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COEFFICIENT BOUNDS FOR SAKAGUCHI KIND OF FUNCTIONS ASSOCIATED WITH SINE FUNCTION

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ABSTRACT. In this paper, we introduce a new general subclass of analytic functions with respect to symmetric points in the domain of sine function. We obtain sharp coefficient bounds and upper bounds for the Fekete-Szegö functional. Also we get sharp bounds for the logarithmic coefficients of functions belonging to this new class.

Key words and phrases: Analytic function; Subordination; Logarithmic coefficient; Fekete-Szegö inequality.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of all functions f which are holomorphic in the region $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. Therefore, for $f \in \mathcal{A}$, one has

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{D}) . \tag{1.1}$$

We write $g_1 \prec g_2$, if there is an analytic function ν in $\mathbb{D}$, with limitations $\nu(0) = 0$ and $|\nu(z)| < 1$, such that $g_1(z) = g_2(\nu(z)),\ (z \in \mathbb{D})$. In case of univalency of g_2 in $\mathbb{D}$, the following relation holds:

$$g_1(z) \prec g_2(z),\ (z \in \mathbb{D}) \iff g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{D}) \subset g_2(\mathbb{D}).$$

By varying the function right hand side of subordinations, we can define some subclasses of the set $\mathcal{S}$ which have several interesting geometric properties, (see [2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14]). From among these subfamilies we recall here the families that are associated with trigonometric function as follows:

$$\mathcal{K}_{sin} = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \sin(z),\ z \in \mathbb{D} \right\}, \tag{1.2}$$

$$\mathcal{S}^*_{sin} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin(z),\ z \in \mathbb{D} \right\}, \tag{1.3}$$

$$\mathcal{R}_{sin} = \left\{ f \in \mathcal{A} : f'(z) \prec 1 + \sin(z),\ z \in \mathbb{D} \right\}. \tag{1.4}$$

The classes $\mathcal{S}^*_{sin}$ and $\mathcal{R}_{sin}$ were established by Cho *et al.* [3] and Khan *et al.* [17], respectively. In the light of the above definitions, we introduce a new general subclass of $\mathcal{A}$ with respect to symmetric points.

Definition 1.1. The function $f \in \mathcal{A}$ is in the class $\mathcal{SK}_s^{\lambda,\mu}(\sin)$ if

$$\frac{2z\, F'_{\lambda,\mu}(z)}{F_{\lambda,\mu}(z) - F_{\lambda,\mu}(-z)} \prec 1 + \sin(z) \qquad (z \in \mathbb{D}), \tag{1.5}$$

where

$$F_{\lambda,\mu}(z) = (1 - \lambda + \mu)\, f(z) + (\lambda - \mu)\, zf'(z) + \lambda\mu z^2 f''(z) \tag{1.6}$$

and $0 \le \mu \le \lambda \le 1$.

Remark 1.1. (i) For $\mu = 0$, we get the following new class $\mathcal{SK}_s^{\lambda}(\sin)$,

$$\mathcal{SK}_s^{\lambda}(\sin) = \left\{ f \in \mathcal{A} : \frac{2\left[\lambda z^2 f''(z) + zf'(z)\right]}{\lambda z\left[f'(z) + f'(-z)\right] + (1-\lambda)\left[f(z) - f(-z)\right]} \prec 1 + \sin(z),\ z \in \mathbb{D} \right\}.$$

(ii) For $\mu = 0$ and $\lambda = 1$, we get the following new class $\mathcal{K}_s(\sin)$,

$$\mathcal{K}_s(\sin) = \left\{ f \in \mathcal{A} : \frac{2\left[z^2 f''(z) + zf'(z)\right]}{z\left[f'(z) + f'(-z)\right]} \prec 1 + \sin(z),\ z \in \mathbb{D} \right\}.$$

(iii) For $\mu = 0$ and $\lambda = 0$, we get the class $\mathcal{S}_s(\sin)$, [18],

$$\mathcal{S}_s(\sin) = \left\{ f \in \mathcal{A} : \frac{2zf'(z)}{f(z) - f(-z)} \prec 1 + \sin(z),\ z \in \mathbb{D} \right\}.$$

2. A SET OF LEMMAS

Let $\mathcal{P}$ be the family of functions p that are holomorphic in $\mathbb{D}$ with $\Re\,(p(z)) > 0$ and the power series form as follows:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in \mathbb{D}). \tag{2.1}$$

Lemma 2.1. [9, 13] *If $p \in \mathcal{P}$ be expressed in series expansion (2.1), then*

$$|c_k| \le 2 \qquad \textit{for} \quad k \ge 1, \tag{2.2}$$

and for complex number γ, we have

$$\left|c_2 - \gamma c_1^2\right| \le 2\,\max\{1, |2\gamma - 1|\}, \tag{2.3}$$

Lemma 2.2. [1] *Let $p \in \mathcal{P}$ has power series expansion (2.1), then*

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \le 2\,|J| + 2\,|K - 2J| + 2\,|J - K + L|\,.$$

Lemma 2.3. [15] *Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$, and*

$$8r(1-r)\left[(mn-2l)^2 + (m(r+m)-n)^2\right] + m(1-m)(n-2rm)^2 \le 4m^2(1-m)^2 r(1-r).$$

If $p \in \mathcal{P}$ and has power series expansion (2.1), then

$$\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2}nc_1^2c_2 - c_4\right| \le 2.$$

3. COEFFICIENTS ESTIMATES AND FEKETE-SZEGÖ INEQUALITY

Theorem 3.1. *If the function f of the form (1.1) belongs to $\mathcal{SK}_s^{\lambda,\mu}(\sin)$, then*

$$|a_k| \le \frac{1}{k\,\left[1 + (k-1)\,(\lambda - \mu + k\,\lambda\mu)\right]}, \qquad k = 2, 4$$

and

$$|a_k| \le \frac{1}{(k-1)\,\left[1 + (k-1)\,(\lambda - \mu + k\,\lambda\mu)\right]}, \qquad k = 3, 5.$$

The bounds are sharp.

Proof. Let the function $f \in \mathcal{SK}_s^{\lambda,\mu}(\sin)$ be of the form (1.1). Let us define the function $q(z)$ by

$$q(z) = \frac{2z\,F'_{\lambda,\mu}(z)}{F_{\lambda,\mu}(z) - F_{\lambda,\mu}(-z)} = 1 + q_1 z + q_2 z^2 + \cdots \qquad (z \in \mathbb{D})\,. \tag{3.1}$$

The LHS of the equation (3.1) gives

$$2z\,F'_{\lambda,\mu}(z) = \left[F_{\lambda,\mu}(z) - F_{\lambda,\mu}(-z)\right]\,q(z) \qquad (z \in \mathbb{D})\,.$$

Since $a_1 = 1$, in view of the above equality, we obtain

$$2j\varphi_{2j}a_{2j} = q_{2j-1} + q_{2j-3}\varphi_3 a_3 + \cdots + q_1\varphi_{2j-1}a_{2j-1} = \sum_{n=1}^{j} q_{2n-1}\varphi_{2j-2n+1}a_{2j-2n+1} \tag{3.2}$$

and

$$2j\varphi_{2j+1}a_{2j+1} = q_{2j} + q_{2j-2}\varphi_3 a_3 + \cdots + q_2\varphi_{2j-1}a_{2j-1} = \sum_{n=1}^{j} q_{2n}\varphi_{2j-2n+1}a_{2j-2n+1} \tag{3.3}$$

where

$$\varphi_1 = 1, \quad \varphi_k = 1 + (k-1)\,(\lambda - \mu + k\,\lambda\mu) \qquad (k \ge 2)\,. \tag{3.4}$$

Furthermore by Definition 1.1, we get

$$q(z) \prec 1+\sin(z) \qquad (z\in\mathbb{D}).$$

On the other hand, from the definition of subordination there exists a Schwarz function $\nu(z)$ with $\nu(0)=0$ and $|\nu(z)|<1$, in such a way that

$$q(z)=1+\sin\left(v\left(z\right)\right) \qquad (z\in\mathbb{D}), \tag{3.5}$$

where q is defined by (3.1), or equivalently

$$\frac{2\left[\lambda\mu z^3 f'''(z)+(\lambda-\mu+2\lambda\mu)z^2 f''(z)+zf'(z)\right]}{\lambda\mu z^2\left[f''(z)-f''(-z)\right]+(\lambda-\mu)z\left[f'(z)+f'(-z)\right]+(1-\lambda+\mu)\left[f(z)-f(-z)\right]} \tag{3.6}$$
$$= 1+\sin\left(v(z)\right) \qquad (z\in\mathbb{D}).$$

Now, define a function h with

$$h(z)=\frac{1+\nu(z)}{1-\nu(z)}=1+c_1z+c_2z^2+\cdots \qquad (z\in\mathbb{D}).$$

Clearly, we have $h\in\mathcal{P}$ and

$$\nu(z)=\frac{h(z)-1}{h(z)+1}=\frac{c_1z+c_2z^2+c_3z^3+\cdots}{2+c_1z+c_2z^2+c_3z^3+\cdots}.$$

This gives

$$\begin{aligned} 1+\sin(\nu(z)) &= 1+\frac{1}{2}c_1z+\left(-\frac{c_1^2}{4}+\frac{c_2}{2}\right)z^2+\left(\frac{5c_1^3}{48}-\frac{c_1c_2}{2}+\frac{c_3}{2}\right)z^3 \\ &\quad +\left(-\frac{1}{32}c_1^4+\frac{5}{16}c_1^2c_2-\frac{1}{2}c_1c_3-\frac{1}{4}c_2^2+\frac{1}{2}c_4\right)z^4+\cdots. \end{aligned} \tag{3.7}$$

From (3.1), (3.5) and (3.7), we get

$$\begin{aligned} q_1 &= \frac{1}{2}c_1, \\ q_2 &= \frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right), \\ q_3 &= \frac{1}{2}\left(c_3-c_2c_1+\frac{5}{24}c_1^3\right), \\ q_4 &= \frac{1}{2}\left(c_4-c_3c_1-\frac{1}{2}c_2^2+\frac{5}{8}c_2c_1^2-\frac{1}{16}c_1^4\right) \end{aligned}$$

which imply by (3.2) and (3.3) that

$$a_2=\frac{1}{4\varphi_2}c_1 \tag{3.8}$$

$$a_3=\frac{1}{4\varphi_3}\left(c_2-\frac{1}{2}c_1^2\right) \tag{3.9}$$

$$a_4=\frac{1}{8\varphi_4}\left(c_3-\frac{3}{4}c_2c_1+\frac{1}{12}c_1^3\right) \tag{3.10}$$

$$a_5=\frac{1}{8\varphi_5}\left(c_4-c_3c_1-\frac{1}{4}c_2^2+\frac{3}{8}c_2c_1^2\right) \tag{3.11}$$

Now implementing (2.2) in (3.8), we obtain

$$|a_2| \leq \frac{1}{2\varphi_2}. \tag{3.12}$$

Now implementing (2.3) in (3.9), we obtain

$$|a_3| \leq \frac{1}{2\varphi_3}. \tag{3.13}$$

Implementation of triangle inequality and Lemma 2.2 in (3.10), leads us to

$$|a_4| \leq \frac{1}{4\varphi_4}. \tag{3.14}$$

By applying Lemma 2.3 in (3.11), it provides

$$|a_5| \leq \frac{1}{4\varphi_5}. \tag{3.15}$$

The equalities in (3.12)-(3.15) hold for the functions f given by (3.6) with $v(z) = z$, $v(z) = z^2$, $v(z) = z^3$ and $v(z) = z^4$, respectively. ∎

Conjecture 3.2. *If the function f of the form (1.1) belongs to $\mathcal{SK}_s^{\lambda,\mu}(\sin)$, then*

$$|a_k| \leq \begin{cases} \frac{1}{k\varphi_k} & , \quad k \text{ even} \\ \frac{1}{(k-1)\varphi_k} & , \quad k \text{ odd} \end{cases},$$

where φ_k is defined by (3.4).

Remark 3.1. The above conjecture has been verified for the values $n = 2, 3, 4, 5$ by the Theorem 3.1.

Letting $\mu = 0$, $\lambda = 0$ in Theorem 3.1, we get the following result.

Corollary 3.3. [16] *If the function f of the form (1.1) belongs to $\mathcal{S}_s(\sin)$, then*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{2}, \quad |a_4| \leq \frac{1}{4}, \quad |a_5| \leq \frac{1}{4}.$$

The bounds are sharp.

Remark 3.2. It is worthy to note that if $f \in \mathcal{S}_s(\sin)$, then Conjecture 3.2 holds for the value $n = 6$, see [16].

Theorem 3.4. *If the function f of the form (1.1) belongs to $\mathcal{SK}_s^{\lambda,\mu}(\sin)$, then for any complex number ρ*

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{2\left[1 + 2\left(\lambda - \mu + 3\lambda\mu\right)\right]} \max\left\{1, \frac{1 + 2\left(\lambda - \mu + 3\lambda\mu\right)}{2\left[1 + \left(\lambda - \mu + 2\lambda\mu\right)\right]^2}|\rho|\right\} \tag{3.16}$$

Proof. From (3.8) and (3.9), we get

$$\left|a_3 - \rho a_2^2\right| = \left|\frac{1}{4\varphi_3}\left(c_2 - \frac{1}{2}c_1^2\right) - \rho\frac{c_1^2}{16\varphi_2^2}\right| = \frac{1}{4\varphi_3}\left|c_2 - \left(\frac{1}{2} - \rho\frac{\varphi_3}{4\varphi_2^2}\right)c_1^2\right|.$$

Application of (2.3) leads us to (3.16). ∎

Letting $\mu = 0$, $\lambda = 0$ in Theorem 3.4, we get the following consequence.

Corollary 3.5. [18] *If the function f of the form (1.1) belongs to $\mathcal{S}_s\,(\sin)$, then for any complex number ρ*

$$\left|a_3-\rho a_2^2\right|\leq\frac{1}{2}\max\left\{1,\frac{|\rho|}{2}\right\}$$

For $\rho=1$ in Theorem 3.4, we obtain the following result.

Corollary 3.6. *If the function f of the form (1.1) belongs to $\mathcal{SK}_s^{\lambda,\mu}\,(\sin)$, then for any complex number ρ*

$$\left|a_3-a_2^2\right|\leq\frac{1}{2\left[1+2\left(\lambda-\mu+3\,\lambda\mu\right)\right]}.$$

4. LOGARITHMIC COEFFICIENTS

For a function $f\in\mathcal{S}$, the logarithmic coefficients $\delta_n\ (n\in\mathbb{N})$ are defined by

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\delta_nz^n\qquad(z\in\mathbb{D})\,, \tag{4.1}$$

and play a central role in the theory of univalent functions.

Theorem 4.1. *Let $f\in\mathcal{SK}_s^{\lambda,\mu}\,(\sin)$ be given by (1.1) and the coefficients of $\log\left(f(z)/z\right)$ be given by (4.1). Then*

$$|\delta_1|\leq\frac{1}{4\varphi_2},\qquad|\delta_2|\leq\frac{1}{4\varphi_3},\qquad|\delta_3|\leq\frac{1}{8\varphi_4},$$

where φ_k is defined by (3.4). The bounds are sharp.

Proof. Firstly, for a function f given by (1.1), by differentiating (4.1) and equating coefficients, we have

$$\delta_1=\frac{1}{2}a_2,$$

$$\delta_2=\frac{1}{2}\left(a_3-\frac{1}{2}a_2^2\right),$$

$$\delta_3=\frac{1}{2}\left(a_4-a_2a_3+\frac{1}{3}a_2^3\right).$$

Substituting for a_2,a_3 and a_4 from (3.8)-(3.10) we obtain

$$\delta_1=\frac{1}{8\varphi_2}c_1, \tag{4.2}$$

$$\delta_2=\frac{1}{8}\left[c_2-\left(\frac{1}{2}+\frac{\varphi_3}{8\varphi_2^2}\right)c_1^2\right], \tag{4.3}$$

$$\delta_3=\frac{1}{16}\left[c_3-\left(\frac{3}{4}+\frac{\varphi_4}{2\varphi_2\varphi_3}\right)c_2c_1+\left(\frac{1}{12}+\frac{\varphi_4}{4\varphi_2\varphi_3}+\frac{\varphi_4}{24\varphi_2^3}\right)c_1^3\right].$$

Using Lemma 2.1 and Lemma 2.2, we get the desired results. The equalities in Theorem 4.1 hold for the functions f given by (3.6) with $v\,(z)=z$, $v\,(z)=z^2$, $v\,(z)=z^3$ and $v\,(z)=z^4$, respectively. ∎

Conjecture 4.2. *If the function f of the form* (1.1) *belongs to* $\mathcal{SK}_s^{\lambda,\mu}(\sin)$*, then*

$$|\delta_k| \leq \begin{cases} \frac{1}{2k\varphi_{k+1}} & , \quad k \textit{ even} \\ \frac{1}{2(k+1)\varphi_{k+1}} & , \quad k \textit{ odd} \end{cases},$$

where φ_k is defined by (3.4).

Letting $\mu = 0$, $\lambda = 0$ in Theorem 4.1, we get the following result.

Corollary 4.3. [16] *Let $f \in \mathcal{S}_s(\sin)$ be given by* (1.1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (4.1). *Then*

$$|\delta_1| \leq \frac{1}{4}, \qquad |\delta_2| \leq \frac{1}{4}, \qquad |\delta_3| \leq \frac{1}{8}.$$

The bounds are sharp.

Theorem 4.4. *Let $f \in \mathcal{SK}_s^{\lambda,\mu}(\sin)$ be given by* (1.1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (4.1). *Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|\delta_2 - \gamma\delta_1^2\right| \leq \frac{1}{4}\max\left\{1, \frac{|\gamma + \varphi_3|}{4\varphi_2^2}\right\},$$

where φ_k is defined by (3.4).

Proof. By using (4.2) and (4.3), the desired result is obtained from the equality

$$\delta_2 - \gamma\delta_1^2 = \frac{1}{8}\left[c_2 - \left(\frac{1}{2} + \frac{\varphi_3}{8\varphi_2^2} + \frac{\gamma}{8\varphi_2^2}\right)c_1^2\right] \qquad (\gamma \in \mathbb{C})$$

and Lemma 2.1. ∎

Letting $\mu = 0$, $\lambda = 0$ in Theorem 4.4, we get the following result.

Corollary 4.5. *Let $f \in \mathcal{S}_s(\sin)$ be given by* (1.1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (4.1). *Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|\delta_2 - \gamma\delta_1^2\right| \leq \frac{1}{4}\max\left\{1, \frac{|\gamma + 1|}{4}\right\}.$$

If we take $\gamma = 1$ in Theorem 4.4, then we get the following consequence.

Corollary 4.6. *Let $f \in \mathcal{SK}_s^{\lambda,\mu}(\sin)$ be given by* (1.1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (4.1). *Then*

$$\left|\delta_2 - \delta_1^2\right| \leq \frac{1}{4}.$$

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