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FRACTIONAL INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR P-CONVEX AND QUASI-CONVEX STOCHASTIC PROCESS

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ABSTRACT. In this paper we consider the class of P-convex and Quasi-convex stochastic processes on witch we apply a general class of generalized fractional integral operator in order to establish new integral inequalities of Hermite-Hadammard type. then we obtain some results for well known types of fractional integrals. Results obtained in this paper may be starting point as well as a useful source of inspiration for further research in convex analysis.

Key words and phrases: Stochastic process, Quasi convex, P-convex, Fractional Integrals, Fractional Operators, Hermite Hadamard Inequality.

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1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction.

The study of convex functions is interesting for the mathematical analysis based on the properties which are deduced from this concept. Due to the requirements of generalization of the concept of convexity in order to obtain new equations and applications, in recent years a great deal of work has been done in the study and development of this subject.
A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$ the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds.

The following inequality for convex functions :

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

known as the Hermite-Hadamard's inequality, is one of the most important results for convex functions, and a compendium of its history can be found in the work of D.S. Mitrinovic and I.B. Lackovic (see [1], [2]).
Numerous works of investigation have been realized extending results on inequalities for convex functions towards others much more generalized, using new concepts such as $E-$convexity (see [3]), quasi-convexity (see [4], [5]), $s-$convexity (see [4],[6]) and others,making the inequality a very useful tool in the Theory of Probability and Optimization (see [3]).

The study on convex stochastic processes began in 1974 when B. Nagy [7], applied a characterization of measurable stochastic processes to solve a generalization of the (additive) Cauchy functional equation. In 1980 Nikodem [8] considered convex stochastic processes. In 1995 Skowronski [9] obtained some further results on convex stochastic processes, which generalize some known properties of convex functions. In the year 2014, E. Tomar in [10] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense. For other results related to stochastic processes (see [11], [12], [13], [14], [15] and [16]), where further references are given.

Fractional calculus, was introduced at the end of the nineteenth century by Liouville and Riemann,the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. In 2011, U. Katugampola presented a new fractional integral operator (see [17]), which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, and various researchers have made use of this result in the field of convexity, generalized convexity and others.
Many research have been done about the Hermite-Hadamard inequalities involving some kinds of fractional integrals and remarkable varieties of refinements can be found in the literature (see [18],[19] and[20]).
For example, K. Qiu and J. Wang (see [21]) investigated two Hermite-Hadamard type inequalities involving right-sided Riemann-Liouville fractional integrals for convex functions.
Z. Lin and J. Wang (see [22]) studied some Riemann-Liouville fractional Hermite-Hadamard inequalities via r-convex and geometric-arithmetically r-convex function.
Yuming Feng (see [23]) gave a refinements of Hermite-Hadamard integral inequality by using

two parameters.

The main purpose of this paper is to introduce a more general integral definition which generalizes some significant well known fractional integral operators such as RiemannLiouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, using a general class of this generalized fractional integral operator, to establish new Hermite-Hadamard type inequalities for P-convex and Quasi-convex stochastic processes.

1.2. Preliminaries and definitions.

We begin by establishing definitions for stochastic processes, then we give some estimates of the right-hand side inequality of Hermite-Hadamard.

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a random variable if $\mathcal{A}$ is measurable.

A stochastic processes is defined as function $X : I \times \Omega \to \mathbb{R}$ where $I \subset \mathbb{R}$ is an interval if for every $t \in I$, the function $X(t, \cdot)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called :

(1) Continuous in probability in interval I, if for all $t_0 \in \mathrm{I} : \mathrm{P} - \lim\limits_{t \to t_0} X(t, \cdot) = X(t_0, \cdot)$. Where $\mathrm{P} - \lim$ denotes the limit in probability.

(2) Mean square continuous in the interval I, if for all $t_0 \in \mathrm{I}$: $\lim\limits_{t \to t_0} \mathrm{E}\left[(X(t, \cdot) - X(t_0, \cdot))^2\right] = 0.$ Where $\mathrm{E}[X(t, \cdot)]$ denote the expectation value of the random variable $X(t, \cdot)$.

(3) Mean-square differentiable at a point $t \in I$, if there is a random variable $X'(t, \cdot) : \Omega \to \mathbb{R}$ such that for all $t_0 \in I$ we have :

$$\lim_{t \to t_0} E\left[\frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0} - X'(t, \cdot)\right]^2 = 0$$

Definition 1.1. (see [11], [12])

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $\mathrm{E}\left[X(t; \cdot)^2\right] \prec \infty$ for all $t \in I$. Let $[a; b] \subset \mathrm{I}; a = t_0 \prec t_1 \prec t_2 \prec t_3 \prec \ldots\ldots \prec t_n = b$ be a partition of $[a; b]$ and $\lambda_k \in [t_{k-1}; t_k]$ for $k = 1, \ldots, n$

A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a; b]$, if we have $\lim\limits_{n \to \infty} \mathrm{E}\left[\left(\sum\limits_{k=1}^{k=n} X(\lambda_k, \cdot)(t_k - t_{k-1}) - Y(\cdot)\right)^2\right] = 0$ for all normal sequence of partitions of the interval $[a; b]$ and for all $\lambda_k \in [t_{k-1}; t_k]$ for all $k = 1, \ldots, n$. Then, we write

$$Y(\cdot) = \int_a^b X(t, \cdot) dt.$$

Definition 1.2. (see [11], [12])

Let $(\Omega; \mathcal{A}; P)$ be a probability space and $I \subset \mathbb{R}$ and $h : [0, 1] \to \mathbb{R}$ be a non-negative function $, h \neq 0$

• The stochastic process $X : I \times \Omega \to \mathbb{R}$ is P-convex if for all $\lambda \in [0; 1]$ and $a; b \in I$ the inequality :

$$X(\lambda a + (1 - \lambda)b, \cdot) \leq X(a, \cdot) + X(b, \cdot)$$

is satisfied.

• The stochastic process $X : I \times \Omega \to \mathbb{R}$ is quasi-convex if for all $\lambda \in [0;1]$ and $a;b \in I$ the inequality :

$$\mathrm{X}(\lambda \mathrm{a} + (1-\lambda)\mathrm{b}, \cdot) \leq \max\{\mathrm{X}(\mathrm{a}, \cdot); \mathrm{X}(\mathrm{b}, \cdot)\}$$

is satisfied.

Theorem 1.1. *: (Hermite-Hadamard inequality for Jensen convex stochastic process (see* [10]*))*
Let $X : \mathrm{I} \times \Omega \to \mathbb{R}$ be a Jensen-convex, mean-square continuous in the interval I stochastic process. Then for any $a, b \in I$, we have the following Hermite-Hadamard inequality :

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{b-a}\int_a^b X(t,\cdot)dt \leq \frac{X(a,\cdot)+X(b,\cdot)}{2} \tag{1.3}$$

In this section, we state the following new integral definition which are useful in the proofs of main theorems :

Let's define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty \tag{1.4}$$

$$\frac{1}{A_1} \leq \frac{\varphi(s)}{\varphi(r)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.5}$$

$$\frac{\varphi(r)}{r^2} \leq A_2 \frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{1.6}$$

$$\left|\frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2}\right| \leq A_3 |r-s| \frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.7}$$

Where $A_1, A_2, A_3 > 0$ are independent of $r, s > 0$. If $\varphi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies (1.4) - (1.7), (see[24]). Therefore, we define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$${}_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a \tag{1.8}$$

$${}_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b \tag{1.9}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (1.4) and (1.7) are mentioned below.

i) If we take $\varphi(t) = t$, the operators (1.8) and (1.9) reduce to the Riemann integral as follows :

$$I_{a+}f(x) = \int_a^x f(t)dt, \quad x > a \tag{1.10}$$

$$I_{b^-}f(x) = \int_x^b f(t)dt, \quad x < b \tag{1.11}$$

ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operators (1.8) and (1.9) reduce to the Riemann Liouville fractional integral as follows:

$$I^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a \tag{1.12}$$

$$I^\alpha_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b \tag{1.13}$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$, the operators (1.8) and (1.9) reduce to the k Riemann-Liouville fractional integral as follows:

$$I^\alpha_{a^+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \quad x > a \tag{1.14}$$

$$I^\alpha_{b^-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt, \quad x < b \tag{1.15}$$

where:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \quad \mathcal{R}(\alpha) > 0 \tag{1.16}$$

and:

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0 \tag{1.17}$$

are given by Mubeen and Habibullah (see [25]).

iv) If we take $\varphi(t) = t(x-t)^{\alpha-1}$, the operator (1.8) reduces the conformable fractional operators as follows :

$$I^\alpha_a f(x) = \int_a^x t^{\alpha-1}f(t)dt = \int_a^x f(t)d_\alpha t, \quad x > a, \alpha \in (0,1) \tag{1.18}$$

is given by Khalil et.al (see [26]).

v) If we take $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$ in the operators (1.8) and (1.9), then they reduce to the right-sided and left-sided fractional integral operators with exponential kernel for $\alpha \in (0,1)$, as follows :

$$\mathcal{I}^{\alpha}_{a^+} f(x) = \frac{1}{\alpha}\int_a^x \exp\left(-\frac{1-\alpha}{\alpha}(x-t)\right) f(t)dt, \quad a < x \tag{1.19}$$

$$\mathcal{I}^{\alpha}_{b^-} f(x) = \frac{1}{\alpha}\int_x^b \exp\left(-\frac{1-\alpha}{\alpha}(t-x)\right) f(t)dt, \quad x < b \tag{1.20}$$

are defined by Kirane and Torebek(see [27]).

2. MAIN RESULTS: GENERALIZED FRACTIONAL INTEGRALS FOR STOCHASTIC PROCESS AND HERMITE-HADAMARD INEQUALITIES

Throughout the study we define :

$$\Psi(y) = \int_0^y \frac{\varphi((b-a)u)}{u}du < \infty \tag{2.1}$$

Theorem 2.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process on $I^\circ (I \subset \mathbb{R})$, and $a, b \in I^\circ$. If x is P_-convex on $[a, b]$, with $a < b$ then we have the following inequalities for generalized fractional integral.*

$$X\left(\frac{a+b}{2}, \cdot\right) \le \frac{1}{\Psi(1)}\left[{}_{a^+}I_\varphi X(b,\cdot) +_{b^-} I_\varphi X(a,\cdot)\right] \le 2(X(a,\cdot) + X(b,\cdot)) \tag{2.2}$$

Proof. For $t \in [0,1]$, let $x = ta + (1-t)b$ and $y = (1-t)a + tb$. Since X is P-convex, we have :

$$X\left(\frac{a+b}{2}, \cdot\right) = X\left(\frac{x+y}{2}, \cdot\right) \le\ X(x,\cdot) + X(y,\cdot)$$

So,

$$X\left(\frac{a+b}{2}, \cdot\right) \le\ X(ta + (1-t)b,\cdot) + X(tb + (1-t)a,\cdot)$$

By multiplying both sides by $\frac{\varphi((a-b)t)}{t}$, then integrating the resulting inequality with respect to t over $[0,1]$ we obtain :

$$X\left(\frac{a+b}{2}, \cdot\right)\int_0^1 \frac{\varphi((b-a)t)}{t}dt \le \int_0^1 \frac{\varphi(b-a)t)}{t}X(ta+(1-t)b,\cdot)dt + \int_0^1 \frac{\varphi(b-a)t)}{t}X(tb+(1-t)a,\cdot)dt$$

We have :

$$\int_0^1 \frac{\varphi((b-a)t)}{t}X(ta+(1-t)b,\cdot)dt = \int_a^b \frac{\varphi(b-x)}{(b-x)}X(x,\cdot)dx =_{b^-} I_\varphi X(a,\cdot)$$

And:

$$\int_0^1 \frac{\varphi((b-a)t)}{t}X(tb+(1-t)a,\cdot)dt = \int_a^b \frac{\varphi(y-a)}{(y-a)}X(y,\cdot)dy =_{a^+} I_\varphi X(b,\cdot)$$

To prove the second half of the inequality, we have X is P-convex then for every $t \in [0,1]$:

$$X(ta+(1-t)b,\cdot)+X((1-t)a+tb) \leq 2X(a,\cdot)+2X(b,\cdot)$$

And by multiplying both sides of the inequality by $\frac{\varphi((b-a)t)}{t}$ and integrating the resulting inequality with respect to $t \in [0,1]$ we obtain :

$$\left[{}_{a^+}I_\varphi X(b,\cdot) +_{b^-} I_\varphi X(a,\cdot)\right] \leq 2(X(a,\cdot)+X(b,\cdot)) \int_0^1 \frac{\varphi(b-a)t)}{t} dt$$

∎

Remark 2.1.
- For $\varphi(t)=t$ we get :

(2.3) $$X\left(\frac{a+b}{2},\cdot\right) \leq \frac{2}{b-a}\int_a^b X(t,\cdot)dt \leq 2(X(a,\cdot)+X(b,\cdot))$$

- For $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ we get :

(2.4)
$$X\left(\frac{a+b}{2},\cdot\right) \leq \frac{\alpha}{(b-a)^\alpha}\int_a^b \left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right] X(t,\cdot)dt \leq 2(X(a,\cdot)+X(b,\cdot))$$

(see [28]).
- For $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ we get :

(2.5)
$$X\left(\frac{a+b}{2},\cdot\right) \leq \frac{\alpha}{k(b-a)^{\frac{\alpha}{k}}}\int_a^b \left[(b-t)^{\frac{\alpha}{k}-1}+(t-a)^{\frac{\alpha}{k}-1}\right] X(t,\cdot)dt \leq 2(X(a,\cdot)+X(b,\cdot))$$

A similar result was proven by Hussain et A. Ali for convex functions(see [29]).

Corollary 2.2. *Under the assumption of Theorem 2.1 with $\varphi(t)=t(b-t)^{\alpha-1}$, and X is a symmetric to $\frac{(a+b)}{2}$, then we have:*

(2.6) $$X\left(\frac{a+b}{2},\cdot\right) \leq \frac{2\alpha}{b^\alpha-a^\alpha}\int_a^b t^{\alpha-1}X(t,\cdot)dt \leq 2\left(X(a,\cdot)+X(b,\cdot)\right)$$

Corollary 2.3. *Under the assumption of Theorem 2.1 with $\varphi(t)=\frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $\alpha \in [0.1]$,we have :*

(2.7) $$X\left(\frac{a+b}{2},\cdot\right) \leq \frac{\alpha-1}{\alpha(\exp(A)-1)}\left[{}_{a^+}I_\varphi X(b,\cdot) +_{b^-} I_\varphi X(a,\cdot)\right] \leq 2\left(X(a,\cdot)+X(b,\cdot)\right)$$

where $A=\frac{\alpha-1}{\alpha}(b-a)$.

This result is given by Kirane and Torebek for convex functions (see [27]).

Theorem 2.4. *Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process on $I^{\circ}(I \subset \mathbb{R})$ and $a, b \in I^{\circ}$. If X is Quasi-convex on $[a, b]$, with $a < b$ then the following inequalities for generalized fractional integral holds :*

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{\Psi(1)} \operatorname{Max}\{{}_{a^+}I_{\varphi}X(b, \cdot), {}_{b^-}I_{\varphi}X(a, \cdot)\} \leq \operatorname{Max}\{X(a, \cdot), X(b, \cdot)\} \tag{2.8}$$

Proof. For $t \in [0, 1]$, let $x = ta + (1-t)b$ and $y = (1-t)a + tb$. Since X is Quasi-convex, we have :

$$X\left(\frac{a+b}{2}, \cdot\right) = X\left(\frac{x+y}{2}, \cdot\right) \leq \operatorname{Max}\{X(x, \cdot), X(y, \cdot)\}$$

So,

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \operatorname{Max}\{X(ta + (1-t)b, \cdot), X(tb + (1-t)a, \cdot)\}$$

By multiplying both sides by $\frac{\varphi((a-b)t)}{t}$, then integrating the resulting inequality with respect to t over $[0, 1]$ we obtain :

$$\begin{aligned} X\left(\frac{a+b}{2}, \cdot\right) \int_0^1 \frac{\varphi((b-a)t)}{t} dt &\leq \int_0^1 \frac{\varphi(b-a)t)}{t} dt \times \operatorname{Max}\{X(ta + (1-t)b, .)dt + X(tb + (1-t)a, .)\} \\ &\leq \operatorname{Max}\{{}_{a^+}I_{\varphi}X(b, .), {}_{b^-}I_{\varphi}X(a, \cdot)\} \end{aligned}$$

To prove the second half of the inequality, We have, for $t \in [0, 1]$.

$$\operatorname{Max}\{X(x, \cdot), X(y, \cdot)\} = \operatorname{Max}\{X(\text{ta} + (1-t)b, \cdot), X(tb + (1-t)a, \cdot)\}$$

Using the fact that X is Quasi-convex we get :

$$X(ta + (1-t)b, \cdot) \leq \operatorname{Max}\{X(a, \cdot), X(b, \cdot)\}$$

and

$$X\,(tb + (1-t)a, \cdot) \leq \operatorname{Max}\{X(a, \cdot), X(b, \cdot)\}$$

We conclude that :

$$\operatorname{Max}\{X(x, \cdot), X(y, \cdot)\} \leq \operatorname{Max}\{X(a, \cdot), X(b, \cdot)\}$$

And by multiplying both sides of the inequality by $\frac{\varphi((b-a)t)}{t}$ and integrating the resulting inequality with respect to $t \in [0, 1]$ we obtain.

$$\operatorname{Max}\{{}_{a^+}I_{\varphi}X(b, \cdot), {}_{b^-}I_{\varphi}X(a, \cdot)\} \leq \operatorname{Max}\{(X(a, \cdot), X(b, \cdot)\} \times \int_0^1 \frac{\varphi(b-a)t)}{t} dt$$

∎

References about the following results can be found in [22], [25] and [27].

Remark 2.2.

• For $\varphi(t) = t$, we get :

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{b-a} \int_a^b X(t, \cdot) dt \leq \operatorname{Max}\{(X(a, \cdot), X(b, \cdot)\} \tag{2.9}$$

• For $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, we get :

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{\alpha}{(b-a)^{\alpha}} \operatorname{Max}\{\int_a^b (b-t)^{\alpha-1} X(t, \cdot) dt, \int_a^b (t-a)^{\alpha-1} X(t, \cdot) dt\} \leq \operatorname{Max}\{(X(a, \cdot), X(b, \cdot)\} \tag{2.10}$$

• For $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we get :

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{\alpha}{k(b-a)^{\frac{\alpha}{k}}} \operatorname{Max}\{\int_a^b (b-t)^{\frac{\alpha}{k}-1} X(t,\cdot)dt, \int_a^b (t-a)^{\frac{\alpha}{k}-1} X(t,\cdot)dt\} \leq \operatorname{Max}\{(X(a,\cdot), X(b,\cdot)\} \tag{2.11}$$

Corollary 2.5. *Under the assumption of Theorem 2.4 with $\varphi(t) = t(b-t)^{\alpha-1}$, and X is a symmetric to $\frac{(a+b)}{2}$, then we have :*

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b t^{\alpha-1} X(t,\cdot)dt \leq \operatorname{Max}\{(X(a,\cdot), X(b,\cdot)\} \tag{2.12}$$

Corollary 2.6. *Under the assumption of Theorem 2.4 with $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$ and α in $[0.1]$, we have :*

(2.13)

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{\alpha-1}{\alpha(\exp(A)-1)} \operatorname{Max}\{_{a^+}I_\varphi X(b,\cdot),_{b^-} I_\varphi X(a,\cdot)\} \leq \operatorname{Max}\{(X(a,\cdot), X(b,\cdot)\}$$

where $A = \frac{\alpha-1}{\alpha}(b-a)$.

3. SOME TRAPEZOID TYPE INEQUALITIES FOR STOCHASTIC PROCESS

Lemma 3.1. *Let $X : I\times\Omega \to \mathbb{R}$ mean-square differentiable on $I^\circ(I \subset \mathbb{R})$ and $a, b \in I^0, a < b$. Then we have the following inequalities for generalized fractional integrals :*

(3.1)

$$X(a,\cdot)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[_{a^+}I_\varphi X(b,\cdot) +_{b^-} I_\varphi X(a,\cdot)\right] = \frac{(b-a)}{\Psi(1)}\int_0^1 \Psi(t)\left[X'(tb+(1-t)a,\cdot) - X'(ta+(1-t)b,\cdot)\right]dt$$

and :

$$X(a,\cdot)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[_{a^+}I_\varphi X(b,\cdot) +_{b^-} I_\varphi X(a,\cdot)\right] = \frac{(b-a)}{\Psi(1)}\int_0^1 \left[\Psi(1-t) - \Psi(t)\right] X'(ta+(1-t)b,\cdot)dt \tag{3.2}$$

Proof. By applying the integration by parts in integrals of the right part of our equality, we get :

$S_1 = \int_0^1 \left[\int_0^t \frac{\varphi((b-a)u)}{u}du\right] X'\left(tb+(1-t)a,\cdot\right)dt$

$= \left[\frac{X((tb+(1-t)a,\cdot)}{b-a}\int_0^t \frac{\varphi((b-a)u)}{u}du\right]_0^1 - \frac{1}{b-a}\int_a^b \frac{\varphi(x-a)}{x-a}X(x,\cdot)dx$

$= \frac{X(b,\cdot)}{(b-a)}\int_0^1 \frac{\varphi((b-a)u)}{u}du - \frac{1}{(b-a)}{}_{b^-}I_\varphi X(a,\cdot)$

And similarly :

$S_2 = \int_0^1 \left[\int_0^t \frac{\varphi((b-a)u)}{u}du\right] X'\left(ta+(1-t)b,\cdot\right)dt$

$= \left[\frac{X((ta+(1-t)b,\cdot)}{b-a}\int_0^t \frac{\varphi((b-a)u)}{u}du\right]_0^1 + \frac{1}{b-a}\int_a^b \frac{\varphi(b-x)}{b-x}X(x,\cdot)dx$

$= \frac{-X(a,\cdot)}{(b-a)}\int_0^1 \frac{\varphi((b-a)u)}{u}du + \frac{1}{(b-a)}{}_{a^+}I_\varphi X(b,\cdot)$

By subtracting S_2 from S_1 we obtain :

$\int_0^1 \Psi(t)\left[X'(tb+(1-t)a,\cdot) - X'(ta+(1-t)b,\cdot)\right]$

$= \Psi(1)\frac{(X(b,\cdot)+X(a,\cdot))}{(b-a)} - \frac{1}{(b-a)}\left[_{a^+}I_\varphi X(b,\cdot) +_{b^-} I_\varphi X(a,\cdot)\right]$

By dividing by $\Psi(1)$ and multiplying by $(b-a)$ we get the result.

∎

Remark 3.1.
• For $\varphi(t)=t$, we obtain :

$$X(a,\cdot)+X(b,\cdot)-\frac{2}{b-a}\int_a^b X(t,\cdot)dt=(b-a)\int_0^1(1-2t)X'(ta+(1-t)b,\cdot)dt$$
$$=(b-a)\int_0^1 t\left[X'(tb+(1-t)a,\cdot)-X'(ta+(1-t)b,\cdot)\right]dt$$

(see [30]).

• For $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, we obtain :
$X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{(b-a)^\alpha}\left[\int_a^b\left((b-t)^{\alpha-1}+(a-t)^{\alpha-1}\right)X(t,\cdot)dt\right]$
$=(b-a)\int_0^1 t^\alpha\left(X'(tb+(1-t)a,\cdot)-X'(ta+(1-t)b,\cdot)\right)dt$
$=(b-a)\int_0^1\left((1-t)^\alpha-t^\alpha\right)X(ta+(1-t)b,\cdot)dt$
(see [31]).

• For $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we get :
$X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{k(b-a)^{\frac{\alpha}{k}}}\int_a^b\left((b-t)^{\frac{\alpha}{k}-1}+(t-a)^{\frac{\alpha}{k}-1}\right)X(t,\cdot)dt$
$=(b-a)\int_0^1 t^{\frac{\alpha}{k}}\left(X'(tb+(1-t)a,\cdot)-X'(ta+(1-t)b,\cdot)\right)dt$
$=(b-a)\int_0^1\left((1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right)X'(ta+(1-t)b,\cdot)dt$
(see [29]).

Corollary 3.2. *Under the assumption of lemma 3.1 with $\varphi(t)=t(b-t)^{\alpha-1}$ and X is a symmetric to $\frac{(a+b)}{2}$, then we have :*
$X(a,\cdot)+X(b,\cdot)-\frac{2\alpha}{b^\alpha-a^\alpha}\int_a^b t^{\alpha-1}X(t,\cdot)dt$
$=\frac{(b-a)}{(b^\alpha-a^\alpha)}\int_0^1\left([b-(b-a)(1-t)]^\alpha-[b-(b-a)t]^\alpha\right)X'(ta+(1-t)b,\cdot)dt$
$=\frac{(b-a)}{(b^\alpha-a^\alpha)}\int_0^1\left(b^\alpha-[b-(b-a)t]^\alpha\right)\left[X'(tb+(1-t)a,\cdot)-X'(ta+(1-t)b,\cdot)\right]dt$

Corollary 3.3. *If in lemma 3.1, we get $\varphi(t)=\frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right),\alpha\in[0,1]$, then we have the following identity :*
$X(a,\cdot)+X(b,\cdot)-\frac{\alpha-1}{(\exp(A)-1)}\left[{}_{a^+}I_\varphi X(b,\cdot)+{}_{b^-}I_\varphi X(a,\cdot)\right]$
$=\frac{(b-a)}{(\exp(A)-1)}\int_0^1[\exp(A(1-t)-\exp(At)]X'(ta+(1-t)b,\cdot)dt$
Where $A=\frac{\alpha-1}{\alpha}(b-a)$. (see [26]*)*

Theorem 3.4. *Let $X:I\times\Omega-\mathbb{R}$ be a stochastic process on I° $(I\subset\mathbb{R})anda,b\in I^\circ$ with $a<b$. X is mean square differentiable, and $|X'|$ is P-convex on $[a,b]$, then we have the following inequality :*

$$\left|X(a,\cdot)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[{}_{a^+}I_\varphi X(b,\cdot)+{}_{b^-}I_\varphi X(a,\cdot)\right]\right|\leq\frac{b-a}{\Psi(1)}\int_0^1|\Psi(1-t)-\Psi(t)|dt\left(\left|X'(a,\cdot)\right|+\left|X'(b,\cdot)\right|\right)\tag{3.3}$$

Proof. By using the lemma 3.1 we get :

$$\left|X(a,\cdot)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[{}_{a^+}I_\varphi X(b,\cdot)+{}_{b^-}I_\varphi X(a,\cdot)\right]\right|$$
$$\leq\frac{(b-a)}{\Psi(1)}\int_0^1\left|\left[\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right]\right||X'(ta+(1-t)b,\cdot)|\,dt$$

And by using the is P-convexity of $|X'|$, the proof is complete.

∎

References for these results can be found in [27] and [29].

Remark 3.2.
- For $\varphi(t) = t$, we have :

$$\left| X(a,\cdot) + X(b,\cdot) - \frac{2}{b-a}\int_a^b X(t,\cdot)dt \right| \leq \frac{(b-a)}{2}\left(|X'(a,\cdot)| + |X'(b,\cdot)|\right)$$

- For $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, we have :

$\left| X(a,\cdot) + X(b,\cdot) - \frac{\alpha}{(b-a)^\alpha}\int_a^b \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] X(t,\cdot)\, dt \right|$
$\leq \frac{2(b-a)}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\left(|X'(a,\cdot)| + |X'(b,\cdot)|\right)$

- For $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have :

$\left| X(a,\cdot) + X(b,\cdot) - \frac{\alpha}{k(b-a)^{\frac{\alpha}{k}}}\int_a^b \left[(b-t)^{\frac{\alpha}{k}-1} + (t-a)^{\frac{\alpha}{k}-1}\right] X(t,\cdot)\, dt \right|$
$\leq \frac{2(b-a)}{\left(\frac{\alpha}{k}+1\right)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right)\left(|X'(a,\cdot)| + |X'(b,\cdot)|\right)$

Corollary 3.5. *Under the assumption of Theorem 3.4 with $\varphi(t) = t(b-t)^{\alpha-1}$ and X is a symmetric to $\frac{(a+b)}{2}$, we have :*

$\left| X(a,\cdot) + X(b,\cdot) - \frac{2\alpha}{b^\alpha - a^\alpha}\int_a^b X(t,\cdot)dt \right|$
$\leq \frac{2}{(b-a)(\alpha+1)}\left[a^{\alpha+1} + b^{\alpha+1} - \frac{(a+b)^{\alpha+1}}{2^\alpha}\right]\left[|X'(a,\cdot)| + |X'(b,\cdot)|\right]$

A similair result in the case of $\alpha = 1$ can be found in the work of Dragomir and Agarwal (see [30]).

Corollary 3.6. *Using theorem 3.4, and for $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$ where $\alpha \in [0,1]$, we obtain :*

$\left| X(a,\cdot) + X(b,\cdot) - \frac{\alpha-1}{(1-\exp(A))}\left[{}_{a^+}I_\varphi X(b,\cdot) +{}_{b^-} I_\varphi X(a,\cdot)\right] \right|$
$\leq \frac{8(b-a)\exp(\frac{A}{2})}{A(\exp(A)-1)}\sinh^2\left(\frac{A}{4}\right)\left(|X'(a,\cdot)| + |X'(b,\cdot)|\right)$
With $A = \frac{\alpha-1}{\alpha}(b-a)$.(see [26]*)*

Theorem 3.7. *Let $X : I \times \Omega - \mathbb{R}$ be a stochastic process on I° $(I \subset \mathbb{R}) anda, b \in I^\circ$ with $a < b$. X is mean square differentiable, and $|X'|$ is Quasi-convex on $[a,b]$, then we have the following inequality :*

(3.4)
$$\left| X(a,\cdot) + X(b,\cdot) - \frac{1}{\Psi(1)}\left[{}_{a^+}I_\varphi X(b,\cdot) +{}_{b^-} I_\varphi X(a,\cdot)\right] \right| \leq \frac{b-a}{\Psi(1)}\int_0^1 |\Psi(1-t) - \Psi(t)|dt\, \text{Max}\{|X'(a,\cdot)|, |X'(b,\cdot)|\}$$

Proof.

By using the lemma 3.1 we get :

$$\left| X(a,\cdot) + X(b,\cdot) - \frac{1}{\Psi(1)}\left[{}_{a^+}I_\varphi X(b,\cdot) +{}_{b^-} I_\varphi X(a,\cdot)\right] \right|$$
$$\leq \frac{(b-a)}{\Psi(1)}\int_0^1 \left| \left[\int_0^{1-t} \frac{\varphi((b-a)u)}{u}du - \int_0^t \frac{\varphi((b-a)u)}{u}du\right] \right| |X'(ta + (1-t)b,\cdot)|\, dt$$

And by using the quasi-convexity of $|X'|$, we get the result.
∎

Remark 3.3.
- For $\varphi(t) = t$, we have :

$\left| X(a,\cdot) + X(b,\cdot) - \frac{2}{b-a}\int_a^b X(t,\cdot)dt \right| \leq \frac{(b-a)}{2}\text{Max}\{|X'(a,\cdot)|, |X'(b,\cdot)|\}$

• For $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, we have :
$\left|X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{(b-a)^\alpha}\int_a^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]X\left(t,\cdot\right)dt\right|$
$\leq\frac{2(b-a)}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\operatorname{Max}\{\left|X'(a,\cdot)\right|,\left|X'(b,\cdot)\right|\}$
• For $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we get :
$\left|X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{k(b-a)^{\frac{\alpha}{k}}}\int_a^b\left[(b-t)^{\frac{\alpha}{k}-1}+(t-a)^{\frac{\alpha}{k}-1}\right]X\left(t,\cdot\right)dt\right|$
$\leq\frac{2(b-a)}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\operatorname{Max}\{\left|X'(a,\cdot)\right|,\left|X'(b,\cdot)\right|\}$

Corollary 3.8. *Under the assumption of theorem 3.7 with $\varphi(t)=t(b-t)^{\alpha-1}$ and X is a symmetric to $\frac{(a+b)}{2}$, we get :*
$\left|X(a,\cdot)+X(b,\cdot)-\frac{2\alpha}{b^\alpha-a^\alpha}\int_a^b X(t,\cdot)dt\right|$
$\leq\frac{2}{(b-a)(\alpha+1)}\left[a^{\alpha+1}+b^{\alpha+1}-\frac{(a+b)^{\alpha+1}}{2^\alpha}\right]\operatorname{Max}\{\left|X'(a,\cdot)\right|,\left|X'(b,\cdot)\right|\}$

Corollary 3.9. *Under the assumption of theorem 3.7, we get for $\varphi(t)=\frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, with $\alpha\in[0,1]$, we obtain :*
$\left|X(a,\cdot)+X(b,\cdot)-\frac{\alpha-1}{(1-\exp(A))}\left[{}_{a^+}I_\varphi X(b,\cdot)+{}_{b^-}I_\varphi X(a,\cdot)\right]\right|$
$\leq\frac{8(b-a)\exp(\frac{A}{2})}{A(\exp(A)-1)}\sinh^2\left(\frac{A}{4}\right)\operatorname{Max}\{\left|X'(a,\cdot)\right|,\left|X'(b,\cdot)\right|\}$
With $A=\frac{\alpha-1}{\alpha}(b-a)$.

Theorem 3.10. *Let $X:I\times\Omega\to\mathbb{R}$ be a mean square differentiable stochastic process on I° and $a,b\in I$ with $a<b$. If $|X'|^q$ is P-convex on $[a,b]$ for some $q>1$, then the following inequality for fractional integral holds :*

(3.5)
$$\left|X(a,\cdot)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[{}_{a^+}I_\varphi X(b,\cdot)+{}_{b^-}I_\varphi X(a,\cdot)\right]\right|\leq\frac{(b-a)}{\Psi(1)}\left(\int_0^1|\Psi(1-t)-\Psi(t)|^p dt\right)^{\frac{1}{p}}\left(\left|X'(a,\cdot)\right|^q+\left|X'(b,\cdot)\right|^q\right)^{\frac{1}{q}}$$

Where : $\frac{1}{p}+\frac{1}{q}=1$

Proof. Using Lemma 3.1 and HÃűlder's inequality, we get :
$$\left|X\left(a,\cdot\right)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[{}_{a^+}I_\varphi X(b,\cdot)+{}_{b^-}I_\varphi X(a,\cdot)\right]\right|$$
$$\leq\frac{(b-a)}{\Psi(1)}\left(\int_0^1\left|\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right|^p\right)^{\frac{1}{p}}\left|X'(ta+(1-t)b,\cdot)\right|^{\frac{1}{q}}dt$$
with the P-convexity of $|X'|^q$ we get the result.
∎

Remark 3.4.
• For $\varphi(t)=t$, we have :

$$\left|X(a,\cdot)+X(b,\cdot)-\frac{2}{b-a}\int_a^b X(t,\cdot)dt\right|\leq(b-a)\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}\left(\left|X'(a,\cdot)\right|^q+\left|X'(b,\cdot)\right|^q\right)^{\frac{1}{q}}$$
$$\leq\frac{(b-a)}{(p+1)^{\frac{1}{p}}}\left(\left|X'(a,\cdot)\right|^q+\left|X'(b,\cdot)\right|^q\right)^{\frac{1}{q}}$$

• For $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, we have :

$\left|x(a,\cdot)+X(b,\cdot)-\frac{\alpha}{(b-a)^{\alpha}}\int_a^b\left((b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right)X(t,\cdot)dt\right|$
$\leq(b-a)\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(|X'(a,\cdot)|^q+|X'(b,\cdot)|^q\right)^{\frac{1}{q}}$

And by using that for $\alpha\in[0,1],\forall t_1,t_2\in[0,1]$ we have : $|t_1^{\alpha}-t_2^{\alpha}|\leq|t_1-t_2|^{\alpha}$

Therefor we obtain :

$\left|X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{(b-a)^{\alpha}}\int_a^b\left((b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right)X(t,\cdot)dt\right|$
$\leq(b-a)\left(\int_0^1|1-2t|^{2p}dt\right)^{\frac{1}{p}}\left(|X'(a,\cdot)|^q+|X'(b,\cdot)|^q\right)^{\frac{1}{q}}$
$\leq\frac{(b-a)}{(\alpha p+1)^{\frac{1}{p}}}\left(|X'(a,\cdot)|^q+|X'(b,\cdot)|^q\right)^{\frac{1}{q}}$

• For $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have :

$\left|X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{k(b-a)^{\frac{\alpha}{k}}}\int_a^b\left((b-t)^{\frac{\alpha}{k}-1}+(t-a)^{\frac{\alpha}{k}-1}\right)X(t,\cdot)dt\right|$
$\leq\frac{(b-a)}{\left(\frac{\alpha p}{k}+1\right)^{\frac{1}{p}}}\left(|X'(a,\cdot)|^q+|X'(b,\cdot)|^q\right)^{\frac{1}{q}}$

Corollary 3.11. *Under the assumption of Theorem 3.10 with $\varphi(t)=t(b-t)^{\alpha-1}$ and X is a symmetric to $\frac{(a+b)}{2}$, then we have :*

$\left|X(a,\cdot)+X(b,\cdot)-\frac{2\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b t^{\alpha-1}X(t,\cdot)dt\right|$
$\leq\frac{2^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}{(b^{\alpha}-a^{\alpha})(\alpha p+1)^{\frac{1}{p}}}\left[a^{\alpha p+1}+b^{\alpha p+1}-\frac{(a+b)^{\alpha p+1}}{2^{\alpha p}}\right]^{\frac{1}{p}}\left(|X'(a,\cdot)|^q+|X'(b,\cdot)|^q\right)^{\frac{1}{q}}$

Proof. We take $\varphi(t)=t(b-t)^{\alpha-1}$ and X is a symmetric to $\frac{(a+b)}{2}$, then we get :

$\left|X(a,\cdot)+X(b,\cdot)-\frac{2\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b t^{\alpha-1}X(t,\cdot)dt\right|$
$\leq\frac{(b-a)\alpha}{(b^{\alpha}-a^{\alpha})}\left(\int_0^1|\Psi(1-t)-\Psi(t)|^p dt\right)^{\frac{1}{p}}\left(|X'(a,\cdot)|^q+|X'(b,\cdot)|^q\right)^{\frac{1}{q}}$

By computing the right-side integral in the previous inequality, using the inequality $A\geq B>0$ and $q>1$, $(A-B)^q\leq A^q-B^q$, we get :

$$\begin{aligned}\int_0^1|\Psi(1-t)-\Psi(t)|^p dt&=\frac{1}{\alpha^p}\int_0^1\left|[b-(b-a)(1-t)]^{\alpha}-[b-(b-a)t]^{\alpha}\right|^p dt\\&=\frac{1}{\alpha^p(b-a)}\int_a^b\left|u^{\alpha}-[a+b-u]^{\alpha}\right|^p du\end{aligned}$$

$=\frac{1}{\alpha^p(b-a)}\int_a^{\frac{a+b}{2}}\left([a+b-u]^{\alpha}-u^{\alpha}\right)^p du+\frac{1}{\alpha^p(b-a)}\int_{\frac{a+b}{2}}^b\left(u^{\alpha}-[a+b-u]^{\alpha}\right)^p du$
$\leq\frac{1}{\alpha^p(b-a)}\int_a^{\frac{a+b}{2}}\left([a+b-u]^{\alpha p}-u^{\alpha p}\right)du+\frac{1}{\alpha^p(b-a)}\int_{\frac{a+b}{2}}^b\left(u^{\alpha p}-[a+b-u]^{\alpha p}\right)du$
$=\frac{2}{\alpha^p(b-a)(\alpha p+1)}\left[a^{\alpha p+1}+b^{\alpha p+1}-\frac{(a+b)^{\alpha p+1}}{2^{\alpha p}}\right]$

Witch gives us the result.

∎

Theorem 3.12. *Let $X:I\times\Omega\to\mathbb{R}$ be a mean square differentiable stochastic process on I° and $a,b\in I$ with $a<b$. If $|X'|^q$ is Quasi-convex on $[a,b]$ for some $q>1$, then the following inequality for fractional integral holds :*

$\left|X(a,\cdot)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[{}_{a^+}I_{\varphi}X(b,\cdot)+{}_{b^-}I_{\varphi}X(a,\cdot)\right]\right|$
$\leq\frac{(b-a)}{\Psi(1)}\left(\int_0^1|\Psi(1-t)-\Psi(t)|^p dt\right)^{\frac{1}{p}}\mathrm{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$

Where : $\frac{1}{p}+\frac{1}{q}=1.$

Proof. Using lemma 3.1 and HÃűlder's inequality, we get :

$$\left|X(a,\cdot)+X(b,\cdot)-\frac{1}{\Psi(1)}\left[{}_{a^+}I_{\varphi}X(b,\cdot)+{}_{b^-}I_{\varphi}X(a,\cdot)\right]\right|$$
$$\leq \frac{(b-a)}{\Psi(1)}\left(\int_0^1\left|\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right|^p\right)^{\frac{1}{p}}|X'(ta+(1-t)b,\cdot)|^{\frac{1}{q}}dt$$

with the Quasi-convexity of $|X'|^q$ we get the result.
∎

Remark 3.5.
• For $\varphi(t)=t$, we obtain :

$$\left|X(a,\cdot)+X(b,\cdot)-\frac{2}{b-a}\int_a^b X(t,\cdot)dt\right|\leq(b-a)\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$$
$$\leq\frac{(b-a)}{(p+1)^{\frac{1}{p}}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$$

• For $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, we obtain :

$\left|X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{(b-a)^\alpha}\int_a^b\left((b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right)X(t,\cdot)dt\right|$

$\leq(b-a)\left(\int_0^1|(1-t)^\alpha-t^\alpha|^p\,dt\right)^{\frac{1}{p}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$

And by using that for $\alpha\in[0,1],\forall t_1,t_2\in[0,1]$ we have : $|t_1^\alpha-t_2^\alpha|\leq|t_1-t_2|^\alpha$

Therefor we obtain :

$\left|x(a,\cdot)+X(b,\cdot)-\frac{\alpha}{(b-a)^\alpha}\int_a^b\left((b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right)X(t,\cdot)dt\right|$

$\leq(b-a)\left(\int_0^1|1-2t|^{2p}dt\right)^{\frac{1}{p}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$

$\leq\frac{(b-a)}{(\alpha p+1)^{\frac{1}{p}}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$

• For $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we obtain :

$\left|X(a,\cdot)+X(b,\cdot)-\frac{\alpha}{k(b-a)^{\frac{\alpha}{k}}}\int_a^b\left((b-t)^{\frac{\alpha}{k}-1}+(t-a)^{\frac{\alpha}{k}-1})X(t,\cdot)\right)dt\right|$

$\leq\frac{(b-a)}{\left(\frac{\alpha p}{k}+1\right)^{\frac{1}{p}}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$

Corollary 3.13. *Under the assumption of theorem 3.12 with* $\varphi(t)=t(b-t)^{\alpha-1}$ *and* X *is a symmetric to* $\frac{(a+b)}{2}$*, then we have :*

$\left|X(a,\cdot)+X(b,\cdot)-\frac{2\alpha}{b^\alpha-a^\alpha}\int_a^b t^{\alpha-1}X(t,\cdot)dt\right|$

$\leq\frac{2^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}{(b^\alpha-a^\alpha)(\alpha p+1)^{\frac{1}{p}}}\left[a^{\alpha p+1}+b^{\alpha p+1}-\frac{(a+b)^{\alpha p+1}}{2^{\alpha p}}\right]^{\frac{1}{p}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$

Proof. We take $\varphi(t)=t(b-t)^{\alpha-1}$, then we get :

$\left|X(a,\cdot)+X(b,\cdot)-\frac{2\alpha}{b^\alpha-a^\alpha}\int_a^b t^{\alpha-1}X(t,\cdot)dt\right|$

$\leq\frac{(b-a)\alpha}{(b^\alpha-a^\alpha)}\left(\int_0^1|\Psi(1-t)-\Psi(t)|^p dt\right)^{\frac{1}{p}}\text{Max}\{|X'(a,\cdot)|,|X'(b,\cdot)|\}$

By computing right-side integral in the previous inequality, using the inequality $A\geq B>0$ and $q>1$, $(A-B)^q\leq A^q-B^q$, we obtain :

$$\int_0^1|\Psi(1-t)-\Psi(t)|^p dt=\frac{1}{\alpha^p}\int_0^1|[b-(b-a)(1-t)]^\alpha-[b-(b-a)t]^\alpha|^p\,dt$$

$$= \frac{1}{\alpha^p(b-a)} \int_a^b |u^\alpha - [a+b-u]^\alpha|^p \, du$$
$$= \frac{1}{\alpha^p(b-a)} \int_a^{\frac{a+b}{2}} ([a+b-u]^\alpha - u^\alpha)^p \, du + \frac{1}{\alpha^p(b-a)} \int_{\frac{a+b}{2}}^b (u^\alpha - [a+b-u]^\alpha)^p \, du$$
$$\leq \frac{1}{\alpha^p(b-a)} \int_a^{\frac{a+b}{2}} ([a+b-u]^{\alpha p} - u^{\alpha p}) \, du + \frac{1}{\alpha^p(b-a)} \int_{\frac{a+b}{2}}^b (u^{\alpha p} - [a+b-u]^{\alpha p}) \, du$$
$$= \frac{2}{\alpha^p(b-a)(\alpha p+1)} \left[a^{\alpha p+1} + b^{\alpha p+1} - \frac{(a+b)^{\alpha p+1}}{2^{\alpha p}}\right]$$

witch ends the proof.

∎

4. CONCLUSION

In this article, we have established new Hermite–Hadamard type inequalities via P-convex and Quasi-convex stochastic processes in the framework of a generalized fractional integral operator, as well as some other results for well known fractional integral operators, that estimate the difference between the value of the fractional integral and the right side of such inequality, as well as a refinement of the aforementioned inequality hoping that it will inspire as starting point, interested the researchers, to derive some other type of inequalities for other type of convex stochastic processes.

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