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A DETERMINANTAL REPRESENTATION OF CORE EP INVERSE

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ABSTRACT. The notion of Core EP inverse is introduced by Prasad in the article "Core - EP inverse" and proved its existence and uniqueness. Also, a formula for computing the Core EP inverse is obtained from particular linear combination of minors of a given matrix. Here a determinantal representation for Core EP inverse of a matrix A with the help of rank factorization of A is obtained.

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1. INTRODUCTION

1.1. Preliminaries.

Definition 1. *(Core-EP inverse) A matrix G is a core-EP inverse of A if G is an outer inverse of A satisfying*

$$C(G) = R(G) = C(A^d)$$

where d is the index of A.

Manjunatha Prasad and Mohana [3] proved the existence and uniqueness of core EP inverse over a real or complex field.

Theorem 1. *Given a square matrix A of index d, the core-EP inverse is unique whenever it exists and is given by*

$$A^{\textcircled{\dagger}} = A^d(A^{*d}A^{d+1})^{-}A^{*d} \tag{1.1}$$

Also it is observed that when the index of the matrix is one, the definition reduces to the definition of Core-EP generalized inverse.

Definition 2. *(Core-EP Generalized Inverse) For a square matrix A, the matrix X satisfying the conditions $(2),(3)$ and (1^k) for $k = 1$ is called the core-EP generalized inverse of A and denoted by $A^{\textcircled{\ddagger}}$.*

It is given by the formula $A(A^*A^2)^-A^*$. The relation between the core EP inverse of a matrix of index d and the core EP generalized inverse is similar to that of Drazin inverse and group inverse of a matrix A. The following theorem explains the relation between the core EP inverse of a matrix A of index d and the core-EP generalized inverse.

Theorem 2. *Let A be a matrix with $\delta(A) = d$.If $G = A^{\textcircled{\dagger}}$, the core-EP inverse of A exists, then we have the following:*

1. *G^d is a core-EP inverse of A^d*
2. *$G^d = (A^d)^{\{1,2,3,1^k\}}$ for $k = 1$*
3. *If C_A is the core part of core-nilpotent decomposition of A then $G = C_A^{\{1,2,3,1^k\}}$ for $k = 1$.*

The following lemma [3] describes the necessary and sufficient conditions for the existence of core-EP generalized inverse in the case of rank one matrix with reference to its rank factorization.

Lemma 1. *Let A be a square matrix of rank one and with rank factorization $A = xy^\star$, where x and y are suitable column matrices. Then the following statements are equivalent:*

1. *$A^{\textcircled{\ddagger}}$ exists,*
2. *x^*x and $Trace(A)$ are non-zero,*
3. *$A_i^*A_i$ and $Trace(A)$ are non-zero, where A_i is some ith column of A.*

Using this expression a determinantal representation for core-EP generalized inverse $(g_{ij}) = G = A^{\textcircled{\ddagger}}$is given by

$$g_{ij} = (\sum_K (|A_L^K|)^2)^{-1}(TraceC_r(A))^{-1} \sum_{i\epsilon I j\epsilon J} |A_L^I||A_L^J|$$

A determinantal representation of core EP inverse is given by Kyrchei [2] by defining the right and left core - EP inverse. But in this article the approach is different. We attempted to give the determinantal representation of core EP inverse of a matrix A with the rank factorization of A.

2. NOTATIONS

Assume A is an $n \times n$ matrix of rank r. Let $\alpha = \{\alpha_1, \dots \alpha_p\}$ and $\beta = \{\beta_1, \dots \beta_p\}$ be subsets of $\{1, 2, \dots m\}$ and$\{1, 2, \dots n\}$,respectively of the order $1 \leq p \leq min\{m, n\}$. Then $|A^\alpha_\beta|$ denotes the minors of A determined by the rows indexed by α and the columns indexed by β.

For $1 \leq p \leq n$, denote the collection of strictly increasing sequences of p integers chosen from $\{1, 2, \dots n\}$, by

$\mathbb{Q}_{p,n} = \{\alpha : \alpha = (\alpha_1, \dots, \alpha_p), 1 \leq \alpha_1 \leq \dots \leq \alpha_p \leq n\}$.

Let $\mathcal{N} = \mathbb{Q}_{r,m} \times \mathbb{Q}_{r,n}$.For fixed $\alpha \in \mathbb{Q}_{k,m}, \beta \in \mathbb{Q}_{k,n}, 1 \leq k \leq r$, let

$\mathcal{I}(\alpha) = \{I : I \in \mathbb{Q}_{r,m}, I \supseteq \alpha\}$, $\mathcal{J}(\beta) = \{J : J \in \mathbb{Q}_{r,n}, J \supseteq \beta\}$, and $\mathcal{N}(\alpha, \beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta)$.

If A is a square matrix, the coefficient of $|A^\alpha_\beta|$ in the Laplace expansion of $|A|$ is denoted by $\frac{\partial}{\partial |A^\alpha_\beta|}|A|$.

$C_p(A)$ denotes the p^{th} compound matrix of A with rows indexed by r-element subsets of $\{1, 2, \dots m\}$, columns indexed by r-element subsets of $\{1, 2, \dots, n\}$, and the (α, β) entry is defined by $|A^\alpha_\beta|$.

Also we use the following extension of these notations:

$\mathcal{N}_{r_k} = \mathbb{Q}_{r_k,m} \times \mathbb{Q}_{r_k,n}$, where $r_k = rank(A^l)$, $l \geq k = ind(A)$;

For fixed $\alpha, \beta \in \mathbb{Q}_{p,n}.1 \leq p \leq r_k$.let

$\mathcal{I}_{r_k}(\alpha) = \{I : I \in \mathbb{Q}_{r_k,m}, I \supseteq \alpha\}$, $\mathcal{J}_{r_k}(\beta) = \{J : J \in \mathbb{Q}_{r_k,n}, J \supseteq \beta\}$, $\mathcal{N}(\alpha, \beta) = \mathcal{I}_{r_k}(\alpha) \times \mathcal{J}_{r_k}(\beta)$

Also the core EP generalized inverse of a matrix A is denoted by $A^{\textcircled{\ddagger}}$ and the core EP inverse is given by $A^{\textcircled{\dagger}}$.

3. RESULTS

Expression for group inverse and Drazin inverse in terms of its rank factorizations are well known in the literature [1].

Let A be any square complex matrix with index of A is k, then either $A^k = \theta$ (where θ designates the null matrix of appropriate size) or A^k can be written as

$$A^k = \prod_{i=1}^{k} B_i \prod_{i=1}^{k} G_{k+1-i}$$

where each of the matrices $B_1, B_2, \dots B_k$ and $\prod_{i=1}^{k} B_i$ has full column rank and each of the matrices $G_1, G_2, \dots G_k$ and $\prod_{i=1}^{k} G_{k+1-i}$ has full row rank. The matrices B_i and G_{k+1-i} are further determined by the conditions that B_1 and G_1 satisfy

$$A = B_1 G_1$$

and that

$$G_i B_i = B_{i+1} G_{i+1},\ i = 1, 2, \dots k-1$$

where $G_k B_k$ is nonsingular. Then an expression for Drazin inverse is given by

$$A^d = \prod_{i=1}^{k} B_i (G_k B_k)^{-k-1} \prod_{i=1}^{k} G_{k+1-i}$$

A similar kind of expression is obtained here for core-EP inverse.

Assume that A is a square matrix with index k.Let $A = B_1 G_1$ be a full rank factorization for the matrix A.Then as in the above the expression for core-EP inverse is

$$A^{\textcircled{\dagger}} = (B_1 B_2 \dots B_k)(G_k B_k)^{-1}(B_1 B_2 \dots B_k)^{\dagger}$$

In particular when $k = 1$ this expression reduces to the expression for core-EP generalized inverse given by $B_1(G_1B_1)^{-1}B_1^{\dagger}$.

Theorem 3. *Let $A \in C_r^{n\times n}$with $l \geq k = ind(A)$ with an arbitrary full rank factorization $A^k = B_{A^k}G_{A^k}$.Then the core-EP inverse of the matrix can be given by $A^{\textcircled{\dagger}} = B_{A^k}(G^*_{A^k}AB_{A^k})^{-}G^*_{A^k}$.*

Proof. Since $A^k = B_{A^k}G_{A^k}$, $(A^*)^k = G^*_{A^k}B^*_{A^k}$
Now

$$\begin{aligned} A^{\textcircled{\dagger}} &= B_{A^k}G_{A^k}(G^*_{A^k}B^*_{A^k}AB_{A^k}G_{A^k})^{\dagger}G^*_{A^k}B^*_{A^k} \\ &= B_{A^k}G_{A^k}(G_{A^k})^{\dagger}(B^*_{A^k}AB_{A^k})^{-1}(G^*_{A^k})^{\dagger}G^*_{A^k}B^*_{A^k} \\ &= B_{A^k}(B^*_{A^k}AB_{A^k})^{-1}B^*_{A^k} \end{aligned}$$

∎

Theorem 4. *If A is an $n \times n$ complex matrix of index k, and $A^k = B_{A^k}G_{A^k}$ is an arbitrary full rank decomposition of A^k, then*

$$\begin{aligned} (1)(A^{\textcircled{\dagger}})^k &= B_{A^k}(G_{A^k}B_{A^k})^{-1}(B_{A^k})^{\dagger} \\ (2)AA^{\textcircled{\dagger}} &= B_{A^k}(B^*_{A^k}B_{A^k})^{-1}B^*_{A^k} \\ (3)(A^{\textcircled{\dagger}})^{\dagger} &= (B^*_{A^k})^{\dagger}(B^*_{A^k}AB_{A^k})(B_{A^k})^{\dagger} \end{aligned}$$

Proof. (1) By Theorem3 and Lemma1
$(A^k)^{\textcircled{\sharp}} = (B_{A^k}G_{A^k})^{\textcircled{\sharp}} = B_{A^k}(G_{A^k}B_{A^k})^{-1}(B_{A^k})^{\dagger}$
(2) From the definition it is clear that

$$AA^{\textcircled{\dagger}} = A^k(A^k)^{\dagger}(A^{*k})^{\dagger}A^{*k}$$

Now

$$\begin{aligned} AA^{\textcircled{\dagger}} &= B_{A^k}G_{A^k}(G^*_{A^k}B^*_{A^k}B_{A^k}G_{A^k})^{\dagger}G^*_{A^k}B^*_{A^k} \\ &= B_{A^k}G_{A^k}(G_{A^k})^{\dagger}(B^*_{A^k}B_{A^k})^{-1}(G^*_{A^k})^{\dagger}G^*_{A^k}B^*_{A^k} \\ &= B_{A^k}(B^*_{A^k}B_{A^k})^{-1}B^*_{A^k} \end{aligned}$$

(3) Since

$$\begin{aligned} A^{\textcircled{\dagger}} &= B_{A^k}(B^*_{A^k}AB_{A^k})^{-}B^*_{A^k} \\ (A^{\textcircled{\dagger}})^{\dagger} &= (B_{A^k}(B^*_{A^k}AB_{A^k})^{-}B^*_{A^k})^{\dagger} \\ &= (B^*_{A^k})^{\dagger}(B^*_{A^k}AB_{A^k})(B_{A^k})^{\dagger} \end{aligned}$$

∎

Theorem 5. *The core-EP inverse of an arbitrary matrix $A \in C_r^{n\times n}$ possesses the following determinantal representation*

$$A^{\textcircled{\dagger}}_{ij} = \frac{\sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)}|(B_{A^l})^{\beta}||(B^*_{A^l})_{\alpha}|\frac{\partial}{\partial a_{ji}}|A^{\beta}_{\alpha}|}{\sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}}|(B^*_{A^l})_{\gamma}||(B_{A^l})^{\delta}||A^{\gamma}_{\delta}|}$$

where $l \geq k = ind(A)$ and $r_k = rank(A^l)$.

Proof. Let $A = BG$ be arbitrary full rank factorization of A, and $A^l = B_{A^l}G_{A^l}$ is full rank factorization of A^l, where $l \geq ind(A)$

Now

$$A^{\textcircled{\dagger}} = B_{A^l}(B^*_{A^l}AB_{A^l})^{-1}B^*_{A^l}$$

$$= \frac{B_{A^l}adj(B^*_{A^l}AB_{A^l})B^*_{A^l}}{|B^*_{A^l}AB_{A^l}|}$$

Consider

$$\begin{aligned}|B^*_{A^l}AB_{A^l}| &= |B^*_{A^l}BGB_{A^l}| \\ &= \sum_{\epsilon\in\mathbb{Q}_{r_k,r}} |(B^*_{A^l}B)_\epsilon||(GB_{A^l})^\epsilon| \\ &= \sum_{\epsilon\in\mathbb{Q}_{r_k,r}} |(B^*_{A^l}B_{\epsilon)}||(G^\epsilon B_{A^l})|\end{aligned}$$

Again by applying Cauchy - Binet formula,

$$|B^*_{A^l}BGB_{A^l}| = \sum_{\epsilon\in\mathbb{Q}_{r_k,r}}(\sum_{\gamma\in\mathbb{Q}_{r_k,n}}|(B^*_{A^l})_\gamma||B^\gamma_\epsilon|)(\sum_{\delta\in\mathbb{Q}_{r_k,n}}|G^\epsilon_\delta||(B_{A^l})^\delta|)$$

Hence

$$\begin{aligned}|B^*_{A^l}BGB_{A^l}| &= \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}}|(B^*_{A^l})_\gamma||(B_{A^l})^\delta|\sum_{\epsilon\in\mathbb{Q}_{r_k},r}|B^\gamma_\epsilon||G^\epsilon_\delta| \\ &= \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}}|(B^*_{A^l})_\gamma||(B_{A^l})^\delta||B^\gamma G_\delta| \\ &= \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}}|(B^*_{A^l})_\gamma||(B_{A^l})^\delta||A^\gamma_\delta|\end{aligned}$$

Now consider $B_{A^l}adj(B^*_{A^l}AB_{A^l})B^*_{A^l}$.If the submatrix of A generated by deleting i^{th} row of A is denoted by $(A^{\{i\}})'$ and the j^{th} column by $(A_{\{j\}})'$ respectively.
Since $(adj(B^*_{A^l}BGB_{A^l}))_{ij} = (-1)^{i+j}|(B^*_{A^l})^{\{j\}'}BG(B_{A^l})_{\{i\}'}|$

By Cauchy-Binet theorem

$$(adj(B^*_{A^l}BGB_{A^l}))_{ij} = (-1)^{i+j}\sum_{\epsilon'\in\mathbb{Q}_{r_k-1,r}}|G^{\epsilon'}(B_{A^l})_{\{i\}'}||(B^*_{A^l})^{\{j\}'}B_{\epsilon'}|$$

Now applying Cauchy- Binet formula for both the determinants,
$(adj(B^*_{A^l}BGB_{A^l}))_{ij} = (-1)^{i+j}\sum_{\epsilon'\in\mathbb{Q}_{r_k-1,r}}(\sum_{\beta'\in\mathbb{Q}_{r_k-1,n}}|G^{\epsilon'}_{\beta'}||((B_{A^l})_{\{i\}'})^{\beta'}|$
$\times(\sum_{\alpha'\in\mathbb{Q}_{r_k-1,n}}|B^{\epsilon'}_{\beta'}||((B^*_{A^l})^{\{j\}'})_{\alpha'}|$
Therefore,

$$(adj(B^*_{A^l}BGB_{A^l}))_{ij} = \sum_{t=1}^{r_k}(P_{A^l})_{it}(adj(B^*_{A^l}BGB_A))_{tj}$$

$$= \sum_{\epsilon'\in\mathbb{Q}_{r_k-1,r}}\sum_{\beta'\in\mathbb{Q}_{r_k-1,n}}|G^{\epsilon'}_{\beta'}|(\sum_{t=1}^{r_k}(-1)^t((B_{A^l})_{it}|(B_{A^l})_{\{t'\}})^{\beta'}|)\times(\sum_{\alpha'\in\mathbb{Q}_{r_k-1,n}}(-1)^j|B^{\alpha'}_{\epsilon'}||((B^*_{A^l})^{\{j\}'})_\alpha|)$$

If i is contained in the combination $\beta^{'}$ then,

$$\sum_{t=1}^{r_k}(-1)^t(B_{A^l})_{it}|(B_{A^l})_{\{t\}'})^{\beta^{'}}| = 0$$

If the set $\beta^{'}$ does not contain i, then $i = \beta_p$ and the system $\beta^{'}$ is denoted by

$$\beta^{'} = \{1 \leq \beta_1 < \ldots < \beta_{p-1} < \beta_{p+1} < \ldots \leq n\}$$

If the set β denotes the following combination

$$\beta = \{1 \leq \beta_1 < \ldots < \beta_{p-1} < i = \beta_p < \beta_{p+1} < \ldots \leq n\}$$

we obtain the representation for
$(B_{A^l}adj(B^*_{A^l}BGB_{A^l}))_{ij} =$
$\sum_{\epsilon' \in \mathbb{Q}_{r_k-1,r}}(\sum_{\beta\in\mathcal{J}_{r_k}(i)}(-1)^p|G^{\epsilon^{'}}_{\beta\setminus\{i\}}||(B_{A^l})^{\beta}|) \times (\sum_{\alpha'\in\mathbb{Q}_{r_k-1,n}}(-1)^j|B^{\alpha^{'}}_{\epsilon^{'}}||(B^*_{A^l})^{\{j\}^{'}})_{\alpha'}|)$
Continuing in the same way we get the representation for

$$B_{A^l}adj(B^*_{A^l}BGB_{A^l}B^*_{A^l})_{ij} = \sum_{t=1}^{r_k} B_{A^l}adj(B^*_{A^l}BGB_{A^l}))_{it}(B^*_{A^l})_{tj}$$

=

$$\sum_{\epsilon^{'}\in\mathbb{Q}_{r_k-1,r}}(\sum_{\beta\in\mathcal{J}_{r_k}(i)}(-1)^p|G^{\epsilon^{'}}_{\beta\setminus\{i\}}||(B_{A^l})^{\beta}|) \times (\sum_{\alpha^{'}\in\mathbb{Q}_{r_k-1,n}}|B^{\alpha^{'}}_{\epsilon^{'}}|\sum_{t=1}^{r_k}(B^*_{A^l})_{tj}||(B^*_{A^l})^{\{t\}^{'}})_{\alpha'}|)$$

Similarly if j is contained in the combination α', then

$$\sum_{t=1}^{r_k}(-1)^t(B^*_{A^l})_{tj}|(B^*_{A^l})_{\{t\}'})_{\alpha'}| = 0$$

Otherwise $j = \alpha_q$ and

$$\alpha^{'} = \{1 \leq \alpha_1 < \ldots < \alpha_{q-1} < \alpha_{q+1} < \ldots \alpha_{r_k} \leq n\}$$

and

$$\alpha = \{1 \leq \alpha_1 < \ldots < \alpha_{q-1} < j = \alpha_q, \alpha_{q+1} < \ldots \alpha_{r_k} \leq n\}$$

Therefore the (i,j) th element of $B_{A^l}adj(B^*_{A^l}BGB_{A^l})B^*_{A^l}$ is equal to

$$\sum_{\epsilon^{'}\in\mathbb{Q}_{r_k-1,r}}(\sum_{\beta\in\mathcal{J}_{r_k}(i)}(-1)^p|G^{\epsilon^{'}}_{\beta\setminus\{i\}}||(B_{A^l})^{\beta}|) \times (\sum_{\alpha\in\mathcal{J}_{r_k}(j)}(-1)^q|B^{\alpha\setminus\{j\}}_{\epsilon^{'}}||(B^*_{A^l})^{\alpha}|)$$

$$\sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)}|(B_{A^l})^{\beta}||(B^*_{A^l})_{\alpha}(\sum_{\epsilon^{'}\in\mathbb{Q}_{r_k-1,r})}(-)^{(p+q)}|B^{\alpha\setminus\{j\}}_{\epsilon^{'}}||G^{\epsilon^{'}}_{\beta\setminus\{j\}}|$$

$$= \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)}|(B_{A^l})^{\beta}||(B^*_{A^l})_{\alpha}|\frac{\partial}{\partial a_{ji}}|A^{\alpha}_{\beta}|$$

∎

Theorem 6. *For a given matrix $A \in C_r^{m\times n}$ with core EP inverse of index k with rank factorization $A = BG$ and $A^k = B_{A^k}G_{A^k}$, we obtain the following representations;*

(i) $(AA^{\textcircled{\dagger}})_{ij} \;=\; \frac{\sum_{\alpha\in\mathcal{J}_{r_k(i)},j\notin\alpha}|(B^*_{A^k})_\alpha|\sum_{t=1}^{r_k}(B_{A^k})_{it}\frac{\partial}{\partial(B_{A^k})_{jt}}|(B_{A^k})^\alpha|}{\sum_{\gamma\in\mathbb{Q}_{r_k,n}}|(B^*_{A^k})_\gamma||(B_{A^k})^\gamma|}$

$= \frac{\sum_{\alpha\in\mathcal{J}_{r_k(i)},j\notin\alpha}|(B^*_{A^k})_\alpha||(B_{A^k})^\alpha|}{\sum_{\gamma\in\mathbb{Q}_{r_k,n}}|(B^*_{A^k})_\gamma||(B_{A^k})^\gamma|}\;(1\le t\le r_k, 1\le j\le n)$

(ii) $(A^{\textcircled{\dagger}}(A^{\textcircled{\dagger}})^\dagger)_{ij} \;=\; \frac{\sum_{\alpha\in\mathcal{J}_{r_k(i)},j\notin\alpha}|(\overline{B_{A^k}})^\alpha||(B_{A^k})^\alpha|}{Tr(C_{r_k}(B_{A^k}(B_{A^k})^*)}$, $1\le i,j\le n$

(iii) $((A^{\textcircled{\dagger}})^\dagger A^{\textcircled{\dagger}})_{ij} \;=\; \frac{\sum_{\alpha\in\mathcal{J}_{r_k(i)},j\notin\alpha}|(\overline{(B^*_{A_k})}_\alpha)||(B^*_{A^k})_\alpha|}{Tr((C_{r_k}(B_{A^k}(B^*_{A^k})))}$

Proof. (i) From the definition

$$(AA^{\textcircled{\dagger}})_{ij} = B_{A^k}(B^*_{A^k}B_{A^k})^{-1}B^*_{A^k}$$

Since

$$((B^*_{A^k}B_{A^k})^{-1}B^*_{A^k})_{tj} = \frac{\sum_{\alpha\in\mathcal{J}_{r_k}(j)}|(B^*_{A^k})_\alpha|\frac{\partial}{\partial(B_{A^k})_{jt}}|(B_{A^k})^\alpha|}{\sum_{\gamma\in\mathbb{Q}_{r_k,n}}|(B^*_{A^k})_\gamma||(B_{A^k})^\gamma|}$$

Now for arbitrary $1\le i,j\le n$ we get

$$(AA^{\textcircled{\dagger}})_{ij} = \sum_{t=1}^{r_k}(B_{A^k})_{it}((B^*_{A^k}B_{A^k})^{-1}B^*_{A^k})_{tj}$$

$$= \frac{\sum_{\alpha\in\mathcal{J}_{r_k(i)},j\notin\alpha}|(B^*_{A^k})_\alpha|\sum_{t=1}^{r_k}(B_{A^k})_{it}\frac{\partial}{\partial(B_{A^k})_{jt}}|(B_{A^k})^\alpha|}{\sum_{\gamma\in\mathbb{Q}_{r_k,n}}|(B^*_{A^k})_\gamma||(B_{A^k})^\gamma|}$$

$$= \frac{\sum_{\alpha\in\mathcal{J}_{r_k}(i),j\notin\alpha}|(B^*_{A^k})_\alpha||(B_{A^k})^\alpha|}{\sum_{\gamma\in\mathbb{Q}_{r_k,n}}|(B^*_{A^k})_\gamma||(B_{A^k})^\gamma|}$$

(ii) Since

$$(A^{\textcircled{\dagger}}(A^{\textcircled{\dagger}})^\dagger)_{ij} = B_{A^k}(B_{A^k})^\dagger = B_{A^k}(B^*_{A^k}B_{A^k})^{-1}(B_{A^k})^*$$

and also,

$$((A^{\textcircled{\dagger}})^\dagger A^{\textcircled{\dagger}})_{ij} = (B^*_{A^k})^\dagger(B^*_{A^k}) = B_{A^k}(B^*_{A^k}B_{A^k})^{-1}B^*_{A^k}$$

the proof follows in the same as in (i). ∎

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