Aust. J. Math. Anal. Appl.
Vol. **20** (2023), No. 1, Art. 12, 33 pp.
AJMAA

COMPLETE ANALYSIS OF GLOBAL BEHAVIOR OF CERTAIN SYSTEM OF PIECEWISE LINEAR DIFFERENCE EQUATIONS

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Received 5 September, 2022; accepted 14 March, 2023; published 28 April, 2023.

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ABSTRACT. Our goal is to study the system of piecewise linear difference equations $x_{n+1} = |x_n| - y_n - b$ and $y_{n+1} = x_n - |y_n| + 1$ where $n \geq 0$ and $b \geq 6$. We can prove that the behavior of the solution can be divided into 2 types depending on the region of initial condition (x_0, y_0). That is, the solution eventually becomes the equilibrium point. Otherwise, the solution eventually becomes the periodic solution of prime period 5. All regions of initial condition for each type of solution are determined.

Key words and phrases: Equilibrium point; Periodic solution; System of piecewise linear difference equation.

2010 *Mathematics Subject Classification.* Primary 39A10. Secondary 65Q10.

ISSN (electronic): 1449-5910

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The first author is supported by Kamnoetvidya Science Academy.

1. INTRODUCTION

The first order system of piecewise difference equation of the form

$$x_{n+1} = |x_n| - ay_n - b \quad \text{and} \quad y_{n+1} = x_n - c\,|y_n| + d \tag{1.1}$$

for $n \geq 0$ with a given initial condition (x_0, y_0) has been considered by several researchers. Especially, Grove, Lapierre, Tikjha and their team, see [13] - [17] and references therein. This system is also the generalization of the Lozi map $x_{n+1} = -a|x_n| + y_n + 1$ and $y_{n+1} = bx_n$, where $a, b \in \mathbb{R}$ (see [2] and [9]) and the Devaney's Gingerbreadman map $x_{n+1} = |x_n| - x_{n-1} + 1$ considered in [7]. $(x_n, y_n)_{n=0}^{\infty}$ is called the solution of (1.1) with a given initial condition (x_0, y_0) provided that the sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ satisfy (1.1) and the given initial conditions for all $n \geq 0$. The solution $(x_n, y_n)_{n=0}^{\infty}$ of (1.1) is said to be eventually becomes the equilibrium point $(\bar{x}, \bar{y})$ of (1.1) if there exist an integer N, real numbers $\bar{x}$ and $\bar{y}$ and an integer N such that $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq N$. In addition, the solution $(x_n, y_n)_{n=0}^{\infty}$ of (1.1) is said to be eventually periodic with prime period p if p is the smallest positive integer such that $(x_{n+p}, y_{n+p}) = (x_n, y_n)$ for all $n \geq N$, for some integer N. If the reader need more information about the system of difference equations and their solutions, please see [4] and [5].

In 2021, Busakorn et al. [1] proved that for $a = c = d = 1$ and $b = 4$, the solution of (1.1) eventually becomes the equilibrium point $(-1, -2)$. Moreover, they also showed that for given the initial condition (x_0, y_0) with large values of $|x_0|$ and $|y_0|$, the solution eventually becomes the equilibrium point or the periodic solution of prime period 5 for $b \geq 5$. Recently, Rewlirdsirikul [10] proved case-by-case that, globally, all solution of (1.1) for $a = c = d = 1$ and $b = 5$ eventually becomes the equilibrium point $(-1, -3)$.

In this paper, we use the recurrence algorithm technique to tackle the behavior of the solution of (1.1) for the case that $a = c = d = 1$ and $b \geq 6$. That is, we consider the following system

$$x_{n+1} = |x_n| - y_n - b \quad \text{and} \quad y_{n+1} = x_n - |y_n| + 1, \tag{1.2}$$

for $n \geq 0$ and $b \geq 6$. Let us first recall the lemmas about the equilibrium and the periodicity of the solution of (1.2) which are proven in [1].

Lemma 1.1. [1] *Let $b \geq 6$.*

(1) The equilibrium point of (1.2) *is* $(-1, -b+2)$.
(2) Let $(x_n, y_n)_{n=0}^{\infty}$ be the solution of (1.2). *Assume that there exists a positive integer N such that $y_N = -x_N - b + 1 \leq 0$ and $x_N \leq 0$. Then, $(x_n, y_n) = (-1, -b+2)$ for all $n > N$.*

Lemma 1.2. [1] *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be the solution of system* (1.2). *Suppose that there exists a positive integer N such that $(x_N, y_N) = (-5, b-4)$. Then, the solution eventually becomes the periodic with period* 5.

The following Lemmas 1.3 - 1.7 consider the region of initial condition in the first quadrant of $\mathbb{R}^2$.

Lemma 1.3. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be the solution of system* (1.2) *with the initial condition $(x_0, y_0) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that $x_0 - y_0 = \frac{b-3}{7}$. Then, the solution eventually becomes the periodic with prime period* 5.

Proof. Note that $b \geq 6$. By direct calculation, we have

$$
\begin{aligned}
x_1 &= |x_0| - y_0 - b = -\frac{6b+3}{7} < 0, & y_1 &= x_0 - |y_0| + 1 = \frac{b+4}{7} > 0.\\
x_2 &= |x_1| - y_1 - b = -\frac{2b+1}{7} < 0, & y_2 &= x_1 - |y_1| + 1 = -b < 0.\\
x_3 &= |x_2| - y_2 - b = \frac{2b+1}{7} > 0, & y_3 &= x_2 - |y_2| + 1 = -\frac{9b-6}{7} < 0.\\
x_4 &= |x_3| - y_3 - b = \frac{4b-5}{7} > 0, & y_4 &= x_3 - |y_3| + 1 = -b+2 < 0.\\
x_5 &= |x_4| - y_4 - b = \frac{4b-19}{7} > 0, & y_5 &= x_4 - |y_4| + 1 = -\frac{3b-16}{7} < 0.\\
x_6 &= |x_5| - y_5 - b = -5, & y_6 &= x_5 - |y_5| + 1 = \frac{b+4}{7} > 0.\\
x_7 &= |x_6| - y_6 - b = -\frac{8b-31}{7} < 0, & y_7 &= x_6 - |y_6| + 1 = -\frac{b+32}{7} < 0.\\
x_8 &= |x_7| - y_7 - b = \frac{2b+1}{7} = x_3, & y_8 &= x_7 - |y_7| + 1 = -\frac{9b-6}{7} = y_3.
\end{aligned}
$$

Thus, by the mathematical induction, the proof is completed. ∎

Lemma 1.4. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be the solution of system* (1.2) *with the initial condition* $(x_0, y_0) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that* $x_0 - y_0 = -\frac{b+4}{7}$. *Then, the solution eventually becomes the periodic with prime period* 5.

Proof. Note that $b \geq 6$. By direct calculation, we obtain

$$
\begin{aligned}
x_1 &= |x_0| - y_0 - b = -\frac{8b+4}{7} < 0, & y_1 &= x_0 - |y_0| + 1 = -\frac{b-3}{7} < 0.\\
x_2 &= |x_1| - y_1 - b = \frac{2b+1}{7} > 0, & y_2 &= x_1 - |y_1| + 1 = -\frac{9b-6}{7} < 0.
\end{aligned}
$$

Similar to Lemma 1.3, we have that $x_7 = x_2$ and $y_7 = y_2$. Thus, by the mathematical induction, the proof is completed. ∎

Lemma 1.5. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be the solution of system* (1.2) *with the initial condition* $(x_0, y_0) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that* $-\frac{b+4}{7} < x_0 - y_0 < \frac{b-3}{7}$. *Then, the solution eventually becomes the equilibrium point* $(-1, -b+2)$.

Proof. First, we have

$$
x_1 = |x_0| - y_0 - b = (x_0 - y_0) - b < 0, \qquad y_1 = x_0 - |y_0| + 1 = (x_0 - y_0) + 1.
$$

Case 1: If $x_0 - y_0 \in \left(-\frac{b+4}{7}, -1\right)$, then $y_1 < 0$. Thus,

$$
\begin{aligned}
x_2 &= |x_1| - y_1 - b = -2(x_0 - y_0) - 1 > 0, & y_2 &= x_1 - |y_1| + 1 = 2(x_0 - y_0) - b + 2 < 0.\\
x_3 &= |x_2| - y_2 - b = -4(x_0 - y_0) - 3 > 0, & y_3 &= x_2 - |y_2| + 1 = -b + 2 < 0.\\
x_4 &= |x_3| - y_3 - b = -4(x_0 - y_0) - 5, & y_4 &= x_3 - |y_3| + 1 = -4(x_0 - y_0) - b < 0.
\end{aligned}
$$

Case 1.1: If $x_0 - y_0 \in \left(-\frac{5}{4}, -1\right)$, then $x_4 < 0$ and

$$
x_5 = |x_4| - y_4 - b = 8(x_0 - y_0) + 5 < 0, \quad y_5 = x_4 - |y_4| + 1 = -8(x_0 - y_0) - b - 4 < 0.
$$

Since $y_5 = -x_5 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 6$.

Case 1.2: $x_0 - y_0 \in \left(-\frac{b+4}{7}, -\frac{5}{4}\right]$. We define sequences of real numbers $\alpha_k, \beta_k, \gamma_k, \omega_k$ and e_k by

$$\alpha_k = \frac{-(b+4)2^{3k-1} + (4b-19)}{7 \cdot 2^{3k-1}}, \quad \beta_k = \frac{(b+4)(1-2^{3k})}{7 \cdot 2^{3k}}, \quad \gamma_k = \frac{-(b+4)2^{3k+1} + (2b+1)}{7 \cdot 2^{3k+1}}$$

$$\omega_k = \frac{-(b+4)2^{3k+2} + (4b-5)}{7 \cdot 2^{3k+2}} \text{ and } e_k = \frac{(b+4)(1-2^{3k})}{7}$$

for $k \in \mathbb{N}$. Note that $-\frac{b+4}{7} < \alpha_{k+1} < \omega_k < \gamma_k < \beta_k < \alpha_k$, $\alpha_k, \beta_k, \gamma_k$ and ω_k tend to $-\frac{b+4}{7}$ as $k \to \infty$,

$$\begin{aligned}
4e_k - 5 &= \frac{-(b+4)2^{3k+2} + (4b-19)}{7} = 2^{3k+2}\alpha_{k+1},\\
8e_k - b - 4 &= \frac{(b+4)(1-2^{3k+3})}{7} = e_{k+1},\\
e_k + b - 6 &= \frac{-(b+4)2^{3k} + (4b-19)}{7} = 2^{3k}\alpha_k,\\
e_k &= 2^{3k}\beta_k,\\
2e_k - 1 &= \frac{-(b+4)2^{3k+1} + (2b+1)}{7} = 2^{3k+1}\gamma_k \text{ and}\\
4e_k - 3 &= \frac{-(b+4)2^{3k+2} + (4b-5)}{7} = 2^{3k+2}\omega_k.
\end{aligned}$$

Consider the following algorithm.

(1) If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \alpha_k\right]$, then $x_{5k-4} \geq 0$ and
$$x_{5k} = -5, \qquad y_{5k} = -2^{3k}(x_0 - y_0) + e_k.$$

(2) If $x_0 - y_0 \in (\beta_k, \alpha_k]$, then $y_{5k} < 0$,
$$x_{5k+1} = 2^{3k}(x_0 - y_0) - e_k - b + 5 < 0, \qquad y_{5k+1} = -2^{3k}(x_0 - y_0) + e_k - 4 < 0$$
and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+2$.

(3) If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \beta_k\right]$, then $y_{5k} \geq 0$ and
$$x_{5k+1} = 2^{3k}(x_0 - y_0) - e_k - b + 5 < 0, \qquad y_{5k+1} = 2^{3k}(x_0 - y_0) - e_k - 4 < 0.$$
$$x_{5k+2} = -2^{3k+1}(x_0 - y_0) + 2e_k - 1, \qquad y_{5k+2} = 2^{3k+1}(x_0 - y_0) - 2e_k - b + 2 < 0.$$

(4) If $x_0 - y_0 \in (\gamma_k, \beta_k]$, then $x_{5k+2} < 0$ and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+3$.

(5) If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \gamma_k\right]$, then $x_{5k+2} \geq 0$ and
$$x_{5k+3} = -2^{3k+2}(x_0 - y_0) + 4e_k - 3, \qquad y_{5k+3} = -b + 2.$$

(6) If $x_0 - y_0 \in (\omega_k, \gamma_k]$, then $x_{5k+3} < 0$,
$$x_{5k+4} = 2^{3k+2}(x_0 - y_0) - 4e_k + 1 < 0, \qquad y_{5k+4} = -2^{3k+2}(x_0 - y_0) + 4e_k - b < 0$$
and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+5$.

(7) If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \omega_k\right]$, then $x_{5k+3} \geq 0$ and
$$x_{5k+4} = -2^{3k+2}(x_0 - y_0) + 4e_k - 5, \qquad y_{5k+4} = -2^{3k+2}(x_0 - y_0) + 4e_k - b < 0.$$

(8) If $x_0 - y_0 \in (\alpha_{k+1}, \omega_k]$, then $x_{5k+4} < 0$,
$$x_{5k+5} = 2^{3k+3}(x_0 - y_0) - 8e_k + 5 < 0, \qquad y_{5k+5} = -2^{3k+3}(x_0 - y_0) + 8e_k - b - 4 < 0$$
and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+6$.

To prove that this algorithm explains the behavior of the solution for (1.2), let $k = 1$, we have

$$\alpha_1 = -\frac{5}{4}, \qquad \beta_1 = -\frac{b+4}{8}, \qquad \gamma_1 = -\frac{2b+9}{16},$$
$$\omega_1 = -\frac{4b+19}{32}, \qquad e_1 = -(b+4), \qquad \text{and} \quad \alpha_2 = -\frac{4b+21}{32}.$$

Each item of the algorithm holds as follows.

- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, -\frac{5}{4}\right]$, then $x_4 \geq 0$ and
$$x_5 = |x_4| - y_4 - b = -5,$$
$$y_5 = x_4 - |y_4| + 1 = -8(x_0 - y_0) - b - 4 = -2^3(x_0 - y_0) + e_1.$$
- If $x_0 - y_0 \in \left(-\frac{b+4}{8}, -\frac{5}{4}\right]$, then $x_4 < 0$. Thus,
$$x_6 = |x_5| - y_5 - b = 8(x_0 - y_0) + 9 = 2^3(x_0 - y_0) - e_1 - b + 5,$$
$$y_6 = x_5 - |y_5| + 1 = -8(x_0 - y_0) - b - 8 = -2^3(x_0 - y_0) + e_1 - 4$$
with $x_6 \leq -1 < 0$ and $y_6 < -4 < 0$. Since $y_6 = -x_6 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 7$.
- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, -\frac{b+4}{8}\right]$, then $y_5 \geq 0$. Thus,
$$x_6 = |x_5| - y_5 - b = 8(x_0 - y_0) + 9 = 2^3(x_0 - y_0) - e_1 - b + 5 \leq -b + 5 < 0,$$
$$y_6 = x_5 - |y_5| + 1 = 8(x_0 - y_0) + b = 2^3(x_0 - y_0) - e_1 - 4 \leq -4 < 0.$$
$$x_7 = |x_6| - y_6 - b = -16(x_0 - y_0) - 2b - 9 = -2^4(x_0 - y_0) + 2e_1 - 1,$$
$$y_7 = x_6 - |y_6| + 1 = 16(x_0 - y_0) + b + 10 = 2^4(x_0 - y_0) - 2e_1 - b + 2 \leq -b + 2 < 0.$$
- If $x_0 - y_0 \in \left(-\frac{2b+9}{16}, -\frac{b+4}{8}\right]$, then $x_7 < 0$. Since $y_7 = -x_7 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 8$.
- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, -\frac{2b+9}{16}\right]$, then $x_7 \geq 0$. Thus,
$$x_8 = |x_6| - y_6 - b = -32(x_0 - y_0) - 4b - 19 = -2^5(x_0 - y_0) + 4e_1 - 3,$$
$$y_8 = x_6 - |y_6| + 1 = -b + 2.$$
- If $x_0 - y_0 \in \left(-\frac{4b+19}{32}, -\frac{2b+9}{16}\right]$, then $x_8 < 0$. Thus,
$$x_9 = |x_8| - y_8 - b = 32(x_0 - y_0) + 4b + 17 = 2^5(x_0 - y_0) - 4e_1 + 1 \leq -1 < 0,$$
$$y_9 = x_8 - |y_8| + 1 = -32(x_0 - y_0) - 5b - 16 = -2^5(x_0 - y_0) + 4e_1 - b < -b + 3 < 0.$$
Since $y_9 = -x_9 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 10$.
- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, -\frac{4b+19}{32}\right]$, then $x_8 \geq 0$. Thus,
$$x_9 = |x_8| - y_8 - b = -32(x_0 - y_0) - 4b - 21 = -2^5(x_0 - y_0) + 4e_1 - 5,$$
$$y_9 = x_8 - |y_8| + 1 = -32(x_0 - y_0) - 5b - 16 = -2^5(x_0 - y_0) + 4e_1 - b$$
with $y_9 < -\frac{3b-16}{7} < 0$.
- If $x_0 - y_0 \in \left(-\frac{4b+21}{32}, -\frac{4b+19}{32}\right]$, then $x_9 < 0$. Thus,
$$x_{10} = |x_9| - y_9 - b = 64(x_0 - y_0) + 8b + 37 = 2^6(x_0 - y_0) - 8e_1 + 5,$$
$$y_{10} = x_9 - |y_9| + 1 = -64(x_0 - y_0) - 9b - 36 = -2^6(x_0 - y_0) + 8e_1 - b - 4.$$
with $x_{10} \leq -1 < 0$ and $y_{10} < -b + 6 < 0$. Since $y_{10} = -x_9 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 11$.

Next, let $k \in \mathbb{N}$ such that the algorithm holds. Consider each item of the algorithm as we replace k by $k+1$.

- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \alpha_{k+1}\right] \subset \left(-\frac{b+4}{7}, \alpha_k\right]$, then
$$\begin{aligned} x_{5k+4} &= -2^{3k+2}(x_0 - y_0) + 4e_k - 5 = -2^{3k+2}((x_0 - y_0) - \alpha_{k+1}) \geq 0, \\ y_{5k+4} &= -2^{3k+2}(x_0 - y_0) + 4e_k - b < 0. \end{aligned}$$
Thus,
$$\begin{aligned} x_{5k+5} &= |x_{5k+4}| - y_{5k+4} - b = -5, \\ y_{5k+5} &= x_{5k+4} - |y_{5k+4}| + 1 = -2^{3k+3}(x_0 - y_0) + 8e_k - b - 4 \\ &= -2^{3k+3}((x_0 - y_0) - \beta_{k+1}). \end{aligned}$$
- If $x_0 - y_0 \in (\beta_{k+1}, \alpha_{k+1}]$, then $y_{5k+5} < 0$. Thus,
$$\begin{aligned} x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = 2^{3k+3}(x_0 - y_0) - e_{k+1} - b + 5, \\ &= 2^{3k+3}((x_0 - y_0) - \alpha_{k+1}) - 1 \leq -1 < 0, \\ y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = -2^{3k+3}(x_0 - y_0) + e_{k+1} - 4, \\ &= -2^{3k+3}((x_0 - y_0) - \beta_{k+1}) - 4 < -4 < 0. \end{aligned}$$
Since $y_{5k+6} = -x_{5k+6} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+7$.
- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \beta_{k+1}\right]$, then $y_{5k+5} \geq 0$. Thus,
$$\begin{aligned} x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = 2^{3k+3}(x_0 - y_0) - e_{k+1} - b + 5 \\ &= 2^{3k+3}((x_0 - y_0) - \beta_{k+1}) - b + 5 \leq -b + 5 < 0, \\ y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = 2^{3k+3}(x_0 - y_0) - e_{k+1} - 4 \\ &= 2^{3k+3}((x_0 - y_0) - \beta_{k+1}) - 4 \leq -4 < 0. \\ x_{5k+7} &= |x_{5k+6}| - y_{5k+6} - b = -2^{3k+4}(x_0 - y_0) + 2e_{k+1} - 1 \\ &= -2^{3k+3}((x_0 - y_0) - \gamma_{k+1}), \\ y_{5k+7} &= x_{5k+6} - |y_{5k+6}| + 1 = 2^{3k+4}(x_0 - y_0) - 2e_{k+1} - b + 2 \\ &= 2^{3k+4}((x_0 - y_0) - \beta_{k+1}) - b + 2 \leq -b + 2 < 0. \end{aligned}$$
- If $x_0 - y_0 \in (\gamma_{k+1}, \beta_{k+1}]$, then $x_{5k+7} < 0$. Since $y_{5k+7} = -x_{5k+7} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+8$.
- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \gamma_{k+1}\right]$, then $x_{5k+7} \geq 0$. Thus,
$$\begin{aligned} x_{5k+8} &= |x_{5k+7}| - y_{5k+7} - b = -2^{3k+5}(x_0 - y_0) + 4e_{k+1} - 3 \\ &= -2^{3k+5}((x_0 - y_0) - \omega_{k+1}), \\ y_{5k+8} &= x_{5k+7} - |y_{5k+7}| + 1 = -b + 2 < 0. \end{aligned}$$
- If $x_0 - y_0 \in (\omega_{k+1}, \gamma_{k+1}]$, then $x_{5k+8} < 0$. Thus,
$$\begin{aligned} x_{5k+9} &= |x_{5k+8}| - y_{5k+8} - b = 2^{3k+5}(x_0 - y_0) - 4e_{k+1} + 1 \\ &= 2^{3k+5}((x_0 - y_0) - \gamma_{k+1}) - 1 \leq -1 < 0, \\ y_{5k+9} &= x_{5k+8} - |y_{5k+8}| + 1 = -2^{3k+5}(x_0 - y_0) + 4e_{k+1} - b, \\ &= -2^{3k+5}((x_0 - y_0) - \omega_{k+1}) - b + 3 < -b + 3 < 0. \end{aligned}$$
Since $y_{5k+9} = -x_{5k+9} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+10$.

- If $x_0 - y_0 \in \left(-\frac{b+4}{7}, \omega_{k+1}\right]$, then $x_{5k+8} \geq 0$. Thus,

$$\begin{aligned} x_{5k+9} &= |x_{5k+8}| - y_{5k+8} - b = -2^{3k+5}(x_0 - y_0) + 4e_{k+1} - 5 \\ &= -2^{3k+5}((x_0 - y_0) - \alpha_{k+2}), \\ y_{5k+9} &= x_{5k+8} - |y_{5k+8}| + 1 = -2^{3k+5}(x_0 - y_0) + 4e_{k+1} - b < -\frac{3b-16}{7} < 0. \end{aligned}$$

- If $x_0 - y_0 \in (\alpha_{k+2}, \omega_{k+1}]$, then $x_{5k+9} < 0$. Thus,

$$\begin{aligned} x_{5k+10} &= |x_{5k+9}| - y_{5k+9} - b = 2^{3k+6}(x_0 - y_0) - 8e_{k+1} + 5 \\ &= 2^{3k+6}((x_0 - y_0) - \omega_{k+1}) - 1 \leq -1 < 0, \\ y_{5k+10} &= x_{5k+9} - |y_{5k+9}| + 1 = -2^{3k+6}(x_0 - y_0) + 8e_{k+1} - b - 4 \\ &= -2^{3k+6}((x_0 - y_0) - \alpha_{k+2}) - b + 6 < -b + 6 < 0. \end{aligned}$$

Since $y_{5k+10} = -x_{5k+10} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+11$.

Hence, by the mathematical induction this algorithm holds for all $k \in \mathbb{N}$. This implies that the solution eventually becomes the equilibrium point $(-1, -b+2)$ for every $x_0 - y_0 \in (\alpha_{k+1}, \alpha_k]$. This is because $(\alpha_{k+1}, \omega_k] \cup (\omega_k, \gamma_k] \cup (\gamma_k, \beta_k] \cup (\beta_k, \alpha_k] = (\alpha_{k+1}, \alpha_k]$. Next, since, $\bigcup_{k\in\mathbb{N}} (\alpha_{k+1}, \alpha_k] = \left(-\frac{b+4}{7}, -\frac{5}{4}\right]$ and $(\alpha_{i+1}, \alpha_i] \cap (\alpha_{j+1}, \alpha_j] = \emptyset$, for every $i, j \in \mathbb{N}$ with $i \neq j$, we have $\{\alpha_k\}_{k\in\mathbb{N}}$ is a partition of $\left(-\frac{b+4}{7}, \frac{5}{4}\right]$. Hence, for every $x_0 - y_0 \in \left(-\frac{b+4}{7}, \frac{5}{4}\right]$, there exists a unique $m \in \mathbb{N}$ such that $x_0 - y_0 \in (\alpha_m, \alpha_{m+1}]$. Thus, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 2: If $x_0 - y_0 \in \left[-1, \frac{b-3}{7}\right)$, then $y_1 \geq 0$. Thus,

$$x_2 = |x_1| - y_1 - b = -2(x_0 - y_0) - 1, \qquad y_2 = x_1 - |y_1| + 1 = -b < 0.$$

Case 2.1: If $x_0 - y_0 \in \left[-1, -\frac{1}{2}\right)$, then $x_2 > 0$. Thus,

$$\begin{aligned} x_3 &= |x_2| - y_2 - b = -2(x_0 - y_0) - 1 > 0, & y_3 &= x_2 - |y_2| + 1 = -2(x_0 - y_0) - b < 0. \\ x_4 &= |x_3| - y_3 - b = -1 < 0, & y_4 &= x_3 - |y_3| + 1 = -b + 2 < 0. \\ x_5 &= |x_4| - y_4 - b = 4(x_0 - y_0) + 1 < 0, & y_5 &= x_4 - |y_4| + 1 = -4(x_0 - y_0) - b < 0. \end{aligned}$$

Since $y_5 = -x_5 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 6$.

Case 2.2: If $x_0 - y_0 \in \left[-\frac{1}{2}, \frac{b-3}{7}\right)$, then $x_2 \leq 0$. Thus,

$$\begin{aligned} x_3 &= |x_2| - y_2 - b = 2(x_0 - y_0) + 1 \geq 0, & y_3 &= x_2 - |y_2| + 1 = -2(x_0 - y_0) - b < 0. \\ x_4 &= |x_3| - y_3 - b = 4(x_0 - y_0) + 1, & y_4 &= x_3 - |y_3| + 1 = -b + 2 < 0. \end{aligned}$$

Next, we define sequences of real numbers $\alpha_k, \beta_k, \gamma_k, \omega_k$ and e_k by

$$\alpha_k = \frac{(b-3)2^{3k-2} - (2b+1)}{7 \cdot 2^{3k-2}}, \qquad \beta_k = \frac{(b-3)2^{3k-1} - (4b-5)}{7 \cdot 2^{3k-1}},$$

$$\gamma_k = \frac{(b-3)2^{3k-1} - (4b-19)}{7 \cdot 2^{3k-1}}, \qquad \omega_k = \frac{(b-3)2^{3k} - (b+4)}{7 \cdot 2^{3k}},$$

and

$$e_k = \frac{(b-3)2^{3k-1} - (4b+9)}{7}$$

for $k \in \mathbb{N}$. Note that $\alpha_k < \beta_k < \gamma_k < \omega_k < \alpha_{k+1} < \frac{b-3}{7}$, $\alpha_k, \beta_k, \gamma_k$ and ω_k tend to $\frac{b-3}{7}$ as $k \to \infty$ and

$$\begin{aligned}
e_k + 1 &= \frac{(b-3)2^{3k-1} - (4b+2)}{7} = 2^{3k-1}\alpha_k,\\
8e_k + 4b + 11 &= \frac{(b-3)2^{3k+2} - (4b-5)}{7} = 2^{3k+2}\beta_{k+1},\\
8e_k + 4b + 9 &= \frac{(b-3)2^{3k+2} - (4b+9)}{7} = e_{k+1},\\
e_k + 2 &= \frac{(b-3)2^{3k-1} - (4b-5)}{7} = 2^{3k-1}\beta_k,\\
e_k + 4 &= \frac{(b-3)2^{3k-1} - (4b-19)}{7} = 2^{3k-1}\gamma_k,\\
2e_k + b + 2 &= \frac{(b-3)2^{3k} - (b+4)}{7} = 2^{3k}\omega_k \text{ and}\\
4e_k + 2b + 5 &= \frac{(b-3)2^{3k+1} - (2b+1)}{7} = 2^{3k+1}\alpha_{k+1}.
\end{aligned}$$

Consider the following algorithm.

(1) If $x_0 - y_0 \in [\alpha_k, \beta_k)$, then $x_{5k-1} < 0$,

$$x_{5k} = -\, 2^{3k-1}(x_0 - y_0) + e_k < 0, \qquad y_{5k} = 2^{3k-1}(x_0 - y_0) - e_k - b + 1 < 0$$

and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+1$.

(2) If $x_0 - y_0 \in \left[\beta_k, \frac{b-3}{7}\right)$, then $x_{5k-1} \geq 0$ and

$$x_{5k} = 2^{3k-1}(x_0 - y_0) - e_k - 4, \qquad y_{5k} = 2^{3k-1}(x_0 - y_0) - e_k - b + 1 < 0.$$

(3) If $x_0 - y_0 \in [\beta_k, \gamma_k)$, then $x_{5k} < 0$,

$$x_{5k+1} = -2^{3k}(x_0 - y_0) + 2e_k + 3 < 0, \qquad y_{5k+1} = 2^{3k}(x_0 - y_0) - 2e_k - b - 2 < 0$$

and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+2$.

(4) If $x_0 - y_0 \in \left[\gamma_k, \frac{b-3}{7}\right)$, then $x_{5k} \geq 0$ and

$$x_{5k+1} = -5, \qquad y_{5k+1} = 2^{3k}(x_0 - y_0) - 2e_k - b - 2.$$

(5) If $x_0 - y_0 \in [\gamma_k, \omega_k)$, then $y_{5k+1} < 0$,

$$x_{5k+2} = -2^{3k}(x_0 - y_0) + 2e_k + 7 < 0, \qquad y_{5k+2} = 2^{3k}(x_0 - y_0) - 2e_k - b - 6 < 0$$

and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+3$.

(6) If $x_0 - y_0 \in \left[\omega_k, \frac{b-3}{7}\right)$, then $y_{5k+1} \geq 0$ and

$$\begin{aligned}
x_{5k+2} &= -2^{3k}(x_0 - y_0) + 2e_k + 7 < 0, &\quad y_{5k+2} &= -2^{3k}(x_0 - y_0) + 2e_k + b - 2 < 0.\\
x_{5k+3} &= 2^{3k+1}(x_0 - y_0) - 4e_k - 2b - 5, &\quad y_{5k+3} &= -2^{3k+2}(x_0 - y_0) + 4e_k + b + 6 < 0.
\end{aligned}$$

(7) If $x_0 - y_0 \in [\omega_k, \alpha_{k+1})$, then $x_{5k+3} < 0$ and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+4$.

(8) If $x_0 - y_0 \in \left[\alpha_{k+1}, \frac{b-3}{7}\right)$, then $x_{5k+3} \geq 0$ and

$$x_{5k+4} = 2^{3k+2}(x_0 - y_0) - 8e_k - 4b - 11, \qquad y_{5k+4} = -b + 2 < 0.$$

To prove that this algorithm explains the behavior of the solution for (1.2), let $k = 1$, we have

$$\alpha_1 = -\frac{1}{2}, \qquad \beta_1 = -\frac{1}{4}, \qquad \gamma_1 = \frac{1}{4},$$
$$\omega_1 = \frac{b-4}{8}, \qquad \alpha_2 = \frac{2b-7}{16}, \qquad \text{and} \quad e_1 = -3.$$

- If $x_0 - y_0 \in \left[-\frac{1}{2}, -\frac{1}{4}\right)$, then $x_4 < 0$ and

$$x_5 = |x_4| - y_4 - b = -4(x_0 - y_0) - 3 = -2^2(x_0 - y_0) + e_1 < -1 < 0,$$
$$y_5 = x_4 - |y_4| + 1 = 4(x_0 - y_0) - b + 4 = 2^2(x_0 - y_0) - e_1 - b + 1 < -b + 3 < 0.$$

Since $y_5 = -x_5 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 6$.

- If $x_0 - y_0 \in \left[-\frac{1}{4}, \frac{b-3}{7}\right)$, then $x_4 \geq 0$. Thus,

$$x_5 = |x_4| - y_4 - b = 4(x_0 - y_0) - 1 = 2^2(x_0 - y_0) - e_1 - 4,$$
$$y_5 = x_4 - |y_4| + 1 = 4(x_0 - y_0) - b + 4 = 2^2(x_0 - y_0) - e_1 - b + 1 < -\frac{3b-16}{7} < 0.$$

- If $x_0 - y_0 \in \left[-\frac{1}{4}, \frac{1}{4}\right)$, then $x_5 < 0$. Thus,

$$x_6 = |x_5| - y_5 - b = -8(x_0 - y_0) - 3 = -2^3(x_0 - y_0) + 2e_1 + 3 \leq -1 < 0,$$
$$y_6 = x_5 - |y_5| + 1 = 8(x_0 - y_0) - b + 4 = 2^3(x_0 - y_0) - 2e_1 - b - 2 \leq -b + 6 \leq 0.$$

Since $y_6 = -x_6 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 7$.

- If $x_0 - y_0 \in \left[\frac{1}{4}, \frac{b-3}{7}\right)$, then $x_5 \geq 0$. thus,

$$x_6 = |x_5| - y_5 - b = -5,$$
$$y_6 = x_5 - |y_5| + 1 = 8(x_0 - y_0) - b + 4 = 2^3(x_0 - y_0) - 2e_1 - b - 2.$$

- If $x_0 - y_0 \in \left[\frac{1}{4}, \frac{b-4}{8}\right)$, then $y_6 < 0$. Thus,

$$x_7 = |x_6| - y_6 - b = -8(x_0 - y_0) + 1 = -2^3(x_0 - y_0) + 2e_1 + 7 \leq -1 < 0,$$
$$y_7 = x_6 - |y_6| + 1 = 8(x_0 - y_0) - b = 2^3(x_0 - y_0) - 2e_1 - b - 6 < -4 < 0.$$

Since $y_7 = -x_6 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 8$.

- If $x_0 - y_0 \in \left[\frac{b-4}{8}, \frac{b-3}{7}\right)$, then $y_6 \geq 0$. Thus,

$$x_7 = |x_6| - y_6 - b = -8(x_0 - y_0) + 1 = -2^3(x_0 - y_0) + 2e_1 + 7 \leq -b + 3 < 0,$$
$$y_7 = x_6 - |y_6| + 1 = -8(x_0 - y_0) + b - 8 = -2^3(x_0 - y_0) + 2e_1 + b - 2 \leq -4 < 0.$$
$$x_8 = |x_7| - y_7 - b = 16(x_0 - y_0) - 2b + 7 = 2^4(x_0 - y_0) - 4e_1 - 2b - 5,$$
$$y_8 = x_7 - |y_7| + 1 = -16(x_0 - y_0) + b - 6 = -2^4(x_0 - y_0) + 4e_1 + b + 6 \leq -b + 2 < 0.$$

- If $x_0 - y_0 \in \left[\frac{b-4}{8}, \frac{2b-7}{16}\right)$, then $x_8 < 0$. Since $y_8 = -x_6 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 9$.
- If $x_0 - y_0 \in \left[\frac{2b-7}{16}, \frac{b-3}{7}\right)$, then $x_8 \geq 0$. Thus,

$$x_9 = |x_8| - y_8 - b = 32(x_0 - y_0) - 4b + 13 = 2^5(x_0 - y_0) - 8e_1 - 4b - 11,$$
$$y_9 = x_8 - |y_8| + 1 = -b + 2 < 0.$$

Next, let $k \in \mathbb{N}$ such that the algorithm holds. Consider each item of the algorithm as we replace k by $k + 1$.

- If $x_0 - y_0 \in [\alpha_{k+1}, \beta_{k+1}) \subset \left[\alpha_{k+1}, \frac{b-3}{7}\right)$, then

$$\begin{aligned} x_{5k+4} &= 2^{3k+2}(x_0 - y_0) - 8e_k - 4b - 11 = 2^{3k+2}((x_0 - y_0) - \beta_{k+1}) < 0, \\ y_{5k+4} &= -b + 2 < 0. \end{aligned}$$

Thus,

$$\begin{aligned} x_{5k+5} &= |x_{5k+4}| - y_{5k+4} - b = -2^{3k+2}(x_0 - y_0) + 8e_k + 4b + 9 \\ &= -2^{3k+2}((x_0 - y_0) - \alpha_{k+1}) - 1 \le -1 < 0, \\ y_{5k+5} &= x_{5k+4} - |y_{5k+4}| + 1 = 2^{3k+2}(x_0 - y_0) - 8e_k - 5b - 8 \\ &= 2^{3k+2}((x_0 - y_0) - \alpha_{k+1}) - b + 2 \le -b + 2 < 0. \end{aligned}$$

Since $y_{5k+5} = -x_{5k+5} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \ge 5k+6$.

- If $x_0 - y_0 \in \left[\beta_{k+1}, \frac{b-3}{7}\right) \subset \left[\alpha_{k+1}, \frac{b-3}{7}\right)$, then

$$\begin{aligned} x_{5k+4} &= 2^{3k+2}(x_0 - y_0) - 8e_k - 4b - 11 = 2^{3k+2}((x_0 - y_0) - \beta_{k+1}) \ge 0, \\ y_{5k+4} &= -b + 2 < 0. \end{aligned}$$

Thus,

$$\begin{aligned} x_{5k+5} &= |x_{5k+4}| - y_{5k+4} - b = 2^{3k+2}(x_0 - y_0) - 8e_k - 4b - 13 \\ &= 2^{3k+2}((x_0 - y_0) - \gamma_{k+1}), \\ y_{5k+5} &= x_{5k+4} - |y_{5k+4}| + 1 = 2^{3k+2}(x_0 - y_0) - 8e_k - 5b - 8 \\ &= 2^{3k+2}(x_0 - y_0) - e_{k+1} - b + 1 < -\frac{3b - 16}{7} < 0. \end{aligned}$$

- If $x_0 - y_0 \in [\beta_{k+1}, \gamma_{k+1})$, then $x_{5k+5} < 0$. Thus,

$$\begin{aligned} x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = -2^{3k+3}(x_0 - y_0) + 2e_{k+1} + 3, \\ &= -2^{3k+3}((x_0 - y_0) - \beta_{k+1}) - 1 \le -1 < 0, \\ y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = 2^{3k+3}(x_0 - y_0) - 2e_{k+1} - b - 2, \\ &= 2^{3k+3}((x_0 - y_0) - \gamma_{k+1}) - b + 6 \le -b + 6 \le 0. \end{aligned}$$

Since $y_{5k+6} = -x_{5k+6} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \ge 5k+7$.

- If $x_0 - y_0 \in \left[\gamma_{k+1}, \frac{b-3}{7}\right)$, then $x_{5k+5} \ge 0$. Since

$$\begin{aligned} x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = -5, \\ y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = 2^{3k+3}(x_0 - y_0) - 2e_{k+1} - b - 2 \\ &= 2^{3k+3}((x_0 - y_0) - \omega_{k+1}). \end{aligned}$$

- If $x_0 - y_0 \in [\gamma_{k+1}, \omega_{k+1})$, then $x_{5k+6} < 0$. Thus,

$$\begin{aligned} x_{5k+7} &= |x_{5k+6}| - y_{5k+6} - b = -2^{3k+3}(x_0 - y_0) + 2e_{k+1} + 7 \\ &= -2^{3k+3}((x_0 - y_0) - \gamma_{k+1}) - 1 \le -1 < 0, \\ y_{5k+7} &= x_{5k+6} - |y_{5k+6}| + 1 = 2^{3k+3}(x_0 - y_0) - 2e_{k+1} - b - 6 \\ &= 2^{3k+3}((x_0 - y_0) - \omega_{k+1}) - 4 < -4 < 0. \end{aligned}$$

Since $y_{5k+7} = -x_{5k+7} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \ge 5k+8$.

- If $x_0 - y_0 \in \left[\omega_{k+1}, \frac{b-3}{7}\right)$, then $y_{5k+6} \geq 0$. Thus,
$$\begin{aligned}
x_{5k+7} &= |x_{5k+6}| - y_{5k+6} - b = -2^{3k+3}(x_0 - y_0) + 2e_{k+1} + 7 \\
&= -2^{3k+5}((x_0 - y_0) - \omega_{k+1}) - b + 5 \leq -b + 5 < 0, \\
y_{5k+7} &= x_{5k+6} - |y_{5k+6}| + 1 = -2^{3k+3}(x_0 - y_0) + 2e_{k+1} + b - 2 \\
&= -2^{3k+3}((x_0 - y_0) - \omega_{k+1}) - 4 \leq -4 < 0. \\
x_{5k+8} &= |x_{5k+7}| - y_{5k+7} - b = 2^{3k+4}(x_0 - y_0) - 4e_{k+1} - 2b - 5 \\
&= 2^{3k+4}((x_0 - y_0) - \alpha_{k+2}), \\
y_{5k+8} &= x_{5k+7} - |y_{5k+7}| + 1 = -2^{3k+4}(x_0 - y_0) + 4e_{k+1} + b + 6 \\
&= -2^{3k+3}((x_0 - y_0) - \omega_{k+1}) - b + 2 \leq -b + 2 < 0.
\end{aligned}$$
- If $x_0 - y_0 \in [\omega_{k+1}, \alpha_{k+2})$, then $x_{5k+8} < 0$. Since $y_{5k+8} = -x_{5k+8} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 5k + 9$.
- If $x_0 - y_0 \in \left[\alpha_{k+2}, \frac{b-3}{7}\right)$, then $x_{5k+8} \geq 0$. Thus,
$$\begin{aligned}
x_{5k+9} &= |x_{5k+8}| - y_{5k+8} - b = 2^{3k+5}(x_0 - y_0) - 8e_{k+1} - 4b - 11, \\
y_{5k+9} &= x_{5k+8} - |y_{5k+8}| + 1 = -b + 2.
\end{aligned}$$

Hence, by the mathematical induction this algorithm holds for $k \in \mathbb{N}$. This implies that the solution eventually becomes the equilibrium point $(-1, -b+2)$ for every $x_0 - y_0 \in [\alpha_k, \alpha_{k+1})$. This is because $[\alpha_k, \beta_k) \cup [\beta_k, \gamma_k) \cup [\gamma_k, \omega_k) \cup [\omega_k, \alpha_{k+1}) = [\alpha_k, \alpha_{k+1})$. Since, $\bigcup_{k \in \mathbb{N}} [\alpha_k, \alpha_{k+1}) = \left[-\frac{1}{2}, \frac{b-3}{7}\right)$, and $[\alpha_i, \alpha_{i+1}) \cap [\alpha_j, \alpha_{j+1}) = \emptyset$, for every $i, j \in \mathbb{N}$ with $i \neq j$, we have $\{\alpha_k\}_{k \in \mathbb{N}}$ is a partition of $\left[-\frac{1}{2}, \frac{b-3}{7}\right)$. Hence, for every $x_0 - y_0 \in \left[-\frac{1}{2}, \frac{b-3}{7}\right)$, there exists a unique $m \in \mathbb{N}$ such that $x_0 - y_0 \in [\alpha_m, \alpha_{m+1})$. Thus, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b + 2)$. ∎

Lemma 1.6. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be the solution of system* (1.2) *with the initial condition* $(x_0, y_0) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that $x_0 - y_0 \in \left(\frac{b-3}{7}, \infty\right)$. Then, the solution eventually becomes the periodic with prime period* 5.

Proof. By direct calculation, we have
$$x_1 = |x_0| - y_0 - b = (x_0 - y_0) - b, \qquad y_1 = x_0 - |y_0| + 1 = (x_0 - y_0) + 1 > 0.$$

Case 1: If $x_0 - y_0 \in [b, \infty)$, then $x_1 \geq 0$. Thus,
$$\begin{aligned}
x_2 &= |x_1| - y_1 - b = -(2b + 1) < 0, & y_2 &= x_1 - |y_1| + 1 = -b < 0. \\
x_3 &= |x_2| - y_2 - b = -2b + 1 > 0, & y_3 &= x_2 - |y_2| + 1 = -3b < 0. \\
x_4 &= |x_3| - y_3 - b = 4b + 1 > 0, & y_4 &= x_3 - |y_3| + 1 = -b + 2 < 0. \\
x_5 &= |x_4| - y_4 - b = 4b - 1 > 0, & y_5 &= x_4 - |y_4| + 1 = 3b + 4 > 0. \\
x_6 &= |x_5| - y_5 - b = -5, & y_6 &= x_5 - |y_5| + 1 = b - 4.
\end{aligned}$$

Hence, by Lemma 1.2, the solution eventually becomes the periodic with prime period 5 for $n \geq 6$.

Case 2: If $x_0 - y_0 \in \left(\frac{b-3}{7}, b\right)$, then $x_1 < 0$. Thus,
$$\begin{aligned}
x_2 &= |x_1| - y_1 - b = -2(x_0 - y_0) - 1 < 0, & y_2 &= x_1 - |y_1| + 1 = -b < 0. \\
x_3 &= |x_2| - y_2 - b = 2(x_0 - y_0) + 1 > 0, & y_3 &= x_2 - |y_2| + 1 = -2(x_0 - y_0) - b < 0. \\
x_4 &= |x_3| - y_3 - b = 4(x_0 - y_0) + 1 > 0, & y_4 &= x_3 - |y_3| + 1 = -b + 2 < 0. \\
x_5 &= |x_4| - y_4 - b = 4(x_0 - y_0) - 1 > 0, & y_5 &= x_4 - |y_4| + 1 = 4(x_0 - y_0) - b + 4.
\end{aligned}$$

Case 2.1: If $x_0 - y_0 \in \left[\frac{b-4}{4}, b\right)$, then $y_5 \geq 0$. Thus,

$$x_6 = |x_5| - y_5 - b = -5, \qquad y_6 = x_5 - |y_5| + 1 = b - 4.$$

Hence, by Lemma 1.2, the solution eventually becomes the periodic with prime period 5 for $n \geq 6$.

Case 2.2: If $x_0 - y_0 \in \left(\frac{b-3}{7}, \frac{b-4}{4}\right)$. We define sequences of real numbers α_k and e_k by

$$\alpha_k = \frac{(b-3)2^{3k-1} + (3b-16)}{7 \cdot 2^{3k-1}} \quad \text{and} \quad e_k = \frac{(-b+3)2^{3k} + (b+4)}{7}$$

for $k \in \mathbb{N}$. It is easy to check that $\frac{b-3}{7} < \alpha_{k+1} < \alpha_k$, and α_k tends to $\frac{b-3}{7}$ as $k \to \infty$. Note that

$$4e_k - b = \frac{(-b+3)^{3k+2} - (3b-16)}{7} = -2^{3k+2}\alpha_{k+1} \text{ and}$$

$$8e_k - b - 4 = \frac{(-b+3)2^{3k+3} + (b+4)}{7} = e_{k+1}.$$

This implies that $e_{k+1} = -2^{3k+2}\alpha_{k+1} + b - 4$. Now, consider the following algorithm.

(1) If $x_0 - y_0 \in \left(\frac{b-3}{7}, \alpha_k\right)$, then $y_{5k} < 0$ and

$$\begin{aligned}
x_{5k+1} &= -5, & y_{5k+1} &= 2^{3k}(x_0 - y_0) + e_k > 0,\\
x_{5k+2} &= -2^{3k}(x_0 - y_0) - e_k - b + 5 < 0, & y_{5k+2} &= -2^{3k}(x_0 - y_0) - e_k - 4 < 0.\\
x_{5k+3} &= 2^{3k+1}(x_0 - y_0) + 2e_k - 1 > 0, & y_{5k+3} &= -2^{3k+1}(x_0 - y_0) - 2e_k - b + 2 < 0.\\
x_{5k+4} &= 2^{3k+2}(x_0 - y_0) + 4e_k - 3 > 0, & y_{5k+4} &= -b + 2.\\
x_{5k+5} &= 2^{3k+2}(x_0 - y_0) + 4e_k - 5 > 0, & y_{5k+5} &= 2^{3k+2}(x_0 - y_0) + 4e_k - b.
\end{aligned}$$

(2) If $x_0 - y_0 \in [\alpha_{k+1}, \alpha_k)$, then $y_{5k+5} \geq 0$ and

$$x_{5k+6} = -5, \qquad y_{5k+6} = b - 4.$$

To prove that this algorithm explains the behavior of the solution for (1.2), let $k = 1$, we have $\alpha_1 = \frac{b-4}{4}, \alpha_2 = \frac{5b-16}{32}$ and $e_1 = -b + 4$. Each item of the algorithm holds as follows.

- If $x_0 - y_0 \in \left(\frac{b-3}{7}, \frac{b-4}{4}\right)$, then $y_5 < 0$. Thus,

$$\begin{aligned}
x_6 &= |x_5| - y_5 - b = -5,\\
y_6 &= x_5 - |y_5| + 1 = 8(x_0 - y_0) - b + 4 = 2^3(x_0 - y_0) + e_1 > \frac{b+4}{7} > 0.\\
x_7 &= |x_6| - y_6 - b = -8(x_0 - y_0) + 1 = -2^3(x_0 - y_0) - e_1 - b + 5 < -\frac{8b-31}{7} < 0,\\
y_7 &= x_6 - |y_6| + 1 = -8(x_0 - y_0) + b - 8 = -2^3(x_0 - y_0) - e_1 - 4 < -\frac{b+32}{7} < 0.\\
x_8 &= |x_7| - y_7 - b = 16(x_0 - y_0) - 2b + 7 = 2^4(x_0 - y_0) + 2e_1 - 1 > \frac{2b+1}{7},\\
y_8 &= x_7 - |y_7| + 1 = -16(x_0 - y_0) + b - 6 = -2^4(x_0 - y_0) - 2e_1 - b + 2 < -\frac{9b-6}{7}.\\
x_9 &= |x_8| - y_8 - b = 32(x_0 - y_0) - 4b + 13 = 2^5(x_0 - y_0) + 4e_1 - 3 > \frac{4b-5}{7} > 0,\\
y_9 &= x_8 - |y_8| + 1 = -b + 2.\\
x_{10} &= |x_9| - y_9 - b = 32(x_0 - y_0) - 4b + 11 = 2^5(x_0 - y_0) + 4e_1 - 5 > \frac{4b-19}{7} > 0,\\
y_{10} &= x_9 - |y_9| + 1 = 32(x_0 - y_0) - 5b + 16 = 2^5(x_0 - y_0) + 4e_1 - b.
\end{aligned}$$

- If $x_0 - y_0 \in \left[\frac{5b-16}{32}, \frac{b-4}{4}\right)$, then $y_{10} \geq 0$. Thus,

$$x_{11} = |x_{10}| - y_{10} - b = -5, \qquad y_{11} = x_{10} - |y_{10}| + 1 = b - 4.$$

Hence, by Lemma 1.2, the solution eventually becomes the periodic with prime period 5 for $n \geq 11$.

Next, let $k \in \mathbb{N}$ such that the algorithm holds. Consider each item of the algorithm as we replace k by $k+1$.

- If $x_0 - y_0 \in \left(\frac{b-3}{7}, \alpha_{k+1}\right) \subset \left(\frac{b-3}{7}, \alpha_k\right)$. Then,

$$\begin{aligned} x_{5k+5} &= 2^{3k+2}(x_0 - y_0) + 4e_k - 5 > 0, \\ y_{5k+5} &= 2^{3k+2}(x_0 - y_0) + 4e_k - b = 2^{3k+2}((x_0 - y_0) - \alpha_{k+1}) < 0. \end{aligned}$$

Thus,

$$\begin{aligned} x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = -5, \\ y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = 2^{3k+3}(x_0 - y_0) + e_{k+1} \\ &= 2^{3k+3}((x_0 - y_0) - \alpha_{k+1}) + b - 4. \end{aligned}$$

Note that $2^{3k+3}((x_0 - y_0) - \alpha_{k+1}) > 2^{3k+3}\left(\frac{b-3}{7} - \alpha_{k+1}\right) = \frac{6b-32}{7}$. Hence, $y_{5k+6} > \frac{b+4}{7} > 0$ and

$$\begin{aligned} x_{5k+7} &= |x_{5k+6}| - y_{5k+6} - b = -2^{3k+3}(x_0 - y_0) - e_{k+1} - b + 5 \\ &= -2^{3k+3}((x_0 - y_0) - \alpha_{k+1}) - 2b + 9 < -\frac{8b-31}{7} < 0, \\ y_{5k+7} &= x_{5k+6} - |y_{5k+6}| + 1 = -2^{3k+3}(x_0 - y_0) - e_{k+1} - 4 \\ &= -2^{3k+3}((x_0 - y_0) - \alpha_{k+1}) - b < -\frac{b+32}{7} < 0. \\ x_{5k+8} &= |x_{5k+7}| - y_{5k+7} - b = 2^{3k+4}(x_0 - y_0) + 2e_{k+1} - 1 \\ &= 2^{3k+4}((x_0 - y_0) - \alpha_{k+1}) + 2b - 9 > \frac{2b+1}{7} > 0, \\ y_{5k+8} &= x_{5k+7} - |y_{5k+7}| + 1 = -2^{3k+4}(x_0 - y_0) - 2e_{k+1} - b + 2 \\ &= -2^{3k+4}((x_0 - y_0) - \alpha_{k+1}) - 3b + 10 < -\frac{9b-6}{7} < 0. \\ x_{5k+9} &= |x_{5k+8}| - y_{5k+8} - b = 2^{3k+5}(x_0 - y_0) + 4e_{k+1} - 3 \\ &= 2^{3k+5}((x_0 - y_0) - \alpha_{k+1}) + 4b - 19 > \frac{4b-5}{7} > 0, \\ y_{5k+9} &= x_{5k+8} - |y_{5k+8}| + 1 = -b + 2 < 0. \\ x_{5k+10} &= |x_{5k+9}| - y_{5k+9} - b = 2^{3k+5}(x_0 - y_0) + 4e_{k+1} - 5 \\ &= 2^{3k+5}((x_0 - y_0) - \alpha_{k+1}) + 4b - 21 > \frac{4b-19}{7} > 0, \\ y_{5k+10} &= x_{5k+9} - |y_{5k+9}| + 1 = 2^{3k+5}(x_0 - y_0) + 4e_{k+1} - b \\ &= 2^{3k+5}((x_0 - y_0) - \alpha_{k+2}). \end{aligned}$$

- If $x_0 - y_0 \in [\alpha_{k+2}, \alpha_{k+1}) \subset \left(\frac{b-3}{7}, \alpha_{k+1}\right)$, then $(x_0 - y_0) - \alpha_{k+2} \geq 0$. Hence, $y_{5k+10} \geq 0$. Thus,

$$x_{5k+11} = |x_{5k+10}| - y_{5k+10} - b = -5, \qquad y_{5k+11} = x_{5k+10} - |y_{5k+10}| + 1 = b - 4.$$

Hence, by Lemma 1.2, the solution of (1.2) eventually becomes periodic with period 5 for $n \geq 5k+11$.

Therefore, by the mathematical induction, the algorithm holds for $k \in \mathbb{N}$. Since $\bigcup_{k\in\mathbb{N}}[\alpha_{k+1}, \alpha_k) = \left(\frac{b-3}{7}, \frac{b-4}{4}\right)$ and $[\alpha_{i+1}, \alpha_i) \cap [\alpha_{j+1}, \alpha_j) = \emptyset$, for every $i, j \in \mathbb{N}$ with $i \neq j$, this implies that $\{\alpha_k\}_{k\in\mathbb{N}}$ is a partition of $\left(\frac{b-3}{7}, \frac{b-4}{4}\right)$. Hence, for every $x_0 - y_0 \in \left(\frac{b-3}{7}, \frac{b-4}{4}\right)$, there exists a unique $m \in \mathbb{N}$ such that $x_0 - y_0 \in [\alpha_{m+1}, \alpha_m)$. Thus, the solution of (1.2) eventually becomes periodic with prime period 5 for $n \geq 5m+6$. This completes the proof. ∎

Lemma 1.7. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be the solution of system* (1.2) *with the initial condition $(x_0, y_0) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that $x_0 - y_0 \in \left(-\infty, -\frac{b+4}{7}\right)$. Then, the solution eventually becomes the periodic with prime period* 5.

Proof. First, we have

$$\begin{aligned}
x_1 &= |x_0| - y_0 - b = (x_0 - y_0) - b < 0, & y_1 &= x_0 - |y_0| + 1 = (x_0 - y_0) + 1 < 0.\\
x_2 &= |x_1| - y_1 - b = -2(x_0 - y_0) - 1 > 0, & y_2 &= x_1 - |y_1| + 1 = 2(x_0 - y_0) - b + 2 < 0.\\
x_3 &= |x_2| - y_2 - b = -4(x_0 - y_0) - 3 > 0, & y_3 &= x_2 - |y_2| + 1 = -b + 2 < 0.\\
x_4 &= |x_3| - y_3 - b = -4(x_0 - y_0) - 5 > 0, & y_4 &= x_3 - |y_3| + 1 = -4(x_0 - y_0) - b.
\end{aligned}$$

Case 1: If $x_0 - y_0 \in \left(-\infty, -\frac{b}{4}\right]$, then $y_4 \geq 0$. Thus,

$$x_5 = |x_4| - y_4 - b = -5, \qquad y_5 = x_4 - |y_4| + 1 = b - 4.$$

Hence, by Lemma 1.2, the solution of (1.2) eventually becomes the periodic with prime period 5 for $n \geq 5$.

Case 2: If $x_0 - y_0 \in \left(-\frac{b}{4}, -\frac{b+4}{7}\right)$, we define sequences of real numbers α_k and e_k by

$$\alpha_k = \frac{-(b+4)2^{3k-1} - (3b-16)}{7 \cdot 2^{3k-1}} \quad \text{and} \quad e_k = \frac{(b+4)(1-2^{3k})}{7},$$

for $k \in \mathbb{N}$. Note that $\alpha_k < \alpha_{k+1} < -\frac{b+4}{7}$, α_k tends to $-\dfrac{b+4}{7}$ as $k \to \infty$,

$$4e_k - b = \frac{-(b+4)2^{3k+2} - (3b-16)}{7} = 2^{3k+2}\alpha_{k+1} \text{ and}$$

$$8e_k - b - 4 = \frac{(b+4)(1-2^{3k+3})}{7} = e_{k+1}.$$

This implies that $e_{k+1} = 2(4e_k - b) + b - 4 = 2^{3k+3}\alpha_{k+1} + b - 4$. Now, consider the following algorithm.

(1) If $x_0 - y_0 \in \left(\alpha_k, -\frac{b+4}{7}\right)$, then $y_{5k-1} < 0$ and

$$\begin{aligned}
x_{5k} &= -5, & y_{5k} &= -2^{3k}(x_0 - y_0) + e_k < 0.\\
x_{5k+1} &= 2^{3k}(x_0 - y_0) - e_k - b + 5 < 0, & y_{5k+1} &= 2^{3k}(x_0 - y_0) - e_k - 4 < 0.\\
x_{5k+2} &= -2^{3k+1}(x_0 - y_0) + 2e_k - 1 > 0, & y_{5k+2} &= 2^{3k+1}(x_0 - y_0) - 2e_k - b + 2 < 0.\\
x_{5k+3} &= -2^{3k+2}(x_0 - y_0) + 4e_k - 3 > 0, & y_{5k+3} &= -b + 2 < 0.\\
x_{5k+4} &= -2^{3k+2}(x_0 - y_0) + 4e_k - 5 > 0, & y_{5k+4} &= -2^{3k+2}(x_0 - y_0) + 4e_k - b.
\end{aligned}$$

(2) If $x_0 - y_0 \in (\alpha_k, \alpha_{k+1}]$, then $y_{5k+4} \geq 0$ and

$$x_{5k+5} = -5, \qquad y_{5k+5} = b - 4.$$

To prove that this algorithm explains the behavior of the solution for (1.2), let $k = 1$, we have $\alpha_1 = -\frac{b}{4}$, $\alpha_2 = -\frac{5b+16}{32}$ and $e_1 = -(b+4)$. Each item of the algorithm holds as follows.

- If $x_0 - y_0 \in \left(-\frac{b}{4}, -\frac{b+4}{7}\right)$, then $y_4 < 0$ and

$$
\begin{aligned}
x_5 &= |x_4| - y_4 - b = -5,\\
y_5 &= x_4 - |y_4| + 1 = -8(x_0 - y_0) - b - 4 = -2^3(x_0 - y_0) + e_1 > \frac{b+4}{7} > 0.\\
x_6 &= |x_5| - y_5 - b = 8(x_0 - y_0) + 9 = 2^3(x_0 - y_0) - e_1 - b + 5 < -\frac{8b-31}{7} < 0,\\
y_6 &= x_5 - |y_5| + 1 = 8(x_0 - y_0) + b = 2^3(x_0 - y_0) - e_1 - 4 < -\frac{b+32}{7} < 0.\\
x_7 &= |x_6| - y_6 - b = -16(x_0 - y_0) - 2b - 9 = -2^4(x_0 - y_0) + 2e_1 - 1 > \frac{2b+1}{7} > 0,\\
y_7 &= x_6 - |y_6| + 1 = 16(x_0 - y_0) + b + 10 = 2^4(x_0 - y_0) - 2e_1 - b + 2 < -\frac{9b-6}{7} < 0.\\
x_8 &= |x_7| - y_7 - b = -32(x_0 - y_0) - 4b - 19 = -2^5(x_0 - y_0) + 4e_1 - 3 > \frac{4b-5}{7} > 0,\\
y_8 &= x_7 - |y_7| + 1 = -b + 2 < 0.\\
x_9 &= |x_8| - y_8 - b = -32(x_0 - y_0) - 4b - 21 = -2^5(x_0 - y_0) + 4e_1 - 5 > \frac{4b-19}{7} > 0,\\
y_9 &= x_8 - |y_8| + 1 = -32(x_0 - y_0) - 5b - 16 = -2^5(x_0 - y_0) + 4e_1 - b.
\end{aligned}
$$

- If $x_0 - y_0 \in \left(-\frac{b}{4}, -\frac{5b+16}{32}\right]$, then $y_9 \geq 0$. Thus,

$$
x_{10} = |x_9| - y_9 - b = -5, \qquad y_{10} = x_9 - |y_9| + 1 = b - 4.
$$

  Hence, by Lemma 1.2, the solution of (1.2) eventually becomes the periodic with prime period 5 for $n \geq 10$.

Next, let $k \in \mathbb{N}$ such that the algorithm holds. Consider each item of the algorithm as we replace k by $k+1$.

- If $x_0 - y_0 \in \left(\alpha_{k+1}, -\frac{b+4}{7}\right) \subset \left(\alpha_k, -\frac{b+4}{7}\right)$. Then,

$$
\begin{aligned}
x_{5k+4} &= -2^{3k+2}(x_0 - y_0) + 4e_k - 5 > 0,\\
y_{5k+4} &= -2^{3k+2}(x_0 - y_0) + 4e_k - b = -2^{3k+3}((x_0 - y_0) - \alpha_{k+1}) < 0.
\end{aligned}
$$

  Thus,

$$
\begin{aligned}
x_{5k+5} &= |x_{5k+4}| - y_{5k+4} - b = -5,\\
y_{5k+5} &= x_{5k+4} - |y_{5k+4}| + 1 = -2^{3k+3}(x_0 - y_0) + 8e_k - b - 4\\
&= -2^{3k+3}((x_0 - y_0) - \alpha_{k+1}) + b - 4.
\end{aligned}
$$

Note that $2^{3k+3}((x_0-y_0)-\alpha_{k+1}) < -2^{3k+3}\left(\frac{b+4}{7}+\alpha_{k+1}\right) = \frac{6b-32}{7}$. Hence, $y_{5k+5} > \frac{b+4}{7} > 0$ and

$$\begin{aligned}
x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = 2^{3k+3}(x_0-y_0) - e_{k+1} - b + 5\\
&= 2^{3k+3}((x_0-y_0)-\alpha_{k+1}) - 2b + 9 < -\frac{8b-31}{7} < 0,\\
y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = 2^{3k+3}(x_0-y_0) - e_{k+1} - 4\\
&= 2^{3k+3}((x_0-y_0)-\alpha_{k+1}) - b < -\frac{b+32}{7} < 0.\\
x_{5k+7} &= |x_{5k+6}| - y_{5k+6} - b = -2^{3k+4}(x_0-y_0) + 2e_{k+1} - 1\\
&= -2^{3k+4}((x_0-y_0)-\alpha_{k+1}) + 2b - 9 > \frac{2b+1}{7} > 0,\\
y_{5k+7} &= x_{5k+6} - |y_{5k+6}| + 1 = 2^{3k+4}(x_0-y_0) - 2e_{k+1} - b + 2\\
&= 2^{3k+4}((x_0-y_0)-\alpha_{k+1}) - 3b + 10 < -\frac{9b-6}{7} < 0.\\
x_{5k+8} &= |x_{5k+7}| - y_{5k+7} - b = -2^{3k+5}(x_0-y_0) + 4e_{k+1} - 3\\
&= -2^{3k+5}((x_0-y_0)-\alpha_{k+1}) + 4b - 19 > \frac{4b-5}{7} > 0,\\
y_{5k+8} &= x_{5k+7} - |y_{5k+7}| + 1 = -b + 2 < 0.\\
x_{5k+9} &= |x_{5k+8}| - y_{5k+8} - b = -2^{3k+5}(x_0-y_0) + 4e_{k+1} - 5\\
&= -2^{3k+5}((x_0-y_0)-\alpha_{k+1}) + 4b - 21 > \frac{4b-19}{7} > 0,\\
y_{5k+9} &= x_{5k+7} - |y_{5k+7}| + 1 = -2^{3k+5}(x_0-y_0) + 4e_{k+1} - b\\
&= -2^{3k+5}((x_0-y_0)-\alpha_{k+2}).
\end{aligned}$$

- If $x_0-y_0 \in (\alpha_{k+1}, \alpha_{k+2}] \subset \left(\alpha_{k+1}, -\frac{b+4}{7}\right)$, then $(x_0-y_0)-\alpha_{k+2} \leq 0$. Hence, $y_{5k+9} \geq 0$. Thus,

  $$x_{5k+10} = -5, \qquad y_{5k+10} = b-4.$$

  Therefore, by Lemma 1.2, the solution of (1.2) eventually becomes periodic with prime period 5 for $n \geq 5k+10$.

Thus, by the mathematical induction, the algorithm holds for $k \in \mathbb{N}$. Since $\bigcup_{k\in\mathbb{N}}(\alpha_k, \alpha_{k+1}] = \left(-\frac{b}{4}, -\frac{b+4}{7}\right)$, and $(\alpha_i, \alpha_{i+1}] \cap (\alpha_j, \alpha_{j+1}] = \emptyset$, for every $i, j \in \mathbb{N}$ with $i \neq j$, this implies that $\{\alpha_k\}_{k\in\mathbb{N}}$ is a partition of $\left(-\frac{b}{4}, -\frac{b+4}{7}\right)$. Hence, for every $x_0 - y_0 \in \left(-\frac{b}{4}, -\frac{b+4}{7}\right)$, there exists a unique $m \in \mathbb{N}$ such that $x_0 - y_0 \in (\alpha_m, \alpha_{m+1}]$. Thus, the solution of (1.2) eventually becomes periodic with prime period 5 for $n \geq 5m+5$. This completes the proof. ∎

The following Lemmas 1.8 - 1.10 consider the region of initial condition on the third quadrant of $\mathbb{R}^2$.

Lemma 1.8. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be the solution of system* (1.2) *with the initial condition* $(x_0, y_0) \in \mathbb{R}^- \times \mathbb{R}^-$ *such that* $x_0 + y_0 = -\frac{9b+1}{7}$. *Then, the solution eventually becomes the periodic with prime period* 5.

Proof. Note that $b \geq 6$ calculation, we obtain

$$x_1 = |x_0| - y_0 - b = \frac{2b+1}{7} > 0, \qquad y_1 = x_0 - |y_0| + 1 = -\frac{9b-6}{7} < 0.$$

Similar to Lemma 1.3, we have that $x_6 = x_1$ and $y_6 = y_1$. Thus, by the mathematical induction, the proof is completed. ∎

Lemma 1.9. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be the solution of system* (1.2) *with the initial condition* $(x_0, y_0) \in \mathbb{R}^- \times \mathbb{R}^-$ *such that* $x_0 + y_0 \in \left(-\infty, -\frac{9b+1}{7}\right)$. *Then, the solution eventually becomes the periodic with prime period* 5.

Proof. Note that $x_0, y_0 < 0$. Thus,

$$\begin{aligned} x_1 &= |x_0| - y_0 - b = -(x_0 + y_0) - b > 0, & y_1 &= x_0 - |y_0| + 1 = (x_0 + y_0) + 1 < 0. \\ x_2 &= |x_1| - y_1 - b = -2(x_0 + y_0) - 2b - 1 > 0, & y_2 &= x_1 - |y_1| + 1 = -b + 2 < 0. \\ x_3 &= |x_2| - y_2 - b = -2(x_0 + y_0) - 2b - 3 > 0, & y_3 &= x_2 - |y_2| + 1 = -2(x_0 + y_0) - 3b + 2. \end{aligned}$$

Case 1: If $x_0 + y_0 \in \left(-\infty, -\frac{3b-2}{2}\right]$, then $y_3 \geq 0$. Thus,

$$x_4 = |x_3| - y_3 - b = -5, \qquad y_4 = x_3 - |y_3| + 1 = b - 4.$$

Hence, by Lemma 1.2, the solution eventually becomes the periodic with prime period 5 for $n \geq 4$.

Case 2: If $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -\frac{9b+1}{7}\right)$, we define sequences of real numbers α_k and e_k by

$$\alpha_k = \frac{-(9b+1)2^{3k-2} - (3b-16)}{7 \cdot 2^{3k-2}} \quad \text{and} \quad e_k = \frac{-(9b+1)2^{3k-1} + (b+4)}{7}$$

for $k \in \mathbb{N}$. Note that $\alpha_k < \alpha_{k+1} < -\frac{9b+1}{7}$, α_k tends to $-\frac{9b+1}{7}$ as $k \to \infty$,

$$\begin{aligned} 4e_k - b &= \frac{-(9b+1)2^{3k+1} - (3b-16)}{7} = 2^{3k+1}\alpha_{k+1} \text{ and} \\ 8e_k - b - 4 &= \frac{-(9b+1)2^{3k+2} + (b+4)}{7} = e_{k+1}. \end{aligned}$$

This implies that $e_{k+1} = 2(4e_k - b) + b - 4 = 2^{3k+2}\alpha_{k+1} + b - 4$. Now, consider the following algorithm.

(1) If $x_0 + y_0 \in \left(\alpha_k, -\frac{9b+1}{7}\right)$, then $y_{5k-2} < 0$ and

$$\begin{aligned} x_{5k-1} &= -5, & y_{5k-1} &= -2^{3k-1}(x_0 + y_0) + e_k > 0. \\ x_{5k} &= 2^{3k-1}(x_0 + y_0) - e_k - b + 5 < 0, & y_{5k} &= 2^{3k-1}(x_0 + y_0) - e_k - 4 < 0. \\ x_{5k+1} &= -2^{3k}(x_0 + y_0) + 2e_k - 1 > 0, & y_{5k+1} &= 2^{3k}(x_0 + y_0) - 2e_k - b + 2 < 0. \\ x_{5k+2} &= -2^{3k+1}(x_0 + y_0) + 4e_k - 3 > 0, & y_{5k+2} &= -b + 2 < 0. \\ x_{5k+3} &= -2^{3k+1}(x_0 + y_0) + 4e_k - 5 > 0, & y_{5k+3} &= -2^{3k+1}(x_0 + y_0) + 4e_k - b. \end{aligned}$$

(2) If $x_0 + y_0 \in (\alpha_k, \alpha_{k+1}]$, then $y_{5k+3} \geq 0$ and

$$x_{5k+4} = -5, \qquad y_{5k+4} = b - 4.$$

To prove that this algorithm explains the behavior of the solution for (1.2), let $k = 1$ and $k = 2$, we have $\alpha_1 = -\frac{3b-2}{2}$, $\alpha_2 = -\frac{21b}{16}$ and $e_1 = -5b$. Each item of the algorithm holds as follows.

- If $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -\frac{9b+1}{7}\right)$, then $y_3 < 0$. Thus,

$$\begin{aligned}
&x_4 = |x_3| - y_3 - b = -5,\\
&y_4 = x_3 - |y_3| + 1 = -4(x_0 + y_0) - 5b = -2^2(x_0 - y_0) + e_1 > \frac{b+4}{7} > 0.\\
&x_5 = |x_4| - y_4 - b = 4(x_0 + y_0) + 4b + 5 = 2^2(x_0 + y_0) - e_1 - b + 5 < -\frac{8b-31}{7} < 0,\\
&y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) + 5b - 4 = 2^2(x_0 + y_0) - e_1 - 4 < -\frac{b+32}{7} < 0.\\
&x_6 = |x_5| - y_5 - b = -8(x_0 + y_0) - 10b - 1 = -2^3(x_0 + y_0) + 2e_1 - 1 > \frac{2b+1}{7} > 0,\\
&y_6 = x_5 - |y_5| + 1 = 8(x_0 + y_0) + 9b + 2 = 2^3(x_0 + y_0) - 2e_1 - b + 2 < -\frac{9b-6}{7} < 0.\\
&x_7 = |x_6| - y_6 - b = -16(x_0 + y_0) - 20b - 3 = -2^4(x_0 + y_0) + 4e_1 - 3 > \frac{4b-5}{7} > 0,\\
&y_7 = x_6 - |y_6| + 1 = -b + 2 < 0.\\
&x_8 = |x_7| - y_7 - b = -16(x_0 - y_0) - 20b - 5 = -2^4(x_0 + y_0) + 4e_1 - 5 > \frac{4b-19}{7} > 0,\\
&y_8 = x_7 - |y_7| + 1 = -16(x_0 - y_0) - 21b = -2^4(x_0 + y_0) + 4e_1 - b.
\end{aligned}$$

- If $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -\frac{21b}{16}\right]$, then $y_8 \geq 0$. Thus,

$$x_9 = |x_8| - y_8 - b = -5, \qquad y_9 = x_8 - |y_8| + 1 = b - 4.$$

Hence, by Lemma 1.2, the solution of (1.2) eventually becomes the periodic with prime period 5 for $n \geq 9$.

Next, let $k \in \mathbb{N}$ such that the algorithm holds. Consider each item of the algorithm as we replace k by $k+1$.

- If $x_0 + y_0 \in \left(\alpha_{k+1}, -\frac{9b+1}{7}\right) \subset \left(\alpha_k, -\frac{9b+1}{7}\right)$. Then,

$$\begin{aligned}
x_{5k+3} &= -2^{3k+1}(x_0 + y_0) + 4e_k - 5 > 0,\\
y_{5k+3} &= -2^{3k+1}(x_0 + y_0) + 4e_k - b = -2^{3k+1}((x_0 + y_0) - \alpha_{k+1}) < 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
x_{5k+4} &= |x_{5k+3}| - y_{5k+3} - b = -5,\\
y_{5k+4} &= x_{5k+3} - |y_{5k+3}| + 1 = -2^{3k+2}(x_0 + y_0) + 8e_k - b - 4 = -2^{3k+2}(x_0 + y_0) + e_{k+1}\\
&= -2^{3k+2}((x_0 + y_0) - \alpha_{k+1}) + b - 4.
\end{aligned}$$

Note that $2^{3k+2}((x_0+y_0)-\alpha_{k+1}) < 2^{3k+2}\left(-\frac{9b+1}{7}-\alpha_{k+1}\right) = \frac{6b-32}{7}$. Hence, $y_{5k+4} > \frac{b+4}{7} > 0$, and

$$\begin{aligned}
x_{5k+5} &= |x_{5k+4}| - y_{5k+4} - b = 2^{3k+2}(x_0+y_0) - e_{k+1} - b + 5\\
&= 2^{3k+2}((x_0+y_0)-\alpha_{k+1}) - 2b + 9 < -\frac{8b-31}{7} < 0,\\
y_{5k+5} &= x_{5k+4} - |y_{5k+4}| + 1 = 2^{3k+2}(x_0+y_0) - e_{k+1} - 4\\
&= 2^{3k+2}((x_0+y_0)-\alpha_{k+1}) - b < -\frac{b+32}{7} < 0.\\
x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = -2^{3k+3}(x_0+y_0) + 2e_{k+1} - 1\\
&= -2^{3k+3}((x_0+y_0)-\alpha_{k+1}) + 2b - 9 > \frac{2b+1}{7} > 0,\\
y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = 2^{3k+3}(x_0+y_0) - 2e_{k+1} - b + 2\\
&= 2^{3k+4}((x_0-y_0)-\alpha_{k+1}) - 3b + 10 < -\frac{9b-6}{7} < 0.\\
x_{5k+7} &= |x_{5k+6}| - y_{5k+6} - b = -2^{3k+4}(x_0+y_0) + 4e_{k+1} - 3\\
&= -2^{3k+4}((x_0+y_0)-\alpha_{k+1}) + 4b - 19 > \frac{4b-5}{7} > 0,\\
y_{5k+7} &= x_{5k+7} - |y_{5k+7}| + 1 = -b + 2 < 0.\\
x_{5k+8} &= |x_{5k+7}| - y_{5k+7} - b = -2^{3k+4}(x_0+y_0) + 4e_{k+1} - 5\\
&= -2^{3k+4}((x_0-y_0)-\alpha_{k+1}) + 4b - 21 > \frac{4b-19}{7} > 0,\\
y_{5k+8} &= x_{5k+7} - |y_{5k+7}| + 1 = -2^{3k+4}(x_0+y_0) + 4e_{k+1} - b\\
&= -2^{3k+4}((x_0+y_0)-\alpha_{k+2}).
\end{aligned}$$

- If $x_0+y_0 \in (\alpha_{k+1},\alpha_{k+2}] \subset \left(\alpha_{k+1}, -\frac{9b+1}{7}\right)$, then $(x_0+y_0)-\alpha_{k+2} \leq 0$. Hence, $y_{5k+8} \geq 0$. Thus,

  $$x_{5k+9} = |x_{5k+8}| - y_{5k+8} - b = -5, \qquad y_{5k+9} = x_{5k+8} - |y_{5k+8}| + 1 = b-4.$$

  Hence, by Lemma 1.2, the solution eventually becomes periodic with prime period 5 for $n \geq 5k+9$.

Thus, by the mathematical induction, the algorithm holds for $k \in \mathbb{N}$. Since $\bigcup_{k\in\mathbb{N}}(\alpha_k,\alpha_{k+1}] = \left(-\frac{3b-2}{2}, -\frac{9b+1}{7}\right)$, and $(\alpha_i,\alpha_{i+1}] \cap (\alpha_j,\alpha_{j+1}] = \emptyset$, for every $i,j \in \mathbb{N}$ with $i \neq j$, this implies that $\{\alpha_k\}_{k\in\mathbb{N}}$ is a partition of $\left(-\frac{3b-2}{2}, -\frac{9b+1}{7}\right)$. Hence, for every $x_0+y_0 \in \left(-\frac{3b-2}{2}, -\frac{9b+1}{7}\right)$, there exists a unique $m \in \mathbb{N}$ such that $x_0+y_0 \in (\alpha_m,\alpha_{m+1}]$. Thus, the solution of (1.2) eventually becomes periodic with prime period 5 for $n \geq 5m+4$. This complete the proof. ∎

Lemma 1.10. *Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be the solution of system* (1.2) *with the initial condition* $(x_0,y_0) \in \mathbb{R}^- \times \mathbb{R}^-$ *such that* $x_0+y_0 \in \left(-\frac{9b+1}{7}, 0\right)$. *Then, the solution eventually becomes the equilibrium point* $(-1,-b+2)$.

Proof. Note that $x_0, y_0 < 0$. Thus,

$$x_1 = |x_0| - y_0 - b = -(x_0+y_0) - b, \qquad y_1 = x_0 - |y_0| + 1 = (x_0+y_0)+1.$$

Case 1: If $x_0 + y_0 \in [-1, 0)$, then $y_1 \geq 0$ and $x_1 < -(b+1) < 0$. Thus,

$$x_2 = |x_1| - y_1 - b = -1, \qquad y_2 = x_1 - |y_1| + 1 = -2(x_0 + y_0) - b < 0.$$
$$x_3 = |x_2| - y_2 - b = 2(x_0 + y_0) + 1, \qquad y_3 = x_2 - |y_2| + 1 = -2(x_0 + y_0) - b < 0.$$

Case 1.1: If $x_0 + y_0 \in \left[-1, -\frac{1}{2}\right)$, then $x_3 < 0$. Since $y_3 = -x_3 - b + 1$, by Lemma 2, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 4$.
Case 1.2: If $x_0 + y_0 \in \left[-\frac{1}{2}, 0\right)$, then $x_3 \geq 0$. Thus,

$$x_4 = |x_3| - y_3 - b = 4(x_0 + y_0) + 1, \qquad y_4 = x_3 - |y_3| + 1 = -b + 2 < 0.$$

Case 1.2.1: If $x_0 + y_0 \in \left[-\frac{1}{2}, -\frac{1}{4}\right)$, then $x_4 < 0$. Thus,

$$x_5 = |x_4| - y_4 - b = -4(x_0 + y_0) - 3 < 0,$$
$$y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) - b + 4 < 0.$$

Since $y_5 = -x_5 - b + 1$, by Lemma 2, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 6$.
Case 1.2.2: If $x_0 + y_0 \in \left[-\frac{1}{4}, 0\right)$, then $x_4 \geq 0$. Thus,

$$x_5 = |x_4| - y_4 - b = 4(x_0 + y_0) - 1 < 0,$$
$$y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) - b + 4 < 0.$$
$$x_6 = |x_5| - y_5 - b = -8(x_0 + y_0) - 3 < 0,$$
$$y_6 = x_5 - |y_5| + 1 = 8(x_0 + y_0) - b + 4 < 0.$$

Since $y_6 = -x_6 - b + 1$, by Lemma 2, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 7$.

Case 2: If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, -1\right)$, then $y_1 < 0$.
Case 2.1: If $x_0 + y_0 \in (-b, -1)$, then $x_1 < 0$. Since $y_1 = -x_1 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 2$.
Case 2.2: If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, -b\right]$, we define sequences of real numbers $\alpha_k, \beta_k, \gamma_k, \omega_k$ and e_k by

$$\alpha_k = \frac{-(9b+1)2^{3k-3} + (2b+1)}{7 \cdot 2^{3k-3}}, \qquad \beta_k = \frac{-(9b+1)2^{3k-2} + (4b-5)}{7 \cdot 2^{3k-2}},$$
$$\gamma_k = \frac{-(9b+1)2^{3k-2} + (4b-19)}{7 \cdot 2^{3k-2}}, \qquad \omega_k = \frac{-(9b+1)2^{3k-1} + (b+4)}{7 \cdot 2^{3k-1}}$$

and

$$e_k = \frac{-(9b+1)2^{3k-2} + (4b-5)}{7}$$

for $k \in \mathbb{N}$. Note that $-\frac{9b+1}{7} < \alpha_{k+1} < \omega_k < \gamma_k < \beta_k < \alpha_k$, $\alpha_k, \beta_k, \gamma_k$ and ω_k tend to $-\frac{9b+1}{7}$ as $k \to \infty$ and

$$\begin{aligned}
e_k + 1 &= \frac{-(9b+1)2^{3k-2} + (4b+2)}{7} = 2^{3k-2}\alpha_k,\\
8e_k - 4b + 5 &= \frac{-(9b+1)2^{3k+1} + (4b-5)}{7} = e_{k+1},\\
e_k &= \frac{-(9b+1)2^{3k-2} + (4b-5)}{7} = 2^{3k-2}\beta_k,\\
e_k - 2 &= \frac{-(9b+1)2^{3k-2} + (4b-19)}{7} = 2^{3k-2}\gamma_k,\\
2e_k - b + 2 &= \frac{-(9b+1)2^{3k-1} + (b+4)}{7} = 2^{3k-1}\omega_k,\\
4e_k - 2b + 3 &= \frac{-(9b+1)2^{3k} + (2b+1)}{7} = 2^{3k}\alpha_{k+1}.
\end{aligned}$$

Consider the following algorithm.

(1) If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, \alpha_k\right]$, then $x_{5k-4} \geq 0$ and

$$x_{5k-3} = -\,2^{3k-2}(x_0 + y_0) + e_k, \qquad y_{5k-3} = -b + 2 < 0.$$

(2) If $x_0 + y_0 \in (\beta_k, \alpha_k]$, then $x_{5k-3} < 0$,

$$x_{5k-2} = 2^{3k-2}(x_0 + y_0) - e_k - 2 < 0, \qquad y_{5k-2} = -2^{3k-2}(x_0 + y_0) + e_k - b + 3 < 0$$

and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k-1$.

(3) If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, \beta_k\right]$, then $x_{5k-3} \geq 0$ and

$$x_{5k-2} = -2^{3k-2}(x_0 + y_0) + e_k - 2, \qquad y_{5k-2} = -2^{3k-2}(x_0 + y_0) + e_k - b + 3 < 0.$$

(4) If $x_0 + y_0 \in (\gamma_k, \beta_k]$, then $x_{5k-2} < 0$,

$$x_{5k-1} = 2^{3k-1}(x_0 + y_0) - 2e_k - 1 < 0, \qquad y_{5k-1} = -2^{3k-1}(x_0 + y_0) + 2e_k - b + 2 < 0$$

and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k$.

(5) If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, \gamma_k\right]$, then $x_{5k-2} \geq 0$ and

$$x_{5k-1} = -5, \qquad y_{5k-1} = -2^{3k-1}(x_0 + y_0) + 2e_k - b + 2.$$

(6) If $x_0 + y_0 \in (\omega_k, \gamma_k]$, then $y_{5k-1} < 0$,

$$x_{5k} = 2^{3k-1}(x_0 + y_0) - 2e_k + 3 < 0, \qquad y_{5k} = -2^{3k-1}(x_0 + y_0) + 2e_k - b - 2 < 0$$

and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+1$.

(7) If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, \omega_k\right]$, then $y_{5k-1} \geq 0$ and

$$x_{5k} = 2^{3k-1}(x_0 + y_0) - 2e_k + 3 < 0, \qquad y_{5k} = 2^{3k-1}(x_0 + y_0) - 2e_k + b - 6 < 0.$$

$$x_{5k+1} = -2^{3k}(x_0 + y_0) + 4e_k - 2b + 3, \qquad y_{5k+1} = 2^{3k}(x_0 + y_0) - 4e_k + b - 2 < 0.$$

(8) If $x_0 + y_0 \in (\alpha_{k+1}, \omega_k]$, then $x_{5k+1} < 0$ and $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+2$.

To prove that this algorithm explains the behavior of the solution for (1.2), let $k = 1$, we have

$$\alpha_1 = -b, \qquad \beta_1 = -\frac{2b+1}{2}, \qquad \gamma_1 = -\frac{2b+3}{2},$$
$$\omega_1 = -\frac{5b}{4}, \qquad \alpha_2 = -\frac{10b+1}{8}, \qquad \text{and} \quad e_1 = -(2b+1).$$

Each item of the algorithm holds as follows.

- If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, -b\right]$, then $x_1 \geq 0$ and

$$x_2 = |x_1| - y_1 - b = -2(x_0 + y_0) - 2b - 1 = -2(x_0 + y_0) + e_1,$$
$$y_2 = x_1 - |y_1| + 1 = -b + 2 < 0.$$

- If $x_0 + y_0 \in \left(-\frac{2b+1}{2}, -b\right]$, then $x_2 < 0$. Thus,

$$x_3 = |x_2| - y_2 - b = 2(x_0 + y_0) + 2b - 1 = 2(x_0 + y_0) - e_1 - 2 \leq -1 < 0,$$
$$y_3 = x_2 - |y_2| + 1 = -2(x_0 + y_0) - 3b + 2 = -2(x_0 + y_0) + e_1 - b + 3 < -b + 3 < 0.$$

Since $y_3 = -x_3 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 4$.

- If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, -\frac{2b+1}{2}\right]$, then $x_2 \geq 0$. Thus,

$$x_3 = |x_2| - y_2 - b = -2(x_0 + y_0) - 2b - 3 = -2(x_0 + y_0) + e_1 - 2,$$
$$y_3 = x_2 - |y_2| + 1 = -2(x_0 + y_0) - 3b + 2 = -2(x_0 + y_0) + e_1 - b + 3 \leq -\frac{3b - 16}{7} < 0.$$

- If $x_0 + y_0 \in \left(-\frac{2b+3}{2}, -\frac{2b+1}{2}\right]$, then $x_3 < 0$. Thus,

$$x_4 = |x_3| - y_3 - b = 4(x_0 + y_0) + 4b + 1 = 2^2(x_0 + y_0) - 2e_1 - 1 \leq -1 < 0$$
$$y_4 = x_3 - |y_3| + 1 = -4(x_0 + y_0) - 5b = -2^2(x_0 + y_0) + 2e_1 - b + 2 < -b + 6 \leq 0.$$

Since $y_4 = -x_4 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 5$.

- If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, -\frac{2b+3}{2}\right]$, then $x_3 \geq 0$. Thus,

$$x_4 = |x_3| - y_3 - b = -5,$$
$$y_4 = x_3 - |y_3| + 1 = -4(x_0 + y_0) - 5b = -2^2(x_0 + y_0) + 2e_1 - b + 2.$$

- If $x_0 + y_0 \in \left(-\frac{5b}{4}, -\frac{2b+3}{2}\right]$, then $y_4 < 0$. Thus,

$$x_5 = |x_4| - y_4 - b = 4(x_0 + y_0) + 4b + 5 = 2^2(x_0 + y_0) - 2e_1 + 3 \leq -1 < 0,$$
$$y_5 = x_4 - |y_4| + 1 = -4(x_0 + y_0) - 5b - 4 = -2^2(x_0 + y_0) + 2e_1 - b - 2 < -4 < 0.$$

Since $y_5 = -x_5 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 6$.

- If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, -\frac{5b}{4}\right]$, then $y_4 \geq 0$. Thus,

$$x_5 = |x_4| - y_4 - b = 4(x_0 + y_0) + 4b + 5 = 2^2(x_0 + y_0) - 2e_1 + 3 \leq -b + 5 < 0,$$
$$y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) + 5b - 4 = 2^2(x_0 + y_0) - 2e_1 + b - 6 \leq -4 < 0.$$
$$x_6 = |x_5| - y_5 - b = -8(x_0 + y_0) - 10b - 1 = -2^3(x_0 + y_0) + 4e_1 - 2b + 3,$$
$$y_6 = x_5 - |y_5| + 1 = 8(x_0 + y_0) + 9b + 2 = 2^3(x_0 + y_0) - 4e_1 + b - 2 \leq -b + 2 < 0.$$

- If $x_0 + y_0 \in \left(-\frac{10b+1}{8}, -\frac{5b}{4}\right]$, then $x_6 < 0$. Since $y_6 = -x_6 - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b + 2)$ for $n \geq 7$.

Next, let $k \in \mathbb{N}$ such that the algorithm holds. Consider each item of the algorithm as we replace k by $k + 1$.

- If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, \alpha_{k+1}\right] \subset \left(-\frac{9b+1}{7}, \alpha_k\right]$, then

$$x_{5k+1} = -2^{3k}(x_0 + y_0) + 4e_k - 2b + 3 = -2^{3k}((x_0 + y_0) - \alpha_{k+1}) \geq 0,$$
$$y_{5k+1} = 2^{3k}(x_0 + y_0) - 4e_k + b - 2 < 0.$$

Thus,

$$\begin{aligned}
x_{5k+2} &= |x_{5k+1}| - y_{5k+1} - b = -2^{3k+1}(x_0+y_0) + 8e_k - 4b + 5\\
&= -2^{3k+1}(x_0-y_0) + e_{k+1},\\
&= -2^{3k+1}((x_0-y_0) - \beta_{k+1}).\\
y_{5k+2} &= x_{5k+1} - |y_{5k+1}| + 1 = -b + 2 < 0.
\end{aligned}$$

- If $x_0+y_0 \in (\beta_{k+1}, \alpha_{k+1}]$, then $x_{5k+2} < 0$. Thus,

$$\begin{aligned}
x_{5k+3} &= |x_{5k+2}| - y_{5k+2} - b = 2^{3k+1}(x_0+y_0) - e_{k+1} - 2\\
&= 2^{3k+1}((x_0+y_0) - \alpha_{k+1}) - 1 \le -1 < 0,\\
y_{5k+3} &= x_{5k+2} - |y_{5k+2}| + 1 = -2^{3k+1}(x_0+y_0) + e_{k+1} - b + 3\\
&= -2^{3k+1}((x_0+y_0) - \alpha_{k+1}) - b + 2 \le -b + 2 < 0.
\end{aligned}$$

Since $y_{5k+3} = -x_{5k+3} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \ge 5k+4$.

- If $x_0+y_0 \in \left(-\frac{9b+1}{7}, \beta_{k+1}\right]$, then $x_{5k+2} \ge 0$. Thus,

$$\begin{aligned}
x_{5k+3} &= |x_{5k+2}| - y_{5k+2} - b = -2^{3k+1}(x_0+y_0) + e_{k+1} - 2\\
&= -2^{3k+1}((x_0+y_0) - \gamma_{k+1}),\\
y_{5k+3} &= x_{5k+2} - |y_{5k+2}| + 1 = -2^{3k+1}(x_0+y_0) + e_{k+1} - b + 3 < -\frac{3b-16}{7} < 0.
\end{aligned}$$

- If $x_0+y_0 \in (\gamma_{k+1}, \beta_{k+1}]$, then $x_{5k+3} < 0$. Thus,

$$\begin{aligned}
x_{5k+4} &= |x_{5k+5}| - y_{5k+5} - b = 2^{3k+2}(x_0+y_0) - 2e_{k+1} - 1\\
&= 2^{3k+2}((x_0+y_0) - \beta_{k+1}) - 1 \le -1 < 0,\\
y_{5k+4} &= x_{5k+5} - |y_{5k+5}| + 1 = -2^{3k+2}(x_0+y_0) + 2e_{k+1} - b + 2\\
&= -2^{3k+2}((x_0+y_0) - \gamma_{k+1}) - b + 6 < -b + 6 < 0.
\end{aligned}$$

Since $y_{5k+4} = -x_{5k+4} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \ge 5k+5$.

- If $x_0+y_0 \in \left(-\frac{9b+1}{7}, \gamma_{k+1}\right]$, then $x_{5k+3} \ge 0$. Thus,

$$\begin{aligned}
x_{5k+4} &= |x_{5k+6}| - y_{5k+6} - b = -5,\\
y_{5k+4} &= x_{5k+3} - |y_{5k+3}| + 1 = -2^{3k+2}(x_0+y_0) + 2e_{k+1} - b + 2\\
&= -2^{3k+2}((x_0+y_0) - \omega_{k+1}).
\end{aligned}$$

- If $x_0+y_0 \in (\omega_{k+1}, \gamma_{k+1}]$, then $y_{5k+4} < 0$. Thus,

$$\begin{aligned}
x_{5k+5} &= |x_{5k+4}| - y_{5k+4} - b = 2^{3k+2}(x_0+y_0) - 2e_{k+1} + 3\\
&= 2^{3k+2}((x_0+y_0) - \gamma_{k+1}) - 1 \le -1 < 0,\\
y_{5k+5} &= x_{5k+4} - |y_{5k+4}| + 1 = -2^{3k+2}(x_0+y_0) + 2e_{k+1} - b - 2\\
&= -2^{3k+2}((x_0+y_0) - \omega_{k+1}) - 4 < -4 < 0.
\end{aligned}$$

Since $y_{5k+5} = -x_{5k+5} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \ge 5k+6$.

- If $x_0 + y_0 \in \left(-\frac{9b+1}{7}, \omega_{k+1}\right]$, then $y_{5k+4} \geq 0$. Thus,

$$\begin{aligned} x_{5k+5} &= |x_{5k+4}| - y_{5k+4} - b = 2^{3k+2}(x_0+y_0) - 2e_{k+1} + 3 \\ &= 2^{3k+2}((x_0+y_0) - \omega_{k+1}) - b + 5 \leq -b + 5 < 0, \\ y_{5k+5} &= x_{5k+4} - |y_{5k+4}| + 1 = 2^{3k+2}(x_0+y_0) - 2e_{k+1} + b - 6 \\ &= 2^{3k+2}((x_0+y_0) - \omega_{k+1}) - 4 \leq -4 < 0. \\ x_{5k+6} &= |x_{5k+5}| - y_{5k+5} - b = -2^{3k+3}(x_0+y_0) + 4e_{k+1} - 2b + 3 \\ &= -2^{3k+3}((x_0+y_0) - \alpha_{k+2}), \\ y_{5k+6} &= x_{5k+5} - |y_{5k+5}| + 1 = 2^{3k+3}(x_0+y_0) - 4e_{k+1} + b - 2 \\ &= 2^{3k+3}((x_0+y_0) - \omega_{k+1}) - b = 2 \leq -b + 2 < 0. \end{aligned}$$

- If $x_0 + y_0 \in (\alpha_{k+2}, \omega_{k+1}]$, then $x_{5k+6} \geq 0$. Since $y_{5k+6} = -x_{5k+6} - b + 1$, by Lemma 1.1, $(x_n, y_n) = (-1, -b+2)$ for $n \geq 5k+7$.

Hence, by the mathematical induction, this algorithm holds for $k \in \mathbb{N}$. This implies that the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$ for every $x_0 + y_0 \in (\alpha_{k+1}, \alpha_k]$. This is because $(\beta_k, \alpha_k] \cup (\gamma_k, \beta_k] \cup (\omega_k, \gamma_k] \cup (\alpha_{k+1}, \omega_k] = (\alpha_{k+1}, \alpha_k]$. Since, $\bigcup_{k\in\mathbb{N}} (\alpha_{k+1}, \alpha_k] = \left(-\frac{9b+1}{7}, -b\right]$, and $(\alpha_{i+1}, \alpha_i] \cap (\alpha_{j+1}, \alpha_j] = \emptyset$, for every $i, j \in \mathbb{N}$ with $i \neq j$, this implies that $\{\alpha_k\}_{k\in\mathbb{N}}$ is a partition of $\left(-\frac{9b+1}{7}, -b\right]$. Hence, for every $x_0 + y_0 \in \left(-\frac{9b+1}{7}, -b\right]$, there exists a unique $m \in \mathbb{N}$ such that $x_0 + y_0 \in (\alpha_m, \alpha_{m+1}]$. Thus, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$. ∎

The following Lemma 1.11 is for the region of initial condition on the fourth quadrant of $\mathbb{R}^2$.

Lemma 1.11. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be the solution of system* (1.2) *with the initial condition* (x_0, y_0) *is in* $\mathbb{R}_0^+ \times \mathbb{R}^-$. *Then, the solution eventually becomes the equilibrium point* $(-1, -b+2)$ *or eventually becomes the periodic with prime period* 5.

Proof. Note that $x_0 - y_0 > 0$. Thus,

$$x_1 = |x_0| - y_0 - b = (x_0 - y_0) - b, \qquad y_1 = x_0 - |y_0| + 1 = (x_0 + y_0) + 1.$$

Case 1: $x_0 - y_0 \in [b, \infty)$ and $x_0 + y_0 \in [-1, \infty)$. Then, $x_1 \geq 0$ and $y_1 \geq 0$. That is $(x_1, y_1) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $x_1 - y_1 = -2y_0 - b - 1 \in (-(b+1), \infty)$.

- if $y_0 \in \left(-\frac{4b+2}{7}, -\frac{6b+3}{14}\right)$, then $-2y_0 - b - 1 \in \left(-\frac{b+4}{7}, \frac{b-3}{7}\right)$. Thus, by Lemma 1.5, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- if $y_0 \in \left(-\infty, -\frac{4b+2}{7}\right] \cup \left[-\frac{6b+3}{14}, 0\right]$, then $-2y_0 - b - 1 \in \left(-(b+1), -\frac{b+4}{7}\right] \cup \left[\frac{b-3}{7}, \infty\right)$. Thus, by Lemmas 1.3, 1.4, 1.6 and 1.7, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 2: $x_0 - y_0 \in [b, \infty)$, $x_0 + y_0 \in (-\infty, -1)$, $x_0 \in \left[\frac{b-2}{2}, \infty\right)$ and $y_0 \in \left(-\infty, -\frac{2b+1}{2}\right]$. Then,

$$x_2 = |x_1| - y_1 - b = -2y_0 - 2b - 1, \qquad y_2 = x_1 - |y_1| + 1 = 2x_0 - b + 2.$$

That is $(x_2, y_2) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $x_2 - y_2 = 2(x_0 + y_0) - b - 3 \in (-(b+1), \infty)$.

- if $x_0 + y_0 \in \left(-\frac{4b+9}{7}, -\frac{6b+17}{14}\right)$, then $-2(x_0+y_0) - b - 3 \in \left(-\frac{b+4}{7}, \frac{b-3}{7}\right)$. Thus, by Lemma 1.5, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- if $x_0 + y_0 \in \left(-\infty, -\frac{4b+9}{7}\right] \cup \left[-\frac{6b+17}{14}, -1\right)$, then $-2(x_0+y_0) - b - 3 \in \left(-(b+1), -\frac{b+4}{7}\right] \cup \left[\frac{b-3}{7}, \infty\right)$. Thus, by Lemmas 1.3, 1.4, 1.6 and 1.7, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 3: $x_0 - y_0 \in \left[\frac{3b-2}{2}, \infty\right)$, $x_0 + y_0 \in \left(-\infty, -\frac{2b+3}{2}\right]$, $x_0 \in \left[0, \frac{b-2}{2}\right)$ and $y_0 \in \left(-\infty, -\frac{2b+1}{2}\right]$. Then, we have the same (x_2, y_2) as shown in Case 2 and

$$x_3 = |x_2| - y_2 - b = -2(x_0 + y_0) - 2b - 3, \quad y_3 = x_2 - |y_2| + 1 = 2(x_0 - y_0) - 3b + 2.$$

That is $(x_3, y_3) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $x_3 - y_3 = -4x_0 + b - 5 \in (-(b+1), b-5]$.

- if $x_0 \in \left(\frac{3b-16}{14}, \frac{8b-31}{28}\right)$, then $-4x_0 + b - 5 \in \left(-\frac{b+4}{7}, \frac{b-3}{7}\right)$. Thus, by Lemma 1.5, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- if $x_0 \in \left[0, \frac{3b-16}{14}\right] \cup \left[\frac{8b-31}{28}, \frac{b-2}{2}\right)$, then $-4x_0 + b - 5 \in \left(-(b+1), -\frac{b+4}{7}\right] \cup \left[\frac{b-3}{7}, b-5\right]$. Thus, by Lemmas 1.3, 1.4, 1.6 and 1.7, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 4: $x_0 - y_0 \in \left[\frac{2b+3}{2}, \frac{3b-2}{2}\right)$, $x_0 + y_0 \in \left(-\infty, -\frac{2b+3}{2}\right]$, $x_0 \in \left[0, \frac{b-2}{2}\right)$ and $y_0 \in \left(-\frac{5b}{4}, -\frac{2b+3}{2}\right]$. Then, we have the same (x_2, y_2) and (x_3, y_3) as shown in Case 3 and

$$x_4 = |x_3| - y_3 - b = -4x_0 - 5 < 0, \qquad y_4 = x_3 - |y_3| + 1 = -4y_0 - 5b.$$

That is $(x_4, y_4) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_4 - y_4 = -4(x_0 + y_0) - 5b - 5 \in (-(b+1), 0) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 5: $x_0 - y_0 \in \left[\frac{2b+3}{2}, \frac{3b-2}{2}\right)$, $x_0 + y_0 \in \left(-\infty, -\frac{2b+3}{2}\right]$, $x_0 \in \left[0, \frac{b-2}{2}\right)$ and $y_0 \in \left(-\frac{3b-2}{2}, -\frac{5b}{4}\right]$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 4 with $y_4 \geq 0$ and

$$x_5 = |x_4| - y_4 - b = 4(x_0 + y_0) + 4b + 5 < 0, \quad y_5 = x_4 - |y_4| + 1 = -4(x_0 - y_0) + 5b - 4.$$

Since $x_0 - y_0 \geq -y_0 \geq \frac{5b}{4}$, we have $4(x_0 - y_0) - 5b + 4 \geq 4 > 0$ which implies that $y_5 < 0$. Hence, $(x_5, y_5) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_5 - y_5 = 8y_0 + 9b + 1 \in (-3b + 9, -b + 1]$.

- if $y_0 \in \left(-\frac{9b+1}{7}, -\frac{5b}{4}\right]$, then $8y_0 + 9b + 1 \in \left(-\frac{9b+1}{7}, -b+1\right]$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- if $y_0 \in \left(-\frac{3b-2}{2}, -\frac{9b+1}{7}\right]$, then $8y_0 + 9b + 1 \in \left(-3b+9, -\frac{9b+1}{7}\right]$. Thus, by Lemmas 1.8 and 1.9, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 6: $x_0 - y_0 \in \left(-\infty, \frac{3b-2}{2}\right)$, $x_0 + y_0 \in \left(-\frac{2b+3}{2}, -\frac{b+3}{2}\right)$, $x_0 \in \left[0, \frac{b-2}{2}\right)$ and $y_0 \in \left(-\infty, -\frac{2b+1}{2}\right]$. Then, we have the same (x_2, y_2) and (x_3, y_3) as shown in Case 3 with $y_3 < 0$. That is $(x_3, y_3) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_3 + y_3 = -4y_0 - 5b - 1 \in [-b+1, 0) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of 1.2 eventually becomes the equilibrium point $(-1, -b+2)$.

Case 7: $x_0 - y_0 \in \left[\frac{3b-2}{2}, \frac{4b-1}{2}\right)$, $x_0 + y_0 \in \left(-\frac{2b+3}{2}, -\frac{b+3}{2}\right)$, $x_0 \in \left(\frac{b-4}{4}, \frac{b-2}{2}\right]$ and $y_0 \in \left(-\infty, -\frac{2b+1}{2}\right]$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 4 with $y_4 < 0$. That is $(x_4, y_4) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_4 + y_4 = -4(x_0 - y_0) + 5b - 3 \in (-(3b+1), -b+1]$.

- if $x_0 - y_0 \in \left[\frac{3b-2}{2}, \frac{11b-5}{7}\right)$, then $-4(x_0 - y_0) + 5b - 3 \in \left(-\frac{9b+1}{7}, -b+1\right]$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- if $x_0 - y_0 \in \left[\frac{11b-5}{7}, \frac{4b-1}{2}\right)$, then $-4(x_0 - y_0) + 5b - 3 \in \left(-(3b+1), -\frac{9b+1}{7}\right]$. Thus, by Lemmas 1.8 and 1.9, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 8: $x_0 - y_0 \in \left[\frac{3b-2}{2}, \frac{6b-3}{4}\right)$, $x_0 + y_0 \in \left(-\frac{2b+3}{2}, -\frac{b+3}{2}\right)$, $x_0 \in \left(\frac{b-5}{4}, \frac{b-4}{4}\right]$ and $y_0 \in \left(-\infty, -\frac{2b+1}{2}\right]$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 4 with $y_4 \geq 0$ and

$$x_5 = |x_4| - y_4 - b = 4(x_0 - y_0) - 6b + 3, \quad y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) + 3b + 6.$$

Since $x_0+y_0=2x_0-(x_0-y_0)<2\left(\frac{b-4}{4}\right)-\left(\frac{3b-2}{2}\right)=-b-1$, we obtain $y_5<-b+2<0$. Thus, $(x_5,y_5)\in\mathbb{R}^-\times\mathbb{R}^-$. Notice that

$$x_0-y_0=2x_0-(x_0+y_0)<2\left(\frac{b-4}{4}\right)+\frac{2b+3}{2}=\frac{3b-1}{2} \text{ and}$$

$$y_0=(x_0+y_0)-x_0>-\left(\frac{2b+3}{2}\right)-\left(\frac{b-4}{4}\right)=-\frac{5b+2}{4}.$$

Then, we have $x_5+y_5=8x_0-3b+9\in(-(b+1),-b+1)\subseteq\left(-\frac{9b+1}{7},0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1,-b+2)$.

Case 9: $x_0-y_0\in\left[\frac{6b-3}{4},\frac{3b-1}{2}\right)$, $x_0+y_0\in\left(-\frac{2b+3}{2},-\frac{b+3}{2}\right)$, $x_0\in\left(\frac{b-5}{4},\frac{b-4}{4}\right]$ and $y_0\in\left(-\frac{10b+3}{8},-\frac{5b+1}{4}\right]$. Then, we have the same $(x_2,y_2)-(x_5,y_5)$ as shown in Case 8 with $x_5\geq 0$ and

$$x_6=|x_7|-y_7-b=8y_0-10b-3, \qquad y_6=x_7-|y_7|+1=8x_0-3b+10<0.$$

Then, $x_6<0$ and $y_0=x_0-(x_0-y_0)\leq\left(\frac{b-4}{4}\right)-\left(\frac{6b-3}{4}\right)=-\frac{5b+1}{4}$. Thus, $(x_6,y_6)\in\mathbb{R}^-\times\mathbb{R}^-$ and $x_6+y_6=8(x_0-y_0)-13b+7\in(-b+1,-b+3)\subseteq\left(-\frac{9b+1}{7},0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1,-b+2)$.

Case 10: $x_0-y_0\in\left[\frac{6b-3}{4},\frac{3b-1}{2}\right)$, $x_0+y_0\in\left(-\frac{2b+3}{2},-\frac{b+3}{2}\right)$, $x_0\in\left(\frac{b-5}{4},\frac{b-4}{4}\right]$ and $y_0\in\left(-\frac{5b+2}{4},-\frac{10b+3}{8}\right]$. Then, we have the same $(x_2,y_2)-(x_6,y_6)$ as shown in Case 9 with $x_6\geq 0$ and

$$x_7=|x_6|-y_6-b=-8(x_0+y_0)-8b+13, \quad y_7=x_6-|y_6|+1=8(x_0-y_0)-13b+8<0.$$

Since $x_0+y_0>-\frac{2b+3}{2}$, we have $x_7<-1<0$ which implies that $(x_7,y_7)\in\mathbb{R}^-\times\mathbb{R}^-$ and $x_7+y_7=-16y_0-21b-5\in[-b+1,-b+3)\subseteq\left(-\frac{9b+1}{7},0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1,-b+2)$.

Case 11: $x_0-y_0\in[b,\infty)$, $x_0+y_0\in(-\infty,-1)$, $x_0\in\left[0,\frac{b-2}{2}\right)$ and $y_0\in\left(-\frac{2b+1}{2},-\frac{b+1}{2}\right)$. Then, we have the same (x_2,y_2) as shown in Case 2 with $y_2<0$. Thus, $(x_2,y_2)\in\mathbb{R}^-\times\mathbb{R}^-$ and $x_2+y_2=2(x_0-y_0)-3b+1\in[-b+1,0)\subseteq\left(-\frac{9b+1}{7},0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1,-b+2)$.

Case 12: $x_0-y_0\in[b,\infty)$, $x_0+y_0\in\left(-\frac{b+2}{2},-1\right)$, $x_0\in\left[\frac{b-2}{2},\frac{2b-1}{2}\right)$ and $y_0\in\left(-\frac{2b+1}{2},-\frac{b+1}{2}\right)$. Then, we have the same (x_2,y_2) as shown in Case 2 with $y_2\geq 0$ and

$$x_3=|x_2|-y_2-b=-2(x_0-y_0)+2b-1, \quad y_3=x_2-|y_2|+1=-2(x_0+y_0)-b-2.$$

We have $y_3<0$. Since $x_0-y_0\geq b$, we have $x_3=-2(x_0-y_0)+2b-1\leq -1<0$ and $x_0+y_0>\left(\frac{b-2}{2}\right)-\left(\frac{2b+1}{2}\right)=-\frac{b+3}{2}$. Thus, $(x_3,y_3)\in\mathbb{R}^-\times\mathbb{R}^-$ and $x_3+y_3=-4x_0+b-3\in(-(3b+1),-b+1)$.

- if $x_0\in\left[\frac{b-2}{2},\frac{4b-5}{7}\right)$, then $-4x_0+b-3\in\left(-\frac{9b+1}{7},-b+1\right]$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1,-b+2)$.
- if $x_0\in\left[\frac{4b-5}{7},\frac{2b-1}{2}\right)$, then $-4x_0+b-3\in\left(-(3b+1),-\frac{9b+1}{7}\right)$. Thus, by Lemmas 1.8 and 1.9, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 13: $x_0-y_0\in[b,\infty)$, $x_0+y_0\in\left(-\frac{b+3}{2},-\frac{b+2}{2}\right]$, $x_0\in\left[\frac{b-2}{2},\frac{2b-3}{4}\right)$ and $y_0\in\left(-\frac{2b+1}{2},-\frac{b+1}{2}\right)$. Then, we have the same (x_2,y_2) and (x_3,y_3) as shown in Case 12 with $y_3\geq 0$ and

$$x_4=|x_3|-y_3-b=4x_0-2b+3, \qquad y_4=x_3-|y_3|+1=4y_0+3b+2.$$

Since $y_0=(x_0+y_0)-x_0<-\frac{b-2}{2}-\frac{b-2}{2}=-b$, we obtain $y_4<-b+2<0$. Since

$$x_0=(x_0+y_0)-y_0<-\frac{b+2}{2}+\frac{2b+1}{2}=\frac{b-1}{2} \text{ and}$$

$$x_0 - y_0 = (x_0 + y_0) - 2y_0 < -\frac{b+2}{2} + 2\left(\frac{2b+1}{2}\right) = \frac{3b}{2},$$

we have $x_4 < 0$. Thus, $(x_4, y_4) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_4 + y_4 = 4(x_0 + y_0) + b + 5 \in (-(b+1), -b+1] \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 14: $x_0 - y_0 \in \left(\frac{3b-1}{2}, \frac{6b-1}{4}\right)$, $x_0 + y_0 \in \left(-\frac{b+3}{2}, -\frac{b+2}{2}\right]$, $x_0 \in \left[\frac{2b-3}{4}, \frac{b-1}{2}\right)$ and $y_0 \in \left(-\frac{2b+1}{2}, -\frac{b+1}{2}\right)$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 13 with $x_4 \geq 0$ and

$$x_5 = |x_4| - y_4 - b = 4(x_0 - y_0) - 6b + 1, \quad y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) + b + 6 < 0$$

and $x_0 - y_0 = 2x_0 - (x_0 + y_0) > 2\left(\frac{2b-3}{4}\right) + \frac{b+2}{2} = \frac{3b-1}{2}$. Thus, $x_5 < 0$. Therefore, $(x_5, y_5) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_5 + y_5 = 8x_0 - 5b + 7 \in [-b+1, -b+3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 15: $x_0 - y_0 \in \left[\frac{6b-1}{4}, \frac{3b}{2}\right)$, $x_0 + y_0 \in \left(-\frac{b+3}{2}, -\frac{b+2}{2}\right]$, $x_0 \in \left[\frac{2b-3}{4}, \frac{b-1}{2}\right)$ and $y_0 \in \left(-\frac{2b+1}{2}, -\frac{b+1}{2}\right)$. Then, we have the same $(x_2, y_2) - (x_5, y_5)$ as shown in Case 14 with $x_5 \geq 0$ and

$$x_6 = |x_5| - y_5 - b = -8y_0 - 8b - 5, \qquad y_6 = x_5 - |y_5| + 1 = 8x_0 - 5b + 8 < 0.$$

Since, $y_0 > -\frac{2b+1}{2}$, we have $x_6 < -1$ which implies that $(x_6, y_6) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_6 + y_6 = 8(x_0 - y_0) - 13b + 3 \in [-b+1, -b+3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 16: $x_0 - y_0 \in (0, b)$ and $x_0 + y_0 \in (-b, 1)$. Then, $x_1 < 0$ and $-b < -(x_0 - y_0) \leq x_0 + y_0 < x_0 < x_0 - y_0 < b$. Thus, $y_1 < 0$. Therefore, $(x_1, y_1) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_1 + y_1 = 2x_0 - b + 1 \in [-b+1, 0) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 17: $x_0 - y_0 \in (0, b)$, $x_0 + y_0 \in [-1, b)$ and $y_0 \in \left(-\frac{b}{2}, 0\right)$. Then, $y_1 \geq 0$ and

$$x_2 = |x_1| - y_1 - b = -2x_0 - 1 < 0, \qquad y_2 = x_1 - |y_1| + 1 = -2y_0 - b$$

and $y_0 = \frac{1}{2}((x_0 + y_0) - (x_0 - y_0)) > -\frac{b+1}{2}$. Thus, $y_2 < 0$. So, $(x_2, y_2) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_2 + y_2 = -2(x_0 + y_0) - b - 1 \in (-(3b+1), -b+1]$.

- if $x_0 + y_0 \in \left[-1, \frac{b-3}{7}\right)$, then $-2(x_0 + y_0) - b - 1 \in \left(-\frac{9b+1}{7}, -b+1\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- if $x_0 + y_0 \in \left[\frac{b-3}{7}, b\right)$, then $-2(x_0 + y_0) - b - 1 \in \left(-(3b+1), -\frac{9b+1}{7}\right)$. Thus, by Lemmas 1.8 and 1.9, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 18: $x_0 - y_0 \in (0, b)$, $x_0 + y_0 \in \left[-1, -\frac{1}{2}\right)$ and $y_0 \in \left(-\frac{b+1}{2}, -\frac{b}{2}\right]$. Then, we have the same (x_2, y_2) as shown in Case 17 with $y_2 \geq 0$ and

$$x_3 = |x_2| - y_2 - b = 2(x_0 + y_0) + 1, \qquad y_3 = x_2 - |y_2| + 1 = -2(x_0 - y_0) + b.$$

Since $x_0 - y_0 = (x_0 + y_0) - 2y_0 > -1 + 2\left(\frac{b}{2}\right) = b - 1$, we have $y_3 < -b + 2 < 0$. Since

$$x_0 + y_0 = (x_0 - y_0) + 2y_0 < b + 2\left(-\frac{b}{2}\right) = 0 \text{ and}$$

$$x_0 = (x_0 - y_0) + y_0 < b + \left(-\frac{b}{2}\right) = \frac{b}{2},$$

we have $x_3 < 0$. Thus, $(x_3, y_3) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_3 + y_3 = 4y_0 + b + 1 \in (-(b+1), -b+1] \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 19: $x_0 - y_0 \in (0, b)$, $x_0 + y_0 \in \left[-1, -\frac{1}{2}\right)$, $x_0 \in \left[\frac{b-1}{2}, \frac{2b-1}{4}\right)$ and $y_0 \in \left(-\frac{b+1}{2}, -\frac{b}{2}\right]$. Then, we have the same (x_2, y_2) and (x_3, y_3) as shown in Case 18 with $x_3 \geq 0$ and

$$x_4 = |x_3| - y_3 - b = 4x_0 - 2b + 1, \qquad y_4 = x_3 - |y_3| + 1 = 4y_0 + b + 2 < 0$$

and $x_0 = (x_0 + y_0) - y_0 \geq -\frac{1}{2} + \frac{b}{2} = \frac{b-1}{2}$ which implies $x_4 < 0$. Thus, $(x_4, y_4) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_4 + y_4 = 4(x_0 + y_0) - b + 3 \in [-b + 1, -b + 3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b + 2)$.

Case 20: $x_0 - y_0 \in (0, b)$, $x_0 + y_0 \in \left[-1, -\frac{1}{2}\right)$, $x_0 \in \left[\frac{2b-1}{4}, \frac{b}{2}\right)$ and $y_0 \in \left(-\frac{b+1}{2}, -\frac{b}{2}\right]$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 19 with $x_4 \geq 0$ and

$$x_5 = |x_4| - y_4 - b = 4(x_0 - y_0) - 4b - 1, \quad y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) - b + 4.$$

Since $x_0 - y_0 < b$ and $x_0 + y_0 < 0$, we have $x_5 < -1 < 0$ and $y_5 < -b + 4 < 0$ which implies that $(x_5, y_5) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_5 + y_5 = 8x_0 - 5b + 3 \in [-b + 1, -b + 3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$. ∎

The following Lemma 1.12 is the region for the initial condition in the second quadrant of $\mathbb{R}^2$.

Lemma 1.12. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be the solution of system* (1.2) *with the initial condition* (x_0, y_0) *is in* $\mathbb{R}^- \times \mathbb{R}_0^+$*. Then, the solution eventually becomes the equilibrium point* $(-1, -b + 2)$ *or eventually becomes the periodic with prime period* 5*.*

Proof. Note that $x_0 - y_0 < 0$. Thus,

$$x_1 = |x_0| - y_0 - b = -(x_0 + y_0) - b, \qquad y_1 = x_0 - |y_0| + 1 = (x_0 - y_0) + 1.$$

Case 1: $x_0 + y_0 \in \left(-\infty, -\frac{3b-2}{2}\right]$. Then, $x_1 > 0$. Since $x_0 - y_0 \leq x_0 \leq x_0 + y_0 \leq -\frac{3b-2}{2}$, we have $y_1 \leq -\frac{3b-2}{2} + 1 = -\frac{3b-4}{2}$ and

$$\begin{aligned} x_2 &= |x_1| - y_1 - b = -2x_0 - 2b - 1 > 0, \\ y_2 &= x_1 - |y_1| + 1 = -2y_0 - b + 2 < 0. \\ x_3 &= |x_2| - y_2 - b = -2(x_0 - y_0) - 2b - 3 > 0, \\ y_3 &= x_2 - |y_2| + 1 = -2(x_0 + y_0) - 3b + 2 \geq 0. \end{aligned}$$

This implies that $(x_3, y_3) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $x_3 - y_3 = 4y_0 + b - 5 \in [b - 5, \infty) \subseteq \left[\frac{b-3}{7}, \infty\right)$. Thus, by Lemmas 1.3 and 1.6, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 2: $x_0 + y_0 \in (-b, \infty)$ and $x_0 - y_0 \in (-\infty, -1)$. Then, $y_1 < 0$. Thus, $(x_1, y_1) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_1 + y_1 = -2y_0 - b + 1 \in (-\infty, -b + 1]$.

- If $y_0 \in \left[0, \frac{b+4}{7}\right)$, then $-2y_0 - b + 1 \in \left(-\frac{9b+1}{7}, -b + 1\right]$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b + 2)$.
- If $y_0 \in \left[\frac{b+4}{7}, \infty\right)$, then $-2y_0 - b + 1 \in \left(-\infty, -\frac{9b+1}{7}\right]$. Thus, by Lemmas 1.8 and 1.9, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 3: $x_0 + y_0 \in (-b, \infty)$, $x_0 - y_0 \in [-1, 0)$ and $y_0 \in \left[0, \frac{1}{2}\right)$. Then, $y_1 \geq 0$ and

$$x_2 = |x_1| - y_1 - b = 2y_0 - 1, \qquad y_2 = x_1 - |y_1| + 1 = -2x_0 - b.$$

Since $x_0 \geq y_0 - 1 \geq -1$, we have $y_2 = -2x - b \leq -b + 2 < 0$. Moreover, since $y_0 \leq x_0 + 1 < 1$ and $x_0 + y_0 = 2x_0 - (x_0 - y_0) < 1$, we have $x_2 < 0$. Thus, $(x_2, y_2) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_2 + y_2 = -2(x_0 - y_0) - b - 1 \in (-(b + 1), -b + 1] \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b + 2)$.

Case 4: $x_0 + y_0 \in \left[0, \frac{1}{2}\right)$, $x_0 - y_0 \in [-1, 0)$ and $y_0 \in \left[\frac{1}{2}, 1\right)$. Then, we have the same (x_2, y_2) as shown in Case 2 with $x_2 \geq 0$ and

$$x_3 = |x_2| - y_2 - b = 2(x_0 + y_0) - 1, \qquad y_3 = x_2 - |y_2| + 1 = -2(x_0 - y_0) - b < 0.$$

Thus, $x_0 + y_0 = (x_0 - y_0) + 2y_0 \geq -1 + 2\left(\frac{1}{2}\right) = 0$ and $x_3 < 0$. Therefore, $(x_3, y_3) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_3 + y_3 = 4y_0 - b - 1 \in [-b+1, -b+3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 5: $x_0 + y_0 \in \left[\frac{1}{2}, 1\right)$, $x_0 - y_0 \in [-1, 0)$ and $y_0 \in \left[\frac{1}{2}, 1\right)$. Then, we have the same (x_2, y_2) and (x_3, y_3) as shown in Case 4 with $x_3 \geq 0$ and

$$x_4 = |x_3| - y_3 - b = 4x_0 - 1 < 0, \qquad y_4 = x_3 - |y_3| + 1 = 4y_0 - b < 0.$$

Since $y_0 < 1$, we have $y_4 = 4y_0 - b < -b + 4 < 0$ which implies that $(x_4, y_4) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_4 + y_4 = 4(x_0 + y_0) - b - 1 \in [-b+1, -b+3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 6: $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -b\right]$ and $x_0 \in \left(-\frac{2b+1}{2}, 0\right)$. Then, $x_0 - y_0 \leq x_0 + y_0 \leq -b$ which implies that $x_1 \geq 0$, $y_1 < 0$ and

$$x_2 = |x_1| - y_1 - b = -2x_0 - 2b - 1, \qquad y_2 = x_1 - |y_1| + 1 = -2y_0 - b + 2 < 0.$$

Then, $x_2 < 0$. Thus, $(x_2, y_2) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_2 + y_2 = 2(x_0 + y_0) - 3b + 1 \in [-b+1, -1) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 7: $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -b\right]$, $x_0 - y_0 \in \left(-\frac{2b+3}{2}, -\frac{2b+1}{2}\right]$ and $x_0 \in \left(-\infty, -\frac{2b+1}{2}\right)$. Then, we have the same (x_2, y_2) as shown in Case 6 with $x_2 \geq 0$ and

$$x_3 = |x_2| - y_2 - b = -2(x_0 - y_0) - 2b - 3, \quad y_3 = x_2 - |y_2| + 1 = -2(x_0 + y_0) - 3b + 2 < 0.$$

Since $x_0 - y_0 \leq x_0 \leq -\frac{2b+1}{2}$, we have $x_3 < 0$. Thus, $(x_3, y_3) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_3 + y_3 = -4x_0 - 5b - 1 \in [-b+1, 0) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 8: $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -b\right]$, $x_0 - y_0 \in \left(-\infty, -\frac{2b+3}{2}\right]$, $x_0 \in \left(-\infty, -\frac{5b}{4}\right]$ and $y_0 \in \left[\frac{5}{4}, \infty\right)$. Then, we have the same (x_2, y_2) and (x_3, y_3) as shown in Case 7 with $x_3 \geq 0$ and

$$x_4 = |x_3| - y_3 - b = 4y_0 - 5, \qquad y_4 = x_3 - |y_3| + 1 = -4x_0 - 5b.$$

Since $x_0 = \frac{1}{2}[(x_0 - y_0) + (x_0 + y_0)] \leq \frac{1}{2}\left(-\frac{2b+3}{7} - b\right) = -\frac{4b+3}{4}$, we have $x_4 \geq 0$. Thus, $(x_4, y_4) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $x_4 - y_4 = 4(x_0 + y_0) + 5b - 5 \in (-(b+1), b-5]$.

- If $x_0 + y_0 \in \left(-\frac{36b-31}{28}, -\frac{17b-16}{14}\right)$, then $4(x_0 + y_0) + 5b - 5 \in \left(-\frac{b+4}{7}, \frac{b-3}{7}\right)$. Hence, by Lemma 1.5, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- If $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -\frac{36b-31}{28}\right] \cup \left[-\frac{17b-16}{14}, -b\right]$, then $4(x_0 + y_0) + 5b - 5 \in \left(-(b+1), -\frac{b+4}{7}\right] \cup \left[\frac{b-3}{7}, b-5\right]$. Thus, by Lemmas 1.3, 1.4, 1.6 and 1.7, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 9: $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -\frac{5b-4}{4}\right)$, $x_0 - y_0 \in \left(-\infty, -\frac{2b+3}{2}\right]$, $x_0 \in \left(-\infty, -\frac{5b}{4}\right]$ and $y_0 \in \left[0, \frac{5}{4}\right)$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 8 with $x_4 < 0$ and

$$x_5 = |x_4| - y_4 - b = 4(x_0 - y_0) + 4b + 5 < 0, \quad y_5 = x_4 - |y_4| + 1 = 4(x_0 + y_0) + 5b - 4.$$

Since $x_0 + y_0 < -\frac{5b}{4} + \frac{5}{4} = -\left(\frac{5b-5}{4}\right)$, we have $y_5 < 0$. Thus, $(x_5, y_5) \in \mathbb{R}^- \times \mathbb{R}^-$. Note that

$$-\frac{6b+1}{4} = -\frac{3b-2}{2} - \frac{5}{4} < (x_0 + y_0) - y_0 = x_0 \leq -\frac{5b}{4}$$

which implies that $x_5 + y_5 = 8x_0 + 9b + 1 \in (-(3b+1), -b+1]$.

- If $x_0 \in \left(-\frac{9b+1}{7}, -\frac{5b}{4}\right]$, then $8x_0 + 9b + 1 \in \left(-\frac{9b+1}{7}, -b+1\right]$. Hence, by Lemma 1.9,the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- If $x_0 \in \left(-\frac{6b+1}{4}, -\frac{9b+1}{7}\right]$, then $8x_0 + 9b + 1 \in \left(-(3b+1), -\frac{9b+1}{7}\right]$. Thus, by Lemmas 1.8 and 1.9, the solution of (1.2) eventually becomes the periodic with prime period 5.

Case 10: $x_0 + y_0 \in \left[-\frac{5b-4}{4}, -\frac{5b-5}{4}\right)$, $x_0 - y_0 \in \left(-\infty, -\frac{2b+3}{2}\right)$, $x_0 \in \left(-\frac{10b+1}{8}, -\frac{5b}{4}\right]$ and $y_0 \in \left[0, \frac{5}{4}\right)$. Then, we have the same $(x_2, y_2) - (x_5, y_5)$ as shown in Case 9 with $y_5 \geq 0$ and

$$x_6 = |x_5| - y_5 - b = -8x_0 - 10b - 1, \qquad y_6 = x_5 - |y_5| + 1 = -8y_0 - b + 10.$$

Since $y_0 = (x_0 + y_0) - x_0 \geq -\frac{5b-4}{4} + \frac{5b}{4} = 1$, we have $y_6 \leq -b + 2 < 0$. Moreover, we obtain that

$$x_0 = (x_0 + y_0) - y_0 > -\frac{5b-4}{4} - \frac{5}{4} = -\frac{5b+1}{4} \text{ and}$$

$$x_0 - y_0 = (x_0 + y_0) + 2y_0 > -\frac{5b-4}{4} - 2\left(\frac{5}{4}\right) = -\frac{5b+6}{4}.$$

Hence, $x_6 < 0$. Thus, $(x_6, y_6) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_6 + y_6 = -8(x_0 + y_0) - 11b + 9 \in (-(b + 1), -b+1] \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 11: $x_0 + y_0 \in \left[-\frac{5b-4}{4}, -\frac{5b-5}{4}\right)$, $x_0 - y_0 \in \left(-\frac{10b+11}{8}, -\frac{5b+5}{4}\right]$, $x_0 \in \left(-\frac{5b+1}{4}, -\frac{10b+1}{8}\right]$ and $y_0 \in \left[0, \frac{5}{4}\right)$. Then, we have the same $(x_2, y_2) - (x_6, y_6)$ as shown in Case 10 with $x_6 \geq 0$ and

$$x_7 = |x_6| - y_6 - b = -8(x_0 - y_0) - 10b - 11,$$
$$y_7 = x_6 - |y_6| + 1 = -8(x_0 + y_0) - 11b + 10 < 0.$$

Since $x_0 - y_0 = 2x_0 - (x_0 + y_0) \leq 2\left(-\frac{10b+11}{8}\right) + \frac{5b-4}{4} = -\frac{5b+5}{4}$, we have $x_7 < 0$. Thus, $(x_7, y_7) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_7 + y_7 = -16x_0 - 21b - 1 \in [-b+1, -b+3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 12: $x_0 + y_0 \in \left[-\frac{5b-4}{4}, -\frac{5b-5}{4}\right)$, $x_0 - y_0 \in \left(-\frac{5b+6}{4}, -\frac{10b+11}{8}\right]$, $x_0 \in \left(-\frac{5b+1}{4}, -\frac{10b+1}{8}\right]$ and $y_0 \in \left[0, \frac{5}{4}\right)$. Then, we have the same $(x_2, y_2) - (x_7, y_7)$ as shown in Case 11 with $x_7 \geq 0$ and

$$x_8 = |x_7| - y_7 - b = 16y_0 - 21, \qquad y_8 = x_7 - |y_7| + 1 = -16x_0 - 21b.$$

Since $y_0 < \frac{5}{4}$ and $x_0 > -\frac{5b+1}{4}$, we have $x_8 \leq -1 < 0$ and $y_8 < -b + 4 < 0$ which implies that $(x_8, y_8) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_8 + y_8 = -16(x_0 - y_0) - 21b - 21 \in [-b+1, -b+3) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 13: $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -b\right]$, $x_0 - y_0 \in \left(-\infty, -\frac{2b+3}{2}\right)$, $x_0 \in \left(-\frac{5b}{4}, -\frac{4b+3}{4}\right]$ and $y_0 \in \left[0, \frac{5}{4}\right)$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 8 with $x_4 < 0$. Thus, $(x_4, y_4) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_4 + y_4 = -4(x_0 - y_0) - 5b - 5$. Note that

$$-\frac{5b+5}{4} = -\frac{5b}{4} - \frac{5}{4} < x_0 - y_0 < -\frac{4b+3}{4}.$$

Therefore, $-4(x_0 - y_0) - 5b - 5 \in [-b+1, 0) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 14: $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -b\right]$, $x_0 - y_0 \in \left(-\frac{5b+4}{4}, -\frac{2b+5}{2}\right)$, $x_0 \in \left(-\frac{5b}{4}, -\frac{4b+3}{4}\right]$ and $y_0 \in \left[\frac{5}{4}, \frac{b}{4}\right)$. Then, we have the same $(x_2, y_2) - (x_4, y_4)$ as shown in Case 8 with $x_4 \geq 0$ and

$$x_5 = |x_4| - y_4 - b = 4(x_0 + y_0) + 4b - 5 < 0, \quad y_5 = x_4 - |y_4| + 1 = -4(x_0 - y_0) - 5b - 4.$$

Since $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -b\right]$, $y_0 \in \left[\frac{5}{4}, \frac{b}{4}\right)$ and $x_0 - y_0 = (x_0 + y_0) - 2y_0$, we have $-2b + 1 < -\frac{3b-2}{2} - 2\left(\frac{b}{4}\right) < x_0 - y_0 < -b - 2\left(\frac{5}{4}\right) = -\frac{2b+5}{2}$. Then, $y_5 < 0$. Thus, $(x_5, y_5) \in \mathbb{R}^- \times \mathbb{R}^-$

and $x_5 + y_5 = 8y_0 - b - 9 \in (-b+1, \min\{b-9, 0\}) \subseteq \left(-\frac{9b+1}{7}, 0\right)$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.

Case 15: $x_0 + y_0 \in \left(-\frac{3b-2}{2}, -b\right]$, $x_0 - y_0 \in \left(-2b+1, -\frac{5b+4}{4}\right]$, $x_0 \in \left(-\frac{5b}{4}, -\frac{4b+3}{4}\right]$ and $y_0 \in \left[\frac{5}{4}, \frac{b}{4}\right)$. Then, we have the same $(x_2, y_2) - (x_5, y_5)$ as shown in Case 14 with $y_5 \geq 0$ and

$$x_6 = |x_5| - y_5 - b = -8y_0 + 9 < 0, \qquad y_6 = x_5 - |y_5| + 1 = 8x_0 + 9b.$$

Since $x_0 + y_0 \leq -b$, we have $x_0 = \frac{1}{2}((x_0+y_0) + (x_0-y_0)) \leq \frac{1}{2}\left(-b - \frac{5b+4}{4}\right) = -\frac{9b+4}{8}$. Thus, $y_6 \leq -4 < 0$. Note that $y_0 \geq \dfrac{5}{4}$. Thus, $x_6 \leq -1 < 0$. This implies that $(x_6, y_6) \in \mathbb{R}^- \times \mathbb{R}^-$ and $x_6 + y_6 = 8(x_0 - y_0) + 9b + 9 \in (-7b+17, -b+1]$.

- If $x_0 - y_0 \in \left(-\frac{9b+8}{7}, -\frac{5b+4}{4}\right]$, then $8(x_0-y_0) + 9b + 9 \in \left(-\frac{9b+1}{7}, -b+1\right]$. Hence, by Lemma 1.10, the solution of (1.2) eventually becomes the equilibrium point $(-1, -b+2)$.
- If $x_0 - y_0 \in \left(-2b+1, -\frac{9b+8}{7}\right)$, then $8(x_0-y_0)+9b+9 \in \left(-7b+17, -\frac{9b+1}{7}\right]$. Thus, by Lemmas 1.8 and 1.9, the solution of (1.2) eventually becomes the periodic with prime period 5.

∎

After we go through Lemmas 1.3 - 1.12, we can conclude the global behavior of the solution of (1.2) by the following theorem.

Theorem 1.13. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be the solution of system (1.2) with the initial condition (x_0, y_0) is in $\mathbb{R}^2$. Then, the solution eventually becomes the equilibrium point $(-1, -b+2)$ or eventually becomes the periodic of prime period 5.*

The following Figure 1 shows the shaded region where the initial condition (x_0, y_0) induces the equilibrium behavior and the outside region where the initial condition (x_0, y_0) induces the periodic behavior with prime period 5.

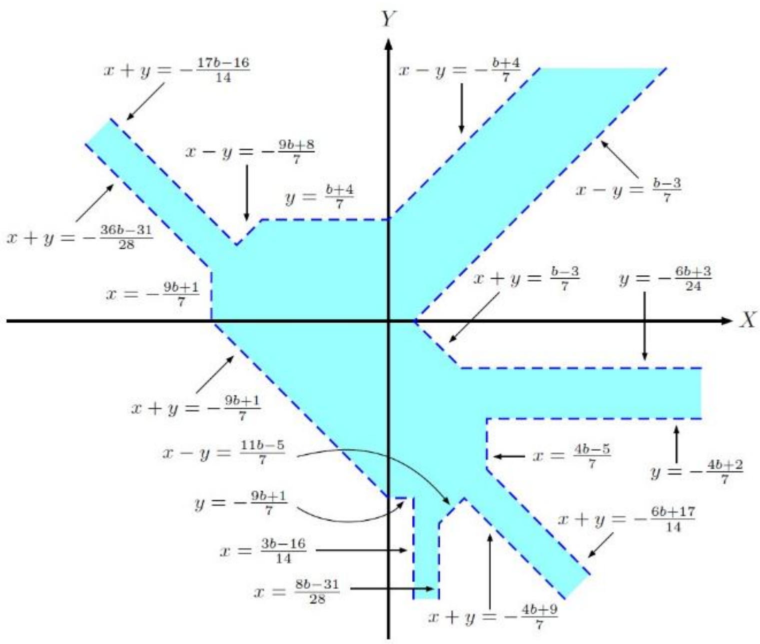

Figure 1: Shaded region where the initial condition (x_0, y_0) induces the equilibrium behavior and outside region where the initial condition (x_0, y_0) induces the periodic behavior with prime period 5

2. CONCLUSION AND DISCUSSION

This manuscript extends the result of [1] where we consider real number b such that $b \geq 6$. It can be seen that by using our approach we cannot conclude the maximum number of iterations where the behavior becomes either equilibrium or periodic with prime period 5. However, it makes our prove more compact. It is also very interesting that the behavior of the solution of (1.2) $b = 4$ or 5 is different from the one when $b \geq 6$. We conjecture that for $4 < b < 5$, the solution of (1.2) eventually becomes equilibrium. We also would like to expose in the future to find the critical value of b where $5 < b < 6$ such that the behavior changes. Some other interests may involve the consideration of a full analysis of the solution of (1.2) where $b < 4$. We also encourage the future interest to explore the behavior of the solution of the system similar to (1.1) with others parameters involved.

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