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LINEAR SYSTEM OF SINGULARLY PERTURBED INITIAL VALUE PROBLEMS WITH ROBIN INITIAL CONDITIONS

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ABSTRACT. On the interval $(0, 1]$, this paper considers an initial value problem for a system of n singularly perturbed differential equations with Robin initial conditions. On a piecewise uniform Shishkin mesh, a computational approach based on a classical finite difference scheme is proposed. This approach is shown to be first-order convergent in the maximum norm uniformly in the perturbation parameters. The theory is illustrated by a numerical example.

Key words and phrases: Singular perturbation problems; Robin initial conditions; Finite difference schemes; Shishkin mesh; Parameter uniform convergence.

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1. INTRODUCTION

We consider a system of n singularly perturbed first-order ordinary differential equations with the Robin initial conditions. Each equation's leading term is multiplied by a small positive parameter, which may or may not be the same. To obtain approximate solutions, a classical finite difference scheme is applied to a Shishkin mesh (which is piecewise uniform). The error estimates and parameter-uniform approximations to its derivatives are presented here. To back up the theory, numerical evidence is provided.
Consider the singularly perturbed linear system and for all $x \in \zeta$,

$$\vec{L}\vec{y}(x) = \begin{cases} (\vec{L}\vec{y})_1(x) = \sigma_1 y_1'(x) + a_{11}(x)y_1(x) + \cdots + a_{1n}(x)y_n(x) = f_1(x) \\ (\vec{L}\vec{y})_2(x) = \sigma_2 y_2'(x) + a_{21}(x)y_1(x) + \cdots + a_{2n}(x)y_n(x) = f_2(x), \\ \vdots \\ (\vec{L}\vec{y})_n(x) = \sigma_n y_n'(x) + a_{n1}(x)y_1(x) + \cdots + a_{nn}(x)y_n(x) = f_n(x) \end{cases} \tag{1.1}$$

where $\zeta = (0, 1]$ and $\overline{\zeta} = [0, 1]$, with the defined initial conditions

$$y_k(0) - \sigma_k y_k'(0) = \wp_k, \quad k = 1, 2, \ldots, n. \tag{1.2}$$

It is expected that the parameters $\sigma_k, \quad k = 1, 2, \cdots, n$ are distinct.

Assumption 1.1. *The functions $a_{kl},\ f_k \in \mathbb{C}^{(2)}(\overline{\zeta}),\ k, l = 1(1)n$ satisfy the following inequalities*

$$\left.\begin{aligned} &(i) \quad a_{kk}(x) > \sum_{\substack{l \neq k \\ k=1}}^{n} |a_{kl}(x)| \ \textit{ for } \ k, l = 1(1)n \\ &(ii) \quad a_{kl}(x) \leq 0 \quad \textit{for } \ k \neq l \ \textit{ and } \ k, l = 1(1)n \end{aligned}\right\} \forall\, x \in \overline{\zeta}. \tag{1.3}$$

Assumption 1.2. *α is a positive integer that satisfies the inequality*

$$0 < \alpha < \min_{\substack{k=1(1)n \\ x \in \overline{\zeta}}} \left\{ \sum_{l=1}^{n} a_{kl}(x) \right\}. \tag{1.4}$$

Assumption 1.3. *For $k = 1(1)n$, $0 < \sigma_k \leq 1$ the singular perturbation parameters $\sigma_1, \sigma_2, \cdots, \sigma_n$ are assumed to be distinct and the ordering $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq 1$ is assumed for convenience.*

The (1.1) and (1.2) problems can alternatively be expressed in operator form

$$\vec{L}\vec{y} = \vec{f} \text{ on } \zeta \tag{1.5}$$

with

$$\vec{\mathbb{B}}\vec{y}(0) = \vec{\wp} \tag{1.6}$$

where $\vec{y}(x) = (y_1(x), y_2(x), \cdots, y_n(x))^T,\ \vec{\wp} = (\wp_1(x), \wp_2(x), \cdots, \wp_n(x))^T$ and the operators $\vec{L}, \vec{\mathbb{B}}$ are defined by

$$\vec{L} = ED + A, \quad \vec{\mathbb{B}} = I - ED,$$

where $E = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}$, I is the identity operator and $D = \dfrac{d}{dx}$ is the first order differential operator.
The aforementioned problem has been disturbed in the following way. The reduced problem

achieved by placing each $\sigma_k = 0, \;\; k = 1, 2, \ldots, n$ in the system (1.1) is the linear algebraic system

$$A(x)\vec{r}(x) = \vec{f}(x) \tag{1.7}$$

where $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ & & \vdots & \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix}$,
$\vec{r}(x) = (r_1(x), r_2(x), \cdots, r_n(x))^T$ and $\vec{f}(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T$.
Because the equation (1.7) has a unique solution for each value of x, the arbitrary robin initial conditions (1.2) cannot be enforced. This shows that there are initial layers in the components of the solution about $x = 0$, where the solution contains overlapping layers.

The layer pattern for the case $\sigma_1, \sigma_2, \cdots, \sigma_n$ is $\vec{y} = (y_1, y_2, \cdots, y_n)^T$. Each component y_k for $k = 1(1)n$ has an initial layer of width $O(\sigma_n)$, whereas the components y_k for $k = 1(1)n-1$ have an extra layer of width $O(\sigma_{n-1})$, and so on.

2. ANALYTICAL RESULTS

The operator $\vec{L}$ complies with the following maximum principle.

Lemma 2.1. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Let $\vec{\psi} = (\psi_1, \psi_2, \cdots, \psi_n)^T$ be a vector-valued function of any type in the domain of $\vec{L}$ such that $\vec{\mathbb{B}}\vec{\psi}(0) \geq \vec{0}$. Then $\vec{L}\vec{\psi}(x) \geq \vec{0}$ on $x \in \zeta$ implies that $\vec{\psi}(x) \geq \vec{0}$ on $x \in \overline{\zeta}$.*

The following stability conclusion is proved as a direct outcome of the preceding lemma:

Lemma 2.2. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Let $\vec{\psi}$ be a vector-valued function of any type in the domain of $\vec{L}$, then for each $k, \; 1 \leq k \leq n$ and $x \in \overline{\zeta}$, then*

$$|\psi_k(x)| \leq \max\{||\vec{\mathbb{B}}\vec{\psi}(0)||, \frac{1}{\alpha}||\vec{L}\vec{\psi}||\}.$$

Lemma 2.3. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Let $\vec{y}$ be the solution of* (1.1), (1.2). *Then, for each $k, \;\; k = 1, 2, \cdots, n$ and $x \in \overline{\zeta}$, there exists a constant C such that*

$$\begin{aligned} |y_k(x)| &\leq C\left\{\| \vec{\wp} \| + \| \vec{f} \|\right\} \\ |y_k'(x)| &\leq C\sigma_k^{-1}\left\{\| \vec{\wp} \| + \| \vec{f} \|\right\} \\ |y_k''(x)| &\leq C\sigma_k^{-2}\left\{\| \vec{\wp} \| + \| \vec{f} \| + \| \vec{f}' \|\right\}. \end{aligned}$$

Consider the Shishkin decomposition of the solution $\vec{y}$ of the initial value problem (1.1) into smooth and singular components,

$$\vec{y} = \vec{r} + \vec{s}. \tag{2.1}$$

Taking into account the sublayers that occur for the components, the smooth component $\vec{r}$ is decomposed further

$$\begin{aligned} r_n &= y_{0,n} + \sigma_n r_{n,n} \\ r_{n-1} &= y_{0,n-1} + \sigma_n r^1_{n-1,n} \\ &\vdots \\ r_1 &= u_{0,1} + \sigma_n r^1_{1,n} \end{aligned} \tag{2.2}$$

as all the components have σ_n layers. Since components except u_n have σ_{n-1} sublayers, the components $r_{n-1}, r_{n-2}, \cdots, r_1$ takes the form

$$\begin{aligned} r_{n-1} &= y_{0,n-1} + \sigma_n(r_{n-1,n} + \sigma_{n-1} r_{n-1,n-1}) \\ r_{n-2} &= y_{0,n-2} + \sigma_n(r_{n-2,n} + \sigma_{n-1} r_{n-2,n-1}) \\ &\vdots \\ r_n &= y_{0,1} + \sigma_n(r_{1,n} + \sigma_{n-1} r_{1,n-1}). \end{aligned} \tag{2.3}$$

Further $y_{n-2}, y_{n-3}, \cdots, y_1$ have σ_{n-2} sublayers and hence that leads to the decomposition,

$$\begin{aligned} r_{n-2} &= y_{0,n-2} + \sigma_n(r_{n-2,n} + \sigma_{n-1}(r_{n-2,n-1} + \sigma_{n-2} r_{n-2,n-2})) \\ r_{n-3} &= y_{0,n-3} + \sigma_n(r_{n-3,n} + \sigma_{n-1}(r_{n-3,n-1} + \sigma_{n-2} v = r_{n-3,n-2})) \\ &\vdots \\ r_1 &= y_{0,1} + \sigma_n(r_{1,n} + \sigma_{n-1}(r_{1,n-1} + \sigma_{n-2} r_{1,n-2})). \end{aligned} \tag{2.4}$$

Continuing in this manner, it is easy to show that

$$\vec{r} = \vec{y}_0(x) + \vec{\eth}(x) \tag{2.5}$$

where $\vec{\eth}(x) = (\eth_1(x), \eth_2(x), \cdots, \eth_n(x))^T$

$$\begin{pmatrix} \eth_1 \\ \eth_2 \\ \vdots \\ \eth_n \end{pmatrix} = \begin{pmatrix} \sigma_1\sigma_2\cdots\sigma_n & \sigma_2\sigma_3\cdots\sigma_n & \cdots & \sigma_n \\ 0 & \sigma_2\sigma_3\cdots\sigma_n & \cdots & \sigma_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix} \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ 0 & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & r_{n,n} \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.6}$$

That is,

$$\eth_l = \vec{\sigma}^{\,l}_l (\vec{v}^{\,l}_l)^T \tag{2.7}$$

$$\begin{aligned} \vec{\sigma}^{\,l}_l &= (0, 0, \ldots, \sigma_l\sigma_{l+1}\ldots\sigma_n, \sigma_{l+1}\sigma_{l+2}\ldots\sigma_n, \ldots, \sigma_{n-1}\sigma_n, \sigma_n) \\ \vec{r}^{\,k}_k &= (0, 0, \ldots, r_{k,k}, r_{k,k+1}, \ldots, r_{k,n}). \end{aligned}$$

Then using (2.1), (2.5) in (1.1), (1.2), the smooth component of the solution $\vec{r}$ is determined to satisfy

$$\vec{L}\vec{r} = \vec{f} \text{ on } \zeta \tag{2.8}$$

with

$$\vec{\mathbb{B}}\vec{r}(0) = \vec{\mathbb{B}}\vec{u}_0(0) + \vec{\mathbb{B}}\vec{\eth}(0) \tag{2.9}$$

and the singular component $\vec{s} = (s_1, s_2, \cdots, s_n)^T$ is the solution of

$$\vec{L}\vec{s}(x) = \vec{0} \text{ for } x \in (0, 1] \tag{2.10}$$

with

$$\vec{\mathbb{B}}\vec{s}(0) = \vec{\wp} - \vec{\mathbb{B}}\vec{r}(0). \tag{2.11}$$

From (2.4), (2.5), (2.6) it is established that the components $r_{k,l},\ k = 1, 2, \ldots, n,\ l = k, k+1, \ldots, n$ satisfy the following systems of equations:

$$\begin{aligned}
a_{11}r_{1,n} + a_{12}r_{2,n} + \cdots + a_{1n}r_{n,n} &= -\frac{\sigma_1}{\sigma_n}y'_{0,1}\\
a_{21}r_{1,n} + a_{22}r_{2,n} + \cdots + a_{2n}r_{n,n} &= -\frac{\sigma_2}{\sigma_n}y'_{0,2}\\
&\vdots\\
a_{n-1\,1}r_{1,n} + a_{n-1\,2}r_{2,n} + \cdots + a_{n-1\,n}r_{n,n} &= -\frac{\sigma_{n-1}}{\sigma_n}y'_{0,n-1}
\end{aligned} \tag{2.12}$$

$$\begin{gathered}
\sigma_n r'_{n,n} + a_{n\,1}r_{1,n} + a_{n\,2}r_{2,n} + \cdots + a_{n\,n}r_{n,n} = -y'_{0,n}\\
r_{n,n}(0) - \sigma_n r'_{n,n}(0) = 0
\end{gathered} \tag{2.13}$$

$$\begin{aligned}
a_{11}r_{1,n-1} + a_{12}r_{2,n-1} + \cdots + a_{1\,n-1}r_{n-1,n-1} &= \frac{-\sigma_1}{\sigma_{n-1}}r'_{1,n}\\
a_{21}r_{1,n-1} + a_{22}r_{2,n-1} + \cdots + a_{2\,n-1}r_{n-1,n-1} &= \frac{-\sigma_2}{\sigma_{n-1}}r'_{2,n}\\
&\vdots\\
a_{n-2\,1}r_{1,n-1} + a_{n-2\,2}r_{2,n-1} + \cdots + a_{n-2\,n-1}r_{n-1,n-1} &= \frac{-\sigma_{n-2}}{\sigma_{n-1}}r'_{n-2,n}
\end{aligned} \tag{2.14}$$

$$\begin{gathered}
\sigma_{n-1}r'_{n-1,n-1} + a_{n-1\,1}r_{1,n-1} + \cdots + a_{n-1\,n-1}r_{n-1,n-1} = -r'_{n-1,n}\\
r_{n-1,n-1}(0) - \sigma_{n-1}r'_{n-1,n-1}(0) = 0
\end{gathered} \tag{2.15}$$

and so on.
Lastly,

$$a_{11}r_{1,2} + a_{12}r_{2,2} = -\frac{\sigma_1}{\sigma_2}r'_{1,3} \tag{2.16}$$

$$\begin{gathered}
\sigma_2 r'_{2,2} + a_{21}r_{1,2} + a_{22}r_{2,2} = -r'_{2,3}\\
r_{2,2}(0) - \sigma_2 r'_{2,2}(0) = 0
\end{gathered} \tag{2.17}$$

and

$$\begin{gathered}
\sigma_1 r'_{1,1} + a_{11}r_{1,1} = -r'_{1,2}\\
r_{1,1}(0) - \sigma_1 r'_{1,1}(0) = 0
\end{gathered} \tag{2.18}$$

The solution's singular component $\vec{u}$ fulfils

$$\begin{gathered}
\vec{L}\vec{s} = \vec{0} \text{ on } \zeta\\
\text{with } \vec{\mathbb{B}}\vec{s}(0) = \vec{\mathbb{B}}(\vec{y} - \vec{r})(0).
\end{gathered} \tag{2.19}$$

From the expressions (2.12) - (2.19) and using lemma 2.3 for $\vec{r}$, it is found that for $k = 1, 2, \ldots, n,\ l = 1, 2, \ldots, n,\ k \le l,\ k = 0, 1, 2$

$$|r^{(m)}_{k,l}(x)| \le C\left(1 + \sigma_l^{-1}\prod_{q=l+1}^{n}\sigma_q^{-1}\right). \tag{2.20}$$

From (2.5), (2.7) and (2.20), the following bounds for $r_k,\ k = 1, 2, \ldots, n,$ hold

$$|r_k^{(m)}| \le C \quad \text{for } m = 0, 1$$
$$|r_k^{(m)}| \le C\sigma_k^{-1} \quad \text{for } m = 2.$$

The layer functions associated with the solution $\vec{y}$ are given by $B_k(x), 1 \le k \le n,$

$$B_k(x) = e^{-\alpha x/\sigma_k}, \qquad x \in [0, \infty). \tag{2.21}$$

The following basic features of these layer functions should be observed for every $1 \le k < l \le n$ and $0 \le x < z \le 1$.

(1) $B_k(x) < B_l(x)$, for all $x > 0$.
(2) $B_k(z) > B_l(x)$, for all $0 \le z < x < \infty$.
(3) $B_k(0) = 1$ and $0 < B_k(x) < 1$ for all $x > 0$.

Next lemma contains bounds on the singular component $\vec{s}$ of $\vec{y}$ and its derivatives.

Lemma 2.4. *Let $A(x)$ satisfy* (1.3) *and* (1.4).*Then there exists a constant C, such that for each $k = 1, 2, \ldots, n,$ and $x \in \overline{\zeta}$,*

$$|s_k(x)| \le CB_n(x)$$
$$|s_k'(x)| \le C \sum_{q=1}^{n} \frac{B_q(x)}{\sigma_q}$$
$$|\sigma_k s_k''(x)| \le C \sum_{q=1}^{n} \frac{B_q(x)}{\sigma_q}.$$

Proof. To derive the bound on $\vec{s}$, define the two functions

$$\theta_k^{\pm}(x) = CB_n(x) + s_k(x), \quad \text{for each } k = 1, 2, \ldots, n \text{ and } x \in \overline{\zeta}.$$

For a proper choice of C,

$$\vec{\mathbb{B}}\vec{\theta}^{\pm}(0) \ge \vec{0}.$$

Also for $x \in \zeta$,

$$(\vec{L}\vec{\theta}^{\pm})_k(x) \ge 0 \quad \text{as} \quad -\frac{\sigma_k}{\sigma_n} > -1.$$

By Lemma 2.1, $\vec{\theta}^{\pm}(x) \ge \vec{0}$ on $\overline{\zeta}$ and it follows that lemma

$$|s_k(x)| \le CB_n(x). \tag{2.22}$$

The bounds on $s_k^{(h)}(x),\ h = 1, 2, \ldots, n$ are now derived by induction on n.
To establish the bounds on $\vec{s}'(x)$, the n^{th} equation of the system (2.19)

$$\sigma_n s_n'(x) + a_{n1}(x)s_1(x) + a_{n2}(x)s_2(x) + \cdots + a_{nn}(x)s_n(x) = 0 \quad \text{is considered.}$$

From this equation, the bounds on $s_n'(x)$ is derived directly. That is,

$$|s_n'(x)| \le C\sigma_n^{-1}B_n(x). \tag{2.23}$$

It is then assumed that the required bounds on s_k', s_k'' hold for all systems upto order $n - 1$. Defining $\tilde{\vec{s}} = (s_1, s_2, \ldots, s_{n-1})$, then $\tilde{\vec{s}}$ satisfies the system

$$\tilde{E}\tilde{\vec{s}}'(x) + \tilde{A}\tilde{\vec{s}}(x) = \vec{g}(x)$$
$$\text{with} \quad \tilde{\vec{\mathbb{B}}}\tilde{\vec{s}}(0) = \tilde{\vec{\mathbb{B}}}\tilde{\vec{y}}(0) - \tilde{\vec{\mathbb{B}}}\tilde{\vec{r}}(0)$$

Here, $\tilde{E}, \tilde{A}$ are the matrices obtained by deleting the last row and last column from E, A respectively and the components of $\vec{g}$ are $g_k = -a_{kn}s_n$ for $1 \leq k \leq n-1$ and $\vec{\tilde{r}} = \vec{\tilde{y}}_0 + \vec{\tilde{\eth}}$ the corresponding components of $\vec{\tilde{r}}$ is similar to (2.5) of $\vec{r}$. Now decompose $\vec{\tilde{s}}$ into smooth and singular components to get $\vec{\tilde{s}} = \vec{p} + \vec{q}$, where $\vec{L}\vec{p} = \vec{g}$, $\vec{\mathbb{B}}\vec{p}(0) = \vec{\mathbb{B}}\vec{\tilde{y}}_0(0) + \vec{\mathbb{B}}\vec{\tilde{\eth}}(0)$ and $\vec{L}\vec{t} = \vec{0}$, $\vec{\mathbb{B}}\vec{t}(0) = \vec{\mathbb{B}}\vec{\tilde{s}}(0) - \vec{\mathbb{B}}\vec{p}(0)$.

Also from the defining equation of $\vec{p}(x)$, $\vec{p}(0) = \vec{0}$.

Introducing the functions, $\vec{\psi}^{\pm}(0) = CB_n(x) \pm \vec{p}(x)$, then clearly, $\vec{\mathbb{B}}\vec{\psi}^{\pm}(0) = CB_n(0) + \vec{\mathbb{B}}\vec{p}(0)$ and for $k = 1, 2, \ldots, n-1$

$$(\vec{L}\vec{\psi}^{\pm})_k(x) = C\left(\sum_{l=1}^{n} a_{kl}(x) - \alpha\left(\frac{\sigma_k}{\sigma_n}\right)\right)B_n(x) \pm \vec{L}\vec{p}(x)$$
$$\geq \vec{0} \quad \text{as} \quad -\frac{\sigma_k}{\sigma_n} \geq -1.$$

Applying Lemma 2.1, it follows that $||\vec{p}(x)|| \leq CB_n(x)$.

Defining the barrier functions, $\vec{\theta}^{\pm}(x) = C\frac{B_n(x)}{\sigma_n} \pm \vec{p}^{\prime(x)}$, and usually Lemma 2.1 for $\vec{\theta}^{\pm}$, the bounds of $\vec{p}'$ are derived.

It is clear from Lemma 2.4, that this is true for the case $n = 2$. It is assumed that the Lemma 2.4 is valid for all systems with $n-1$ equations. Hence the lemma applies to $\vec{t}$ and for $k = 1, 2, \ldots, n-1$

$$|t_k'(x)| \leq C\sum_{q=k}^{n-1} \sigma_q^{-1}B_q(x).$$

Combining the bounds of p_k and t_k, it is clear that

$$|\tilde{s}_k'(x)| \leq C\sum_{q=k}^{n} \sigma_q^{-1}B_q(x).$$

Now from the definition of $\vec{\tilde{s}}$ and using (2.22)

$$|s_k'(x)| \leq C\sum_{q=k}^{n} \sigma_q^{-1}B_q(x).$$

It is thus proved the lemma is true for systems of n equations. To estimate the bound of the second derivative, the k^{th} equation of the system $\vec{L}\vec{s}$ is differentiated to get

$$\sigma_k s_k''(x) = -(A(x)\vec{s}_k'(x) + A'(x)\vec{s}(x))_k$$

and it is seen that the bound on $s_k''(x)$ follows from the bounds of $\vec{s}$ and $\vec{s}'$.

The proof of the lemma is complete. ∎

3. SHISHKIN MESH

A piecewise uniform mesh with N mesh-intervals is created, and mesh points $\{x_j\}_{j=0}^{N}$ are generated by splitting the interval $\overline{\zeta}$ into $n+1$ sub-intervals as shown below

$$\overline{\zeta} = [0, \tau_1] \cup (\tau_1, \tau_2] \cup \cdots (\tau_{n-1}, \tau_n] \cup (\tau_n, 1].$$

The fitted mesh $\overline{\zeta}^N$ is given by $\{x_j\}_0^N$ and $\zeta^N = \{x_j\}_1^N$, where n is the number of transition points between uniform meshes, and

$$\tau_n = \min\left\{\frac{1}{2}, \frac{\sigma_n}{\alpha}\ln N\right\}$$

and τ_k for $k = 1, 2, \dots, n-1$ are presented by

$$\tau_k = \min\left\{\frac{\tau_{k+1}}{2}, \frac{\sigma_k}{\alpha}\ln N\right\}. \tag{3.1}$$

Clearly,

$$0 < \tau_1 < \cdots < \tau_n \le \frac{1}{2}.$$

Then, on the sub-interval $[0, \tau_1]$, a uniform mesh with mesh-intervals of $\frac{N}{2^n}$ is placed. Similarly, a uniform mesh with $\frac{N}{2^{n-k+1}}$ mesh intervals is put on $(\tau_k, \tau_{k+1}],\ 1 \le k \le n-1$, and a uniform mesh with $\frac{N}{2}$ mesh intervals is set on $(\tau_n, 1]$.

This method yields a class of 2^n piecewise uniform Shishkin meshes $M_{\vec{b}}$ where $\vec{b}$ denotes an n-vector with $b_k = 0$ if $\tau_k = \frac{\tau_{k+1}}{2}$ and $b_k = 1$, alternatively. It should be emphasised that any such mesh

$$h_l \le CN^{-1}, \quad \text{for any } l, \quad 1 \le j \le N \tag{3.2}$$

$$\tau_k \le C\sigma_k \ln N \quad \text{for any } k, \quad 1 \le i \le n \tag{3.3}$$

$$B_k(\tau_k) = N^{-1} \quad \text{if } b_k = 1 \tag{3.4}$$

$$\tau_k = 2^{-(l-k+1)}\tau_{l+1}, \quad \text{for } k \le l, \text{ if } b_m = 0,\ m = k, \dots, l. \tag{3.5}$$

4. THE DISCRETE PROBLEM

The Initial Value Problems (1.1) and (1.2) are discretized using a fitted mesh approach consists of a piecewise uniform fitted mesh and a classical finite difference operator. The backward Euler finite difference technique on a piecewise uniform fitted mesh defines the discrete solutions on any $M_{\vec{b}}$. The discrete problem for $l = 1, 2, \dots, N$ is

$$\vec{L}^N\vec{Y}(x_l) = \begin{cases} \sigma_1 D^- Y_1(x_l) + a_{11}(x_l)Y_1(x_l) + \cdots + a_{1n}(x_l)Y_n(x_l) = f_1(x_l) \\ \sigma_2 D^- Y_2(x_l) + a_{21}(x_l)Y_1(x_l) + \cdots + a_{2n}(x_l)Y_n(x_l) = f_2(x_l) \\ \vdots \\ \sigma_n D^- Y_n(x_l) + a_{n1}(x_l)Y_1(x_l) + \cdots + a_{nn}(x_l)Y_n(x_l) = f_n(x_l) \end{cases} \tag{4.1}$$

with

$$\vec{Y}(0) - ED^+\vec{Y}(0) = \vec{\wp}. \tag{4.2}$$

(4.1), (4.2) can also be expressed as an operator form problem

$$\vec{L}^N\vec{Y} = \vec{f} \text{ on } \zeta^N \text{ with}$$

$$\vec{\mathbb{B}}^N\vec{Y}(0) = \vec{\wp}$$

$$\text{where } \vec{L}^N = ED^- + A \text{ with}$$

$$\vec{\mathbb{B}}^N = I - ED^+ I$$

and $D^+,\ D^-$ are the distinguishing operators

$$D^-\vec{Y}(x_l) = \frac{\vec{Y}(x_l) - \vec{Y}(x_{l-1})}{x_l - x_{l-1}}, \quad D^+\vec{Y}(x_l) = \frac{\vec{Y}(x_{l+1}) - \vec{Y}(x_l)}{x_{l+1} - x_l}, \ \ l = 1, 2, \dots, N.$$

The discrete findings that follow are comparable to the continuous results.

Lemma 4.1. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Let $\vec{\Psi} = (\Psi_1, \Psi_2, \dots, \Psi_n)^T$ be any vector-valued mesh function, such that $\vec{\mathbb{B}}^N\vec{\Psi}(0) \geq \vec{0}$. Then $\vec{L}^N\vec{\Psi} \geq \vec{0}$ on ζ^N implies that $\vec{\Psi} \geq \vec{0}$ on $\overline{\zeta}^N$.*

Lemma 4.2. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Let $\vec{\Psi}$ be any vector-valued mesh function on $\overline{\zeta}^N$, then for each $i = 1, 2, \dots, n$,*

$$\Psi_i(x_j) \leq \max\left\{||\vec{\mathbb{B}}^N\vec{\Psi}(0)||, \frac{1}{\alpha}||\vec{L}^N\vec{\Psi}||\right\}, \ \ 0 \leq j \leq N.$$

5. ERROR OF LOCAL TRUNCATION

It can be observed from Lemma 4.2 that in order to bound the error $||\vec{Y} - \vec{y}||$, it is enough to bound $\vec{L}^N(\vec{Y} - \vec{y})$. Notice that, for $x_l \in \zeta^N$

$$\vec{L}^N(\vec{Y}(x_l) - \vec{y}(x_l)) = E(D^- - D)\vec{y}(x_l)$$

and

$$((\vec{L} - \vec{L}^N)y)_k(x_l) = \sigma_k(D^- - D)r_k(x_l) + \sigma_k(D^- - D)s_k(x_l).$$

This is the first derivative truncated locally. The triangle inequality then says

$$|(\vec{L}^N(\vec{Y} - \vec{y}))_k(x_l)| \leq |\sigma_k(D^- - D)v_k(x_l)| + |\sigma_k(D^- - D)w_k(x_l)|.$$

The discrete solution $\vec{Y}$ may be decomposed into $\vec{R}$ and $\vec{S}$, which are specified to be solutions to the following discrete problems, similarly to the continuous example.

$$(\vec{L}^N\vec{V})(x_j) = \vec{f}(x_j) \ \text{ on } \ \zeta^N, \quad \vec{\mathbb{B}}^N\vec{V}(0) = \vec{\mathbb{B}}\vec{v}(0) \tag{5.1}$$

and

$$(\vec{L}^N\vec{S})(x_l) = \vec{0} \ \text{ on } \ \zeta^N, \quad \vec{\mathbb{B}}^N\vec{S}(0) = \vec{\mathbb{B}}\vec{s}(0) \tag{5.2}$$

where $\vec{r}$ and $\vec{s}$ are the solutions of (2.8), (2.9) and (2.10), (2.11) respectively.

Further, for $k = 1, 2, \dots, n$,

$$|(\vec{\mathbb{B}}^N(\vec{R} - \vec{r}))_k(0)| = |(D - D^+)r_k(0)|$$
$$|(\vec{\mathbb{B}}^N(\vec{S} - \vec{s}))_k(0)| = |(D - D^+)s_k(0)|$$

$$|(\vec{L}^N(\vec{R} - \vec{r}))_k(x_l)| = |\sigma_k(D^- - D)r_k(x_l)| \tag{5.3}$$

$$|(\vec{L}^N(\vec{S} - \vec{s}))_k(x_l)| = |\sigma_k(D^- - D)s_k(x_l)|. \tag{5.4}$$

The error at each point $x_l \in \overline{\zeta}^N$ is denoted by $\vec{Y}(x_l) - \vec{y}(x_l)$. Then the local truncation error $L^N(\vec{Y}(x_l) - \vec{y}(x_l))$ has the decomposition

$$\vec{L}^N(\vec{Y} - \vec{y})(x_l) = \vec{L}^N(\vec{R} - \vec{r})(x_l) + \vec{L}^N(\vec{S} - \vec{s})(x_l).$$

As a result, the smooth and singular components' local truncation errors may be dealt individually. In light of this, it's worth noting that the following two independent estimates of the local truncation of a smooth function ψ hold for every smooth function ξ.

$$|(D^- - D)\xi(x_l)| \le 2 \max_{z\in I_l} |\xi'(z)| \tag{5.5}$$

and

$$|(D^- - D)\xi(x_l)| \le \frac{h_l}{2} \max_{z\in I_l} |\xi''(z)| \tag{5.6}$$

where $I_l = x_l - x_{l-1}$.

The next section bounds the error in the smooth and singular components.

6. ERROR ESTIMATE

There are two components to the proof of the theorem on error estimation. To begin, a theorem about the smooth component error is established. The singular component's error is then determined.

Theorem 6.1. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Let $\vec{r}$ denote the smooth component of the solution of* (1.1), (1.2) *and $\vec{R}$ denote the smooth component of the solution of the problem* (4.1), (4.2). *Then*

$$|(\vec{L}^N(\vec{R} - \vec{r}))_k(x_l)| \le CN^{-1}.$$

The following lemmas must be used in order to calculate the error in the singular component of the solution $\vec{y}$.
A comparable estimate for the singular component is generated using the geometry of the 2^{n+1} feasible Shishkin meshes. The preparatory Lemmas listed below are necessary.

Lemma 6.2. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Then for each $k = 1, 2, \dots, n,\ l = 1, 2, \dots, N$, on each mesh $M_{\vec{b}}$*

$$|\sigma_k(D^- - D)s_k(x_l)| \le C\frac{h_l}{\sigma_1}.$$

Proof. From the expression (5.6),

$$|(\mathbb{B}^N(\vec{S} - \vec{s})_k(0)| \le C(x_1 - x_0) \max_{z\in[x_0,x_1]} |s_k''(z)| \le CN^{-1}. \tag{6.1}$$

From (5.6) and Lemma 2.4, we have

$$|\sigma_k(D^- - D)s_k(x_l)| \le Ch_l \max_{z\in I_l} |\sigma_k s_k''(z)| \le Ch_l \sum_{q=1}^{n} \frac{B_q(x_{l-1})}{\sigma_q} \le C\frac{h_l}{\sigma_1}$$

as required. ∎

Lemma 6.3. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Then for each $k = 1, 2, \dots, n,\ l = 1, 2, \dots, N$ and $t = 1, 2, \dots, n-1$, on each mesh $M_{\vec{b}}$ with $b_t = 1$ there exists a decomposition*

$$s_k = \sum_{m=1}^{t+1} s_{k,m} \tag{6.2}$$

$k = 1, 2, \ldots, n,\ t = 1, 2, \ldots, n-1$ for which the following estimates hold for each $m,\ 1 \le m \le r$

$$|\sigma_k s_{k,m}(x)| \le CB_m(x)$$
$$|\sigma_k s'_{k,m}(x)| \le C\sigma'_m B_m(x)$$
$$|\sigma_k s''_{k,m}(x)| \le C \sum_{q=t+1}^{n} \sigma_q^{-1} B_q(x).$$

Furthermore,

$$|\sigma_k(D^- - D)s_k(x_l)| \le C\left(B_t(x_{l-1}) + \frac{h_l}{\sigma_{t+1}}\right).$$

Proof. Since, $b_t = 1$, we have $\sigma_t < \dfrac{\sigma_{t+1}}{2}$ and $x_{k,k+1} \in (0, 1]$, for $k = 1, 2, \ldots, t$.
Defining $s_k = \sum\limits_{m=1}^{t+1} s_{k,m}$, where the components $s_{t,m},\ 1 \le m \le t+1$ are given by

$$w_{k,t+1}(x) = \begin{cases} \sum\limits_{t=0}^{2} \frac{(x-x_{t,t+1})^t}{t!} s_k^{(t)}(x_{t,t+1}), & x \in [0, x_{t,t+1}) \\ s_k(x), & x \in [x_{t,t+1}, 1] \end{cases}$$

and for each $m,\ 2 \le m \le t$

$$s_{k,m}(x) = \begin{cases} \sum\limits_{t=0}^{2} \frac{(x-x_{m-1,m})^t}{t!} s_k^{(t)}(x_{t-1,t}), & x \in [0, x_{m-1,m}) \\ s_k(x) - \sum\limits_{q=m+1}^{t+1} s_{t,q}, & x \in [x_{m-1}, 1] \end{cases}$$

and finally,

$$s_{k,1} = s_k - \sum_{q=2}^{t+1} s_{k,q} \quad \text{on} \quad [0, 1].$$

From the above expressions we note that for each $m,\ 1 \le m \le t,\ s_{k,m} = 0$ on $[x_{m,m-1}, 1]$.
To establish the bounds on the second derivatives we observe that in $[x_{t,t+1}]$, using Lemma 2.4 and $x \ge x_{t,t+1}$, we obtain

$$|\sigma_k s''_{k,t+1}(x)| \le C \sum_{q=1}^{n} \frac{B_q(x)}{\sigma_q} \le \sum_{q=t+1}^{n} \frac{B_q(x)}{\sigma_q}.$$

On $[0, x_{t,t+1}]$, using Lemma 2.4 and $x \le x_{t,t+1}$, we obtain

$$\begin{aligned} |\sigma_k s''_{k,t+1}(x)| = |\sigma_k s''_k(x_{t,t+1})| &\le \sum_{q=1}^{n} \frac{B_q(x_{t,t+1})}{\sigma_q} \\ &\le \sum_{q=t+1}^{n} \frac{B_q(x_{t,t+1})}{\sigma_q} \le \sum_{q=t+1}^{n} \frac{B_q(x)}{\sigma_q} \end{aligned}$$

and for each $m = 2, \ldots, k$, we see that in $[x_{m,m+1}, 1],\ s''_{k,m} = 0$.
On $[x_{m-1,m}, x_{m,m+1}]$, using Lemma 2.4, we obtain

$$|\sigma_k s''_{k,m}(x)| \le |\sigma_k s''_k(x)| + \sum_{q=m+1}^{t+1} |\sigma_k s''_{k,q}(x)| \le C \sum_{q=1}^{n} \frac{B_q(x)}{\sigma_q} \le C\frac{B_m(x)}{\sigma_m}.$$

On $[0, x_{m-1,m}]$, using Lemma 2.4, and $x \le x_{m-1,m}$, we obtain

$$|\sigma_k s''_{k,m}(x)| \le |\sigma_k s''_k(x_{m-1,m})| \le C \sum_{q=1}^{n} \frac{B_q(x_{m-1,m})}{\sigma_q} \le C \frac{B_m(x_{m-1,m})}{\sigma_m} \le C \frac{B_m(x)}{\sigma_m}.$$

On $[x_{1,2}, 1]$, $s''_{k,1} = 0$.
On $[0, x_{1,2}]$, using Lemma 2.4,

$$|\sigma_k s''_{k,1}(x)| \le |\sigma_k s''_k(x)| + \sum_{q=2}^{t+1} |\sigma_k s''_{k,q}(x)| \le C \sum_{q=1}^{n} \frac{B_q(x)}{\sigma_q} \le C \frac{B_1(x)}{\sigma_1}.$$

For the bounds on the first derivative, we observe that for each $m,\ 1 \le m \le t$, on $[x_{m,m+1}, 1]$

$$\int_x^{x_{m,m+1}} \sigma_k s''_{k,m}(s) ds = \sigma_i s'_{k,m}(x_{m,m+1}) - \sigma_k s'_{k,m}(x) = -\sigma_k s'_{k,m}(x)$$

and so,

$$|\sigma_k s'_{k,m}(x)| \le \int_x^{x_{m,m+1}} |\sigma_k s_{k,m}(z)| dz \le C\sigma_m^{-1} \int_x^{x_{m,m+1}} B_m(z) dz \le C B_m(x).$$

Finally, since

$$|\sigma_k(D^- - D)s_k(x_l)| \le \sum_{m=1}^{t} |\sigma_k(D^- - D)s_{k,m}(x_l)| + |\sigma_k(D^- - D)s_{k,t+1}(x_l)|.$$

Using (5.6) on the last term and (5.5) on all other terms on the right hand side, we obtain

$$|\sigma_k(D^- - D)s_k(x_l)| \le C \left(\sum_{m=1}^{t} \max_{z \in I_l} |\sigma_k s'_{k,m}(z)| + h_l \max_{z \in I_l} |\sigma_k s''_{k,t+1}(z)| \right).$$

The proof of the lemma is complete. ∎

The desired result follows by applying the bounds on the derivatives.

Lemma 6.4. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Then, for each $k = 1, 2, \dots, n$ and $l = 1, 2, \dots, N$ on each mesh $M_{\vec{b}}$, we have the estimate*

$$|\sigma_k(D^- - D)s_k(x_l)| \le C B_n(x_{l-1}).$$

Proof. From (5.5) and Lemma 2.4, for each $k = 1, 2, \dots, n$ and $l = 1, 2, \dots, N$, we have

$$|\sigma_k(D^- - D)s_k(x_l)| \le C \max_{z \in I_l} |\sigma_k s'_k(z)| \le C\sigma_k \sum_{q=k}^{n} \frac{B_q(x_{l-1})}{\sigma_q} \le C B_n(x_{l-1})$$

as required. ∎

Using the above preliminary lemmas on appropriate subintervals we obtain the desired estimate of the singular component of the local truncation error in the following lemma.

Lemma 6.5. *Let $A(x)$ satisfy* (1.3) *and* (1.4). *Then, for each $k = 1, 2, \dots, n$ and $l = 1, 2, \dots, N$, we have the estimate*

$$|\sigma_k(D^- - D)s_k(x_l)| \le C N^{-1} \ln N.$$

Proof. We consider each subinterval separately. First, in the subinterval $(0, \tau_1]$ we have $h_l \leq CN^{-1}\tau_1$ and the result follows from Lemma 6.2,

$$|\sigma_k(D^- - D)s_k(x_l)| \leq C\frac{h_l}{\sigma_1} \leq CN^{-1}\ln N.$$

Now, on the interval $(\tau_1, \tau_2]$, we have $\tau_1 \leq x_{l-1}$ and $h_l \leq CN^{-1}\tau_2$. We divide the 2^{n+1} possible meshes into 2 subclasses.
Class (i): On the meshes $M_{\vec{b}}$ with $b_1 = 0$.
b_1 implies $\tau_1 = 2^{-1}\tau_2$. Hence from (3.5) and Lemma 6.2,

$$|\sigma_k(D^- - D)s_k(x_l)| \leq C\frac{h_l}{\sigma_1} \leq \frac{CN^{-1}\tau_1}{\sigma_1} \leq CN^{-1}\ln N.$$

Class (ii): On the meshes $M_{\vec{b}}$ with $b_1 = 1$.
b_1 implies $\tau_1 = \dfrac{\sigma_1}{\alpha}\ln N$. Hence from Lemma 6.3 and using equations (3.3) and (3.4)

$$|\sigma_k(D^- - D)s_k(x_l)| \leq CB_1(x_{l-1}) + \frac{h_l}{\sigma_2} \leq CB_1(x_{l-1}) + \frac{CN^{-1}\tau_2}{\sigma_2} \leq CN^{-1}\ln N.$$

On a general subinterval $(\tau_m, \tau_{m+1}]$ for $2 \leq m \leq n-1$. We have $\tau_m \leq x_{l-1}$ and $h_l \leq CN^{-1}\tau_{m+1}$. We divide $M_{\vec{b}}$ into 3 subclasses:
Class (i): $M^0_{\vec{b}} = \{M_{\vec{b}} : b_1 = b_2 = \cdots = b_m = 0\}$.
From Lemma 6.2 and (3.5),

$$|\sigma_k(D^- - D)s_k(x_l)| \leq C\frac{h_l}{\sigma_1} \leq \frac{CN^{-1}\tau_{m+1}}{\sigma_1} \leq \frac{CN^{-1}\tau_1}{\sigma_1} \leq CN^{-1}\ln N.$$

Class (ii): $M^r_{\vec{b}} = \{M_{\vec{b}} : b_t = 1, b_{t+1} = \cdots = b_m = 0\}$.
From Lemma 6.3 and using equations (3.3), (3.4) and (3.5)

$$\begin{aligned}|\sigma_k(D^- - D)s_k(x_l)| &\leq CB_t(x_{l-1}) + \frac{h_l}{\sigma_{t+1}} \leq CB_t(\tau_m) + \frac{CN^{-1}\tau_{m+1}}{\sigma_{m+1}}\\ &\leq CB_t(\tau_m) + \frac{CN^{-1}\tau_{t+1}}{\sigma_{t+1}} \leq CN^{-1}\ln N.\end{aligned}$$

Class (iii): $M^m_{\vec{b}} = \{M_{\vec{b}} : b_m = 1\}$.
From Lemma 6.3 and using equations (3.3) and (3.4),

$$|\sigma_i(D^- - D)s_k(x_l)| \leq CB_m(x_{l-1}) + \frac{h_l}{\sigma_{m+1}} \leq CB_m(\tau_m) + \frac{CN^{-1}\tau_{m+1}}{\sigma_{m+1}} \leq CN^{-1}\ln N.$$

Finally, on the subintervals $(\tau_n, 1]$, we have $\tau_n \leq x_{l-1}$ and $h_j \leq CN^{-1}$. We divide $M_{\vec{b}}$ into 3 sublasses: $M^0_{\vec{b}} = \{M_{\vec{b}} : b_1 = b_2 = \cdots = b_n = 0\}$, $M^t_{\vec{b}} = \{M_{\vec{b}} : b_t = 1, b_{t+1} = \cdots = b_m = 0 \text{ for some } 1 \leq t \leq n-1\}$ and $M^n_{\vec{b}} = \{M_{\vec{b}} : b_n = 1\}$. On $M^0_{\vec{b}}$, the result follows from (3.3), (3.4) and Lemma 6.2. On $M^t_{\vec{b}}$, the result follows from (3.3), (3.4) and (3.5) and Lemma 6.2. On $M^n_{\vec{b}}$, the result follows from the equation (3.3), (3.4) and Lemma 6.3.
The following result gives the $\vec{\sigma}-$ uniform error estimate. ∎

Theorem 6.6. *Let $\vec{y}$ be the solution of the continuous problem* (1.1), (1.2) *and $\vec{Y}$ be the solution of the discrete problem* (4.1), (4.2). *Then*

$$||(\vec{L}^N(\vec{Y} - \vec{y}))|| \leq CN^{-1}\ln N.$$

Proof. From Lemma 4.2, it is clear that, in order to prove the above theorem it suffices to to prove that $||(\vec{L}^N(\vec{Y} - \vec{y}))|| \leq CN^{-1}\ln N$. But, $||(\vec{L}^N(\vec{Y} - \vec{y}))|| \leq ||(\vec{L}^N(\vec{R} - \vec{r}))|| + ||(\vec{L}^N(\vec{S} - \vec{s}))||$. Hence using theorem 6.1 and the above preliminary lemmas, the above result is derived. ∎

7. NUMERICAL ILLUSTRATION

The numerical technique provided above is explained in this section with an example.

Example 7.1. *Consider the initial value problem*

$$\left.\begin{aligned} \sigma_1 y_1'(x) + (3+x^2)y_1(x) - y_2(x) - y_3(x) &= 1, \\ \sigma_2 y_2'(x) - y_1(x) + (3+2x)y_2(x) - y_3(x) &= 3+x, \\ \sigma_3 y_3'(x) - y_1(x) - y_2(x) + 4y_3(x) &= 2 \end{aligned}\right\} \forall\, x \in (0,1]$$

with

$$\begin{aligned} y_1(0) - \sigma_1 y_1'(0) &= 1 \\ y_2(0) - \sigma_2 y_2'(0) &= 1 \\ y_3(0) - \sigma_3 y_3'(0) &= 1. \end{aligned}$$

Figure 1 depicts the numerical result reached by using the fitted mesh methods (4.1) and (4.2) to the Example 7.1. In Table 1, the convergence order and the error constant are computed and displayed.

Table 7.1:

Values of $D_\sigma^N,\ D^N,\ p^N,\ p^*$ and $C_{p^*}^N$ generated for the example.

η	Number of mesh points N			
	72	144	288	576
0.125E+00	0.302E-01	0.213E-01	0.139E-01	0.853E-02
0.312E-01	0.115E-01	0.705E-02	0.417E-02	0.240E-02
0.781E-02	0.114E-01	0.699E-02	0.414E-02	0.238E-02
0.195E-02	0.114E-01	0.698E-02	0.413E-02	0.238E-02
0.488E-03	0.114E-01	0.697E-02	0.413E-02	0.238E-02
D^N	0.302E-01	0.213E-01	0.139E-01	0.853E-02
p^N	0.502E+00	0.617E+00	0.706E+00	
C_p^N	0.881E+00	0.881E+00	0.813E+00	0.706E+00
The order of $\vec{\sigma}$ -uniform convergence $p^* = 0.5022405E+00$				
Computed $\vec{\sigma}$ -uniform error constant, $C_{p^*}^N = 0.8808367E+00$				

8. CONCLUSION

The initial value problems for a singularly perturbed linear system with robin initial conditions are numerically approximated using the numerical approaches presented in this paper. By resolving a number of initial value problems with robin initial conditions, the solutions to the given singularly perturbed problems are obtained numerically. These techniques need little problem preparation and are relatively simple to use on any computer. We used the classical finite difference approach to resolve the perturbed initial value problems. Any standard analytical or numerical technique can be applied, in fact. We are able to solve the original initial value problem numerically by using the initial condition. To illustrate the applicability of the current strategy, some numerical experiments have been presented. Tables are used to display the results of computations. Although the solutions are computed at all places with mesh size h and the approximation and precise solutions are compared, we have only provided the findings

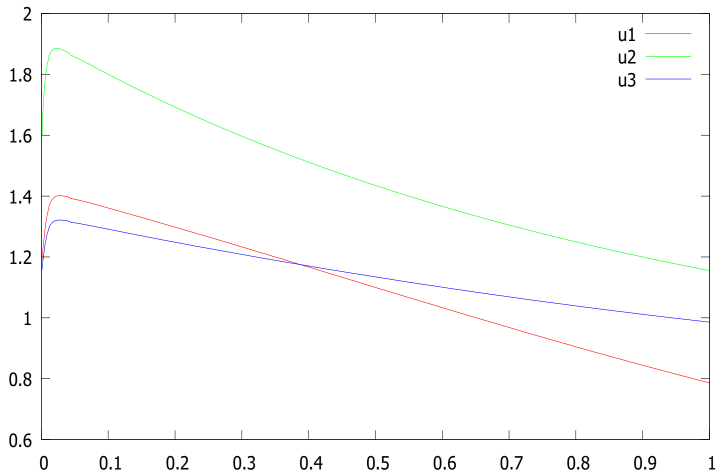

Figure 1: The figure displays the numerical solution for the problem (7.1), computed for N = 1152. The solution components y1(x); y2(x) and y3(x) has initial layers.

for a small subset of the values. The findings reveal that the current approach agrees quite well with the precise answer, demonstrating the approach's effectiveness.

9. FUTURE WORK

Systems with source terms of discontinuity at multiple points need further investigations. The numerical study of the continuation approach for solving semi-linear problems and two-dimensional problems is challenging, and work is ongoing.

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