mathbf{t}\to 0^+} \|\Phi_{\mathbf{t}} - I\|_{\mathbb{D}} = 0.$
Then the family $\mathcal{F} = \{\Phi_{\mathbf{t}} : \mathbf{t} \in \mathbb{D}^+\}$ satisfying above two conditions is called uniformly continuous semigroup of all $\mathbb{D}$-bounded bicomplex linear operators on $X$. If the above two conditions hold for $\mathbf{t}, \mathbf{s} \in \mathbb{D}$, then we call $\mathcal{F} = \{\Phi_{\mathbf{t}} : \mathbf{t} \in \mathbb{D}\}$ a uniformly continuous group of all $\mathbb{D}$-bounded bicomplex linear operators on $X$.

**Definition 3.2.** A family $\Phi_{\mathbf{t}}$ of $\mathbb{D}$-bounded bicomplex linear operators on $X$, indexed by $\mathbf{t} \in \mathbb{D}^+$ is called a strongly continuous semigroup or ($C_0$-semigroup) if it satisfies the following conditions:
(i) For all $\mathbf{t}, \mathbf{s} \in \mathbb{D}^+,\ \Phi_{\mathbf{t}+\mathbf{s}} = \Phi_{\mathbf{t}}\Phi_{\mathbf{s}}$ and $\Phi_0 = I$.
(ii) For all $x \in X$, we have
$\lim\limits_{\mathbf{t}\to 0^+} \Phi_{\mathbf{t}}x = x.$

If these properties hold for $\mathbb{D}$ instead of $\mathbb{D}^+$, we call $\Phi_{\mathbf{t}}$ indexed by $\mathbf{t} \in \mathbb{D}$ a strongly continuous group (or $C_0$-group) on $X$.

**Definition 3.3.** Let $\Phi : \Omega \subset \mathbb{BC} \longrightarrow \mathbb{D}^+$ be a bicomplex function. Then $\Phi$ is said to be a right derivatives, if

$$\Phi^{'}(w^+) := \lim_{z \to w^+} \frac{\Phi(z) - \Phi(w^+)}{z - w^+} = \lim_{\mathbb{NC}_0 \not\ni h \to 0^+} \frac{\Phi(w^+ + h) - \Phi(w^+)}{h}$$

is exist, for $z \in \Omega \subset \mathbb{BC}$ such that $h = z - w^+$ is an invertible bicomplex number.

**Example 3.1.** *Let us consider a product-type $\mathbb{BC}$-function $\Phi : \Omega = \Omega_1\mathbf{e} + \Omega_2\mathbf{e}^\dagger \subset \mathbb{BC} \longrightarrow \mathbb{BC}$ such that*

$$\Phi(z) = ze^{-\frac{1}{z}},\ where\ z \in \Omega.$$

*Then*

$$\Phi^{'}(z^+) = \begin{cases} 0, & \textit{if } z = 0^+\mathbf{e} + 0^+\mathbf{e}^\dagger \\ \left(\frac{\alpha_1 - 1}{\alpha_1}\right) e^{\alpha_1^{-1}} \cdot \mathbf{e}, & \textit{if } z = \alpha_1\mathbf{e} + 0^+\mathbf{e}^\dagger,\ where\ \alpha_1 \notin \mathbb{NC}_0 \\ \left(\frac{\alpha_2 - 1}{\alpha_2}\right) e^{\alpha_2^{-1}} \cdot \mathbf{e}^\dagger, & \textit{if } z = 0^+\mathbf{e} + \alpha_2\mathbf{e}^\dagger,\ where\ \alpha_2 \notin \mathbb{NC}_0. \end{cases}$$

*Thus, $\Phi^{'}(z^+)$ exist i.e., right derivatives of $\Phi(z)$ exist.*

**Example 3.2.** *Let us consider a product-type $\mathbb{BC}$-function $\Phi : \Omega = \Omega_1\mathbf{e} + \Omega_2\mathbf{e}^\dagger \subset \mathbb{BC} \longrightarrow \mathbb{BC}$ such that*

$$\Phi(z) = ze^{\frac{1}{z}},\ where\ z \in \Omega.$$

*Then*

$$\Phi^{'}(z^+) = \begin{cases} doesnot\ exist, & \textit{if } z = 0^+\mathbf{e} + 0^+\mathbf{e}^\dagger \\ doesnot\ exist, & \textit{if } z = \alpha_1\mathbf{e} + 0^+\mathbf{e}^\dagger,\ where\ \alpha_1 \notin \mathbb{NC}_0 \\ doesnot\ exist, & \textit{if } z = 0^+\mathbf{e} + \alpha_2\mathbf{e}^\dagger,\ where\ \alpha_2 \notin \mathbb{NC}_0. \end{cases}$$

*Thus, $\Phi^{'}(z^+)$ doesnot exist i.e., right derivatives of $\Phi(z)$ doesnot exist.*

**Lemma 3.1.** *Let $\mathcal{F} = \{\Phi_{\boldsymbol{t}} : \boldsymbol{t} \in \mathbb{D}^+\}$ be a $C_0$-semigroup of $\mathbb{D}$-bounded bicomplex linear operators on $\mathbb{BC}$-Banach module $X$ and let $x \in X$. If $u : \boldsymbol{t} \mapsto \Phi_{\boldsymbol{t}}x$ is the orbit map, then the following are equivalent:*

(i) *$u$ is differentiable on $\mathbb{D}^+$.*

(ii) *$u$ is right differentiable at $\boldsymbol{t} = 0$.*

*Proof.* We only need to show (ii)⇒(i). Take $h \in \mathbb{D}^+ \setminus \mathbb{NC}_0$. Then $h = h_1\mathbf{e} + h_2\mathbf{e}^\dagger$ with $h_1, h_2 \in \mathbb{R}^+ \setminus \{0\}$.

Then we have

$$\begin{aligned}
u'(\mathbf{t}) &= \lim_{h\to 0^+} \frac{1}{h}\left(u(\mathbf{t}+h)-u(\mathbf{t})\right)\\
&= \lim_{h_1\to 0^+} \frac{1}{h_1}\left(u_1(\mathbf{t}_1+h_1)-u_1(\mathbf{t}_1)\right)\mathbf{e}\\
&\quad + \lim_{h_2\to 0^+} \frac{1}{h_2}\left(u_2(\mathbf{t}_2+h_2)-u_2(\mathbf{t}_2)\right)\mathbf{e}^{\dagger}\\
&= \lim_{h_1\to 0^+} \frac{1}{h_1}\left(\Phi_{1,\mathbf{t}_1+h_1}x_1-\Phi_{1,\mathbf{t}_1}x_1\right)\mathbf{e}\\
&\quad + \lim_{h_2\to 0^+} \frac{1}{h_2}\left(\Phi_{2,\mathbf{t}_2+h_2}x_2-\Phi_{2,\mathbf{t}_2}x_2\right)\mathbf{e}^{\dagger}\\
&= \Phi_{1,\mathbf{t}_1}\lim_{h_1\to 0^+}\left(\frac{1}{h_1}(\Phi_{1,h_1}x_1-x_1)\right)\mathbf{e}\\
&\quad + \Phi_{2,\mathbf{t}_2}\lim_{h_2\to 0^+}\left(\frac{1}{h_2}(\Phi_{2,h_2}x_2-x_2)\right)\mathbf{e}^{\dagger}\\
&= \Phi_{1,\mathbf{t}_1}u_1'(0)\mathbf{e}+\Phi_{2,\mathbf{t}_2}u_2'(0)\mathbf{e}^{\dagger}\\
&= \Phi_{\mathbf{t}}u'(0)
\end{aligned}$$

and hence $u$ is right differentiable on $\mathbb{D}^+$.
On the other hand, for $h\in[-\mathbf{t},0)_{\mathbb{D}}$ we write

$$\begin{aligned}
\frac{1}{h}(\Phi_{\mathbf{t}+h}(x)-\Phi_{\mathbf{t}}(x))-\Phi_{\mathbf{t}}u'(0) &= \Phi_{\mathbf{t}+h}\left(\frac{1}{h}\big(x-\Phi_{-h}(x)\big)-u'(0)\right)\\
&\quad + \Phi_{\mathbf{t}+h}u'(0)-\Phi_{\mathbf{t}}u'(0),\quad h\notin\mathbb{NC}_0.
\end{aligned}$$

By the first part and the boundedness of $\|\Phi_{\mathbf{t}+h}\|_{\mathbb{D}}$ for $h\in[-\mathbf{t},\mathbf{t}]_{\mathbb{D}}$, the first term on the right hand side converges to $0$ as $h\to 0^-$. The other term converges to zero because $\mathcal{F}$ is strongly continuous semigroup. Hence, $u$ is also left differentiable and its derivatives is

$$u'(\mathbf{t})=\Phi_{\mathbf{t}}u'(0)\ \ \forall\, \mathbf{t}\in\mathbb{D}^+.$$

Thus the derivative $u'(0)$ of the orbit map $u(\mathbf{t})=\Phi_{\mathbf{t}}x$ at $\mathbf{t}=0$ determines the derivative at each point $\mathbf{t}\in\mathbb{D}^+$. ∎

**Definition 3.4.** Let $\mathcal{F}$ be a $C_0$-semigroup on $\mathbb{BC}$-Banach module $X$. Then $G: D_G\subseteq X\to X$ is said to be a generator of $\mathcal{F}$ if it satisfies the following condition:

$$\begin{aligned}
Gx &= \Phi_{\mathbf{t}}'x|_{\mathbf{t}=0}\\
&= \lim_{\mathbb{NC}_0\not\ni h\to 0^+}\frac{1}{h}(\Phi_h x-x),
\end{aligned}\tag{3.2}$$

defined for every $x$ in its domain. It's domain is given by

$$D_G = \left\{x\in X:\lim_{\mathbb{NC}_0\not\ni h\to 0^+}\frac{1}{h}(\Phi_h x-x)\ exists\right\}.$$

**Theorem 3.2.** *Suppose $G_1$ and $G_2$ are the generators of $C_0$-semigroups $\mathcal{F}_1=\{\Phi_{1,\boldsymbol{t}_1}:\boldsymbol{t}_1\in\mathbb{R}^+\}$ and $\mathcal{F}_2=\{\Phi_{2,\boldsymbol{t}_2}:\boldsymbol{t}_2\in\mathbb{R}^+\}$ respectively. Then $G: D_G\subseteq X\to X$ is the generator of $C_0$-semigroup $\mathcal{F}=\{\Phi_{\boldsymbol{t}}:\boldsymbol{t}\in\mathbb{D}^+\}$ on $\mathbb{BC}$-Banach module $X$.*

*Proof.* Since $G_m$ is the generator of $C_0$-semigroup $\mathcal{F}_m$. Then for $x_m\in X_m$, we have

$$G_m x_m=\lim_{h_m\to 0^+}\frac{1}{h_m}\left(\Phi_{m,h_m}x_m-x_m\right),$$

where $X_m$ is a Banach space, for $m = 1, 2$. Now, we have to show that $G$ is the generator of $C_0$-semigroup $\mathcal{F} = \{\Phi_{\mathbf{t}} : \mathbf{t} \in \mathbb{D}^+\}$. For this let $x = x_1\mathbf{e} + x_2\mathbf{e}^\dagger \in X$, where $x_1 \in X_1$ and $x_2 \in X_2$.
We can write $G$ as

$$\begin{aligned} Gx &= G_1x_1\mathbf{e} + G_2x_2\mathbf{e}^\dagger \\ &= \lim_{h_1\to 0^+} \frac{1}{h_1}(\Phi_{1,h_1}x_1 - x_1)\mathbf{e} + \lim_{h_2\to 0^+} \frac{1}{h_2}(\Phi_{2,h_2}x_2 - x_2)\mathbf{e}^\dagger \\ &= \lim_{h\to 0^+} \frac{(\Phi_{1,h_1}x_1\mathbf{e} + \Phi_{2,h_2}x_2\mathbf{e}^\dagger) - (x_1\mathbf{e} + x_2\mathbf{e}^\dagger)}{h_1\mathbf{e} + h_2\mathbf{e}^\dagger} \\ &= \lim_{\mathbb{NC}_0 \not\ni h\to 0^+} \frac{(\Phi_h x - x)}{h}. \end{aligned}$$

Thus, the operator $G$ is generator of $C_0$-semigroup $\mathcal{F}$ on $\mathbb{BC}$-Banach module $X$. ∎

**Theorem 3.3.** *Let $G$ be the generator of $C_0$-semigroup $\mathcal{F}$ on $\mathbb{BC}$-Banach module $X$. Then the two operators $G_1 : D_{G_1} \subseteq X_1 \to X_1$ and $G_2 : D_{G_2} \subseteq X_2 \to X_2$ are the generators on $X_1$ and $X_2$ respectively.*

*Proof.* Given that $G$ is the generator of $C_0$-semigroup $\mathcal{F}$ on $\mathbb{BC}$-Banach module $X$. We need to prove that $G_1 : D_{G_1} \subseteq X_1 \to X_1$ and $G_2 : D_{G_2} \subseteq X_2 \to X_2$ are the generators on $X_1$ and $X_2$ respectively.
Since G is the generator of $C_0$-semigroup $\mathcal{F}$, we have

$$Gx = \lim_{\mathbb{NC}_0 \not\ni h\to 0^+} \frac{(\Phi_h x - x)}{h}.$$

We can decompose $G$ as

$$Gx = G_1x_1\mathbf{e} + G_2x_2\mathbf{e}^\dagger.$$

Then

$$\begin{aligned} G_1x_1\mathbf{e} + G_2x_2\mathbf{e}^\dagger &= \lim_{h_1\to 0^+} \frac{1}{h_1}(\Phi_{1,h_1}x_1 - x_1)\mathbf{e} \\ &\quad + \lim_{h_2\to 0^+} \frac{1}{h_2}(\Phi_{2,h_2}x_2 - x_2)\mathbf{e}^\dagger. \end{aligned} \tag{3.3}$$

Multiply (3.3) by e and $\mathbf{e}^\dagger$, we get

$$G_1x_1 = \lim_{h_1\to 0^+} \frac{1}{h_1}(\Phi_{1,h_1}x_1 - x_1), \text{ for } x_1 \in X_1$$

and

$$G_2x_2 = \lim_{h_2\to 0^+} \frac{1}{h_2}(\Phi_{2,h_2}x_2 - x_2),\ x_2 \in X_2.$$

Hence, $G_1$ and $G_2$ are the generators of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. ∎

**Lemma 3.4.** *Let $\mathcal{F}$ be a $C_0$-semigroup on a $\mathbb{BC}$-Banach module and if $G$ is the $C_0$-semigroup's generator. Then we have the following properties:*
(i) *$G : D_G \subseteq X \to X$ is a bicomplex linear operator.*
(ii) *If $x \in D_G$, then $\Phi_{\boldsymbol{t}}x \in D_G$ and*

$$\Phi'_{\boldsymbol{t}}x = \Phi_{\boldsymbol{t}}Gx = G\Phi_{\boldsymbol{t}}x,\ \forall \boldsymbol{t} \in \mathbb{D}^+.$$

*Proof.* (i) Let $G = G_1\mathbf{e} + G_2\mathbf{e}^\dagger$ be the generator on $X$, where $G_1$ and $G_2$ are generators on Banach spaces $X_1$ and $X_2$ respectively.
We know that $G$ is a bicomplex linear operator if and only if $G_1$ and $G_2$ are linear.
(ii) For $\mathbf{t} \in \mathbb{D}^+$ and $x \in D_G$,

$$\begin{aligned}
\Phi_{\mathbf{t}}Gx &= \Phi_{1,\mathbf{t}_1}G_1x_1\mathbf{e} + \Phi_{2,\mathbf{t}_2}G_2x_2\mathbf{e}^\dagger \\
&= \Phi_{1,\mathbf{t}_1}\lim_{h_1\to 0^+}\frac{1}{h_1}[\Phi_{2,h_1}x_1 - x_1]\mathbf{e} \\
&\quad +\Phi_{\mathbf{t}_2,2}\lim_{h_2\to 0^+}\frac{1}{h_2}[\Phi_{h_2,2}x_2 - x_2]\mathbf{e}^\dagger \\
&= \lim_{h_1\to 0^+}\frac{1}{h_1}[\Phi_{1,\mathbf{t}_1}\Phi_{1,h_1}x_1 - \Phi_{1,\mathbf{t}_1}x_1]\mathbf{e} \\
&\quad + \lim_{h_2\to 0^+}\frac{1}{h_2}[\Phi_{2,\mathbf{t}_2}\Phi_{2,h_2}x_2 - \Phi_{2,\mathbf{t}_2}x_2]\mathbf{e}^\dagger \\
&= \lim_{\mathbb{NC}_0\not\ni h\to 0^+}\frac{1}{h}[\Phi_{\mathbf{t}}\Phi_h x - \Phi_{\mathbf{t}}x] = G\Phi_{\mathbf{t}}x.
\end{aligned}$$

So, we have $\Phi_{\mathbf{t}}x \in D_G$ and $\Phi_{\mathbf{t}}Gx = G\Phi_{\mathbf{t}}x$.
Next, we will compute the right derivative of $\Phi_{\mathbf{t}}x$,

$$\Phi'_{\mathbf{t}}x = \lim_{\mathbb{NC}_0\not\ni h\to 0^+}\frac{1}{h}[\Phi_{\mathbf{t}+h}x - \Phi_{\mathbf{t}}x] = \Phi_{\mathbf{t}}\lim_{\mathbb{NC}_0\not\ni h\to 0^+}\frac{1}{h}[\Phi_h x - x] = \Phi_{\mathbf{t}}Gx.$$

Thus,

$$\begin{aligned}
\Phi'_{\mathbf{t}}x &= \Phi_{\mathbf{t}}Gx \\
&= G\Phi_{\mathbf{t}}x,\ \forall\, \mathbf{t} \in \mathbb{D}^+.
\end{aligned}$$

∎

**Lemma 3.5.** *Let $G$ be the generator of $C_0$-semigroup $\mathcal{F}$ on $\mathbb{BC}$-Banach module, then $\int_{[0,\boldsymbol{t}]_{\mathbb{D}}}\Phi_s x ds \in D_G$, for every $\mathbf{t} \in \mathbb{D}^+,\ x \in X$.*

*Proof.* Since $G$ is the generator of $C_0$-semigroup $\mathcal{F}$ on $\mathbb{BC}$-Banach module $X$. We can write $G$ as

$$G = G_1\mathbf{e} + G_2\mathbf{e}^\dagger,$$

where $G_1$ and $G_2$ are the generators on Banach spaces $X_1$ and $X_2$ respectively.
Let $\Phi : [0,\mathbf{t}]_{\mathbb{D}} \to X$ given by $s \to \Phi_s x$ be a continuous function on hyperbolic interval $[0,\mathbf{t}]_{\mathbb{D}}$. Then by [16], the Riemann integral is represented as

$$\int_{[0,\mathbf{t}]_{\mathbb{D}}}\Phi_s x ds = \int_0^{\mathbf{t}_1}\Phi_{1,s_1}x_1 ds_1\mathbf{e} + \int_0^{\mathbf{t}_2}\Phi_{2,s_2}x_2 ds_2\mathbf{e}^\dagger,\ s \in [0,\mathbf{t}]_{\mathbb{D}},\ x \in X.$$

Since $G_m$ is the generator of $C_0$-semigroup $\mathcal{F}_m = \{\Phi_{m,\mathbf{t}_m} : \mathbf{t}_m \in \mathbb{R}^+\}$, then for all $\mathbf{t}_m \in \mathbb{R}^+$ and $x_m \in X_m$, we have

$$\int_0^{\mathbf{t}_m}\Phi_{m,s_m}x_m ds_m \in D_{G_m},$$

where the integral is the Riemann integral of the continuous function $s_m \to \Phi_{m,s_m}x_m$, for $m = 1, 2$ see [2].
Then, clearly we have $\int_{[0,\mathbf{t}]_{\mathbb{D}}}\Phi_s x ds \in D_G$, for every $\mathbf{t} \in \mathbb{D}^+, x \in X$. ∎

**Lemma 3.6.** *Let $\mathcal{F}$ be a $C_0$-semigroup on $\mathbb{BC}$-Banach module $X$ and if $G$ is the generator of $C_0$-semigroup $\mathcal{F}$, then we have the following properties.*

(i) *If* $x \in X$, $\Phi_{\boldsymbol{t}}x - x = G\int_{[0,\boldsymbol{t}]_{\mathbb{D}}} \Phi_s x ds$, *for every* $\boldsymbol{t} \in \mathbb{D}^+$.
(ii) *If* $x \in D_G$, *then* $\Phi_{\boldsymbol{t}}x - x = \int_{[0,\boldsymbol{t}]_{\mathbb{D}}} \Phi_s Gx ds$, *for every* $\boldsymbol{t} \in \mathbb{D}^+$.

*Proof.* (i) Let $x = x_1\mathbf{e} + x_2\mathbf{e}^\dagger \in X$. We can decompose the integral as

$$G\int_{[0,\mathbf{t}]_{\mathbb{D}}} \Phi_s x ds = G_1\int_0^{\mathbf{t}_1} \Phi_{1,s_1}x_1 ds_1\mathbf{e} + G_2\int_0^{\mathbf{t}_2} \Phi_{2,s_2}x_2 ds_2\mathbf{e}^\dagger.$$

If $x_1 \in X_1$ and $x_2 \in X_2$, then

$$G_m\int_0^{\mathbf{t}_m} \Phi_{m,s_m}x_m ds_m = \Phi_{m,\mathbf{t}_m}x_m - x_m,$$

where $G_m$ is the generator of $C_0$-semigroup $\mathcal{F}_m = \{\Phi_{m,\mathbf{t}_m} : \mathbf{t}_m \in \mathbb{R}^+\}$, for $m = 1, 2$. Now,

$$\begin{aligned} G\int_{[0,\mathbf{t}]_{\mathbb{D}}} \Phi_s x ds &= (\Phi_{1,\mathbf{t}_1}x_1 - x_1)\mathbf{e} + (\Phi_{2,\mathbf{t}_2}x_2 - x_2)\mathbf{e}^\dagger \\ &= \Phi_{\mathbf{t}}x - x. \end{aligned}$$

(ii) Let $x = x_1\mathbf{e} + x_2\mathbf{e}^\dagger \in D_G$, where $x_1 \in D_{G_1}$ and $x_2 \in D_{G_2}$.
If $x_m \in D_{G_m}$, then $\int_0^{\mathbf{t}_m} \Phi_{m,s_m}G_m x_m ds_m = \Phi_{m,\mathbf{t}_m}x_m - x_m$, for $m = 1, 2$. Then, clearly $\int_{[0,\mathbf{t}]_{\mathbb{D}}} \Phi_s Gx ds = \Phi_{\mathbf{t}}x - x$. ∎

**Corollary 3.7.** *A* $C_0$*-semigroup* $\mathcal{F} = \{\Phi_{\boldsymbol{t}} : \boldsymbol{t} \in \mathbb{D}^+\}$ *has a* $\mathbb{D}$*-bounded generator* $G$ *if and only if it is uniformly continuous. In this case,*

$$\Phi'_{\boldsymbol{t}}x = \Phi_{\boldsymbol{t}}Gx = G\Phi_{\boldsymbol{t}}x \tag{3.4}$$

*for all* $x \in X$ *and the limit in (3.2) is uniform with respect to the* $\mathbb{D}$*-valued norm in* $X$, *i.e.,*

$$\lim_{\mathbb{NC}_0 \not\ni h \to 0^+} \left\| G - \frac{1}{h}(\Phi_h - I) \right\|_{\mathbb{D}} = 0. \tag{3.5}$$

*Moreover,*

$$\Phi_{\boldsymbol{t}} = exp(\boldsymbol{t}G) = I + \sum_{k=1}^{\infty} \frac{1}{k!}(\boldsymbol{t}G)^k,\ \boldsymbol{t} \in \mathbb{D}^+. \tag{3.6}$$

*It is known that if* $\lim_{\boldsymbol{t}\to 0^+} sup \|\Phi_{\boldsymbol{t}} - I\|_{\mathbb{D}} \prec 1$, *then* $\|\Phi_{\boldsymbol{t}} - I\|_{\mathbb{D}} \to 0$ *and hence the semigroup is uniformly continuous and has the form* $e^{\boldsymbol{t}G}$ *for some* $\mathbb{D}$*-bounded operator* $G$.

## 4. SOME RESULTS ON CLOSED BICOMPLEX LINEAR OPERATORS

In this section, we studied the $C_0$-semigroups of closed bicomplex linear operators on $\mathbb{BC}$-Banach module and find some results in these direction.

**Definition 4.1.** [8] Let $X$ be a $\mathbb{BC}$-Banach module and also let $G : D_G \subset X \to D_G$ be a bicomplex linear map such that

$$D_G = \{x \in X : Gx \in X\}.$$

is a bicomplex submodule in $X$. Then the graph of $G$ is the set of all points in $X \times X$ of the form $(x, Gx)$ with $x \in D_G$.
A bicomplex linear operator $G$ is said to be closed if its graph $G = \{(x, Gx) \mid x \in D_G\}$ is closed in the product spaces $X \times X$ i.e., whenever $x_n \in D_G,\ x_n \to x,\ Gx_n \to y$ implies that $x \in D_G$ and $Gx = y$.
Also the product $X \times X$ of $\mathbb{BC}$-Banach module is a $\mathbb{BC}$-Banach module.

Let $D_G$ be a linear submodule of $X$. Write $X = X_1\mathbf{e} + X_2\mathbf{e}^\dagger$ and $D_G = D_{G_1}\mathbf{e} + D_{G_2}\mathbf{e}^\dagger$, where $D_{G_1}$ and $D_{G_2}$ are linear subspaces of Banach spaces $X_1$ and $X_2$ respectively.
Then the graph norm of $G_1$ and $G_2$ is define by

$$\|x_1\|_{G_1} = \|x_1\|_1 + \|G_1x_1\|_1 \,, \text{ for } x_1 \in D_{G_1}$$

and

$$\|x_2\|_{G_2} = \|x_2\|_2 + \|G_2x_2\|_2 \,, \text{ for } x_2 \in D_{G_2}.$$

Then, indeed $\|\cdot\|_{G_1}$ and $\|\cdot\|_{G_2}$ are norms on $D_{G_1}$ and $D_{G_2}$ respectively.
For $x = x_1\mathbf{e} + x_2\mathbf{e}^\dagger \in X$, we define

$$\|x\|_{G,\mathbb{D}} = \|x_1\|_{G_1}\mathbf{e} + \|x_2\|_{G_2}\mathbf{e}^\dagger, \tag{4.1}$$

where $\|\cdot\|_{G,\mathbb{D}}$ is the $\mathbb{D}$-valued norm on $D_G$. Then the equation (4.1) can be seen as follows:
(i)

$$\begin{aligned} \|x\|_{G,\mathbb{D}} = 0 \;&\Leftrightarrow\; \|x_1\|_{G_1}\mathbf{e} + \|x_2\|_{G_2}\mathbf{e}^\dagger \\ &\Leftrightarrow\; (\|x_1\|_1 + \|G_1x_1\|_1)\,\mathbf{e} + (\|x_2\|_2 + \|G_2x_2\|_2)\,\mathbf{e}^\dagger \\ &\Leftrightarrow\; \|x_1\|_1 + \|G_1x_1\|_1 = 0 \; and \; \|x_2\|_2 + \|G_2x_2\|_2 = 0 \\ &\Leftrightarrow\; x_1 = 0 \text{ and } x_2 = 0. \end{aligned}$$

(ii) Further for any $\mu \in \mathbb{D}$,

$$\begin{aligned} \|\mu x\|_{G,\mathbb{D}} &= \|\mu_1x_1\|_{G_1}\mathbf{e} + \|\mu_2x_2\|_{G_2}\mathbf{e}^\dagger \\ &= (\|\mu_1x_1\|_1 + \|\mu_1G_1x_1\|_1)\,\mathbf{e} + (\|\mu_2x_2\|_2 + \|\mu_2G_2x_2\|_2)\mathbf{e}^\dagger \\ &= |\mu_1|\,(\|x_1\|_1 + \|G_1x_1\|_1)\,\mathbf{e} + |\mu_2|\,(\|x_2\|_2 + \|G_2x_2\|_2)\mathbf{e}^\dagger \\ &= |\mu_1|\,\|x_1\|_{G_1}\mathbf{e} + |\mu_2|\,\|x_2\|_{G_2}\mathbf{e}^\dagger \\ &= |\mu|_{\mathbf{k}}\,\|x\|_{G,\mathbb{D}}\,. \end{aligned}$$

(iii) Let $x = x_1\mathbf{e} + x_2\mathbf{e}^\dagger,\; y = y_1\mathbf{e} + y_2\mathbf{e}^\dagger \in D_G$. Then

$$\begin{aligned} \|x+y\|_{G,\mathbb{D}} &= \|x_1+y_1\|_{G_1}\mathbf{e} + \|x_2+y_2\|_{G_2}\mathbf{e}^\dagger \\ &= \big[\,\|x_1+y_1\|_1 + \|G_1(x_1+y_1)\|_1\,\big]\mathbf{e} \\ &\quad + \big[\,\|x_2+y_2\|_2 + \|G_2(x_2+y_2)\|_2\,\big]\mathbf{e}^\dagger \\ &\preceq \big(\|x_1\|_1\mathbf{e} + \|x_2\|_2\mathbf{e}^\dagger\big) + \big(\|y_1\|_1\mathbf{e} + \|y_2\|_2\mathbf{e}^\dagger\big) \\ &\quad + \big(\|G_1x_1\|_1\mathbf{e} + \|G_2x_2\|_2\mathbf{e}^\dagger\big) + \big(\|G_1y_1\|_1\mathbf{e} + \|G_2y_2\|_2\mathbf{e}^\dagger\big) \\ &= [(\|x_1\|_1 + \|G_1x_1\|_1) + (\|y_1\|_1 + \|G_1y_1\|_1)]\,\mathbf{e} \\ &\quad + [(\|x_2\|_2 + \|G_2x_2\|_2) + (\|y_2\|_2 + \|G_2y_2\|_2)]\,\mathbf{e}^\dagger \\ &= \big(\|x_1\|_{G_1} + \|y_1\|_{G_1}\big)\,\mathbf{e} + \big(\|x_2\|_{G_2} + \|y_2\|_{G_2}\big)\,\mathbf{e}^\dagger \\ &= \|x\|_{G,\mathbb{D}} + \|y\|_{G,\mathbb{D}}\,. \end{aligned}$$

So, we can define $\mathbb{D}$-valued graph norm of $G$ by

$$\|x\|_{G,\mathbb{D}} = \|x\|_{\mathbb{D}} + \|Gx\|_{\mathbb{D}}\,, \quad x \in D_G.$$

**Theorem 4.1.** *Let $G$ be the generator of $C_0$-semigroup $\mathcal{F}$. Then*

(i) *$G$ is closed bicomplex linear operator.*
(ii) *$D_G$ is dense in $X$.*

*Proof.* (i) Let $\{x_n\} \subset D_G$ be a Cauchy sequence in $D_G$ with respect to the $\mathbb{D}$-valued graph norm. Then the inequalities :

$$\|x_n - x_l\|_{\mathbb{D}} \preceq \|x_n - x_l\|_{G,\mathbb{D}}$$

and

$$\|Gx_n - Gx_l\|_{\mathbb{D}} \preceq \|Gx_n - Gx_l\|_{G,\mathbb{D}}$$

hold and so $\{x_n\}$ and $\{Gx_n\}$ are Cauchy sequences in $X$ with respect to $\mathbb{D}$-valued norm $\|\cdot\|_{\mathbb{D}}$. Since $X$ is a $\mathbb{BC}$-Banach module, we see that $x_n \to x$ and $Gx_n \to y$ in $X$ for some $x, y \in X$.
For $\mathbf{t} \in \mathbb{D}^+ \setminus \mathbb{NC}_0$, we have

$$\Phi_{\mathbf{t}} x_n - x_n = \int_{[0,\mathbf{t}]_{\mathbb{D}}} \Phi_s G x_n ds.$$

The uniform convergence of $\Phi_s G x_n$ on $[0, \mathbf{t}]_{\mathbb{D}}$ for $n \to \infty$ implies that

$$\Phi_{\mathbf{t}} x - x = \int_{[0,\mathbf{t}]_{\mathbb{D}}} \Phi_s y ds.$$

Dividing both sides by $\mathbf{t} \in \mathbb{D}^+ \setminus \mathbb{NC}_0$ and let $\mathbf{t} \to 0^+$, we get

$$Gx = \lim_{\mathbf{t}\to 0^+} \frac{\Phi_{\mathbf{t}} x - x}{\mathbf{t}} = \lim_{\mathbf{t}\to 0^+} \frac{1}{\mathbf{t}} \int_{[0,\mathbf{t}]_{\mathbb{D}}} \Phi_s y ds = y,$$

so $x \in D_G$ and $Gx = y$. To conclude, we note that
$\|x - x_n\|_{G,\mathbb{D}} = \|x - x_n\|_{\mathbb{D}} + \|Gx - Gx_n\|_{\mathbb{D}} \to 0$ as $n \to \infty$, i.e., $x_n \to x$ in $\mathbb{D}$-valued graph norm. Thus, $G$ is a closed bicomplex linear operator.
(ii) Let $x \in X$ be arbitrary and define

$$x_{\mathbf{t}} = \frac{1}{\mathbf{t}} \int_{[0,\mathbf{t}]_{\mathbb{D}}} \Phi_s x ds, \text{ where } \mathbf{t} \in \mathbb{D}^+ \setminus \mathbb{NC}_0.$$

By Proposition 3.5, we get $x_{\mathbf{t}} \in D_G$. Since the mapping $s \to \Phi_s x$ is continuous, we have $x_{\mathbf{t}} \to \Phi_0 x = x$ for $\mathbf{t} \to 0^+$.
Hence, $D_G$ is dense in $X$. ∎

**Theorem 4.2.** *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be* $C_0$*-semigroups of* $\mathbb{D}$*-bounded bicomplex linear operators with same generator* $G$*. If* $\mathbf{S} = \mathbf{S}_1\mathbf{e} + \mathbf{S}_2\mathbf{e}^\dagger$, $\mathbf{T} = \mathbf{T}_1\mathbf{e} + \mathbf{T}_2\mathbf{e}^\dagger$ *and* $G = G_1\mathbf{e} + G_2\mathbf{e}^\dagger$, *where* $\mathbf{S}_m, \mathbf{T}_m$ *are* $C_0$*-semigroups over* $X_m$ *with same generator* $G_m$, $m = 1, 2$. *Then*

$$\mathbf{S} = \mathbf{T}$$

*if and only if*

$$\mathbf{S}_1 = \mathbf{T}_1 \ and \ \mathbf{S}_2 = \mathbf{T}_2.$$

*Proof.* Suppose that $\mathbf{S}$ and $\mathbf{T}$ be a $C_0$-semigroups with same generator $G$. Then $\mathbf{S}$ coincide with $\mathbf{T}$. We have to show that $C_0$-semigroups $\mathbf{S}_m$ and $\mathbf{T}_m$ with same generator are coincide, for $m = 1, 2$. Now,

$$\begin{aligned} \mathbf{S}_1 &= \mathbf{Se} \\ &= \mathbf{Te} = \mathbf{T}_1 \end{aligned}$$

implies,

$$\mathbf{S}_1 = \mathbf{T}_1.$$

Similarly,

$$\mathbf{S}_2 = \mathbf{T}_2.$$

Thus, $\mathbf{S}_1$ coincide with $\mathbf{T}_1$ and $\mathbf{S}_2$ coincide with $\mathbf{T}_2$.
Conversely, suppose that $\mathbf{S}_m$ and $\mathbf{T}_m,\ m = 1, 2$ be $C_0$-semigroups with the same generator. Then $\mathbf{S}_1 = \mathbf{T}_1$ and $\mathbf{S}_2 = \mathbf{T}_2$. We have to show that $\mathbf{S} = \mathbf{T}$. Now

$$\begin{aligned} \mathbf{S} &= \mathbf{S}_1\mathbf{e} + \mathbf{S}_2\mathbf{e}^\dagger \\ &= \mathbf{T}_1\mathbf{e} + \mathbf{T}_2\mathbf{e}^\dagger = \mathbf{T}. \end{aligned}$$

Thus, $\mathbf{S}$ coincide with $\mathbf{T}$. ∎

## 5. CONCLUSION

The author are working to extend the semi groups of linear operators with real and complex scalars to bicomplex scalars. The authors establish generators of bicomplex and Hille-Yoshida theorem in the bicomplex framework. Interesting future work will include study of semi groups of linear operators, their generators and Hille-Yoshida theorem in locally convex bicomplex module.

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