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REVERSES OF THE TRIANGLE INEQUALITY IN INNER PRODUCT SPACES

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ABSTRACT. Some new reverses for the generalised triangle inequality in inner product spaces are given. Applications in connection to the Schwarz inequality and for vector-valued integrals are provided as well.

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1. INTRODUCTION

In 1966, J.B. Diaz and F.T. Metcalf [1] proved the following reverse of the triangle inequality:

Theorem 1.1. *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i \in H\backslash\{0\}$, $i \in \{1,\dots,n\}$ satisfy*

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \quad i \in \{1,\dots,n\}. \tag{1.1}$$

Then

$$r\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.2}$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.3}$$

A generalisation of this result for orthonormal families is incorporated in the following result [1].

Theorem 1.2. *Let $a_1,\dots,a_n$ be orthonormal vectors in H. Suppose the vectors $x_1,\dots,x_n \in H\backslash\{0\}$ satisfy*

$$0 \leq r_k \leq \frac{\operatorname{Re}\langle x_i, a_k\rangle}{\|x_i\|}, \quad i \in \{1,\dots,n\},\ k \in \{1,\dots,m\}. \tag{1.4}$$

Then

$$\left(\sum_{k=1}^{m} r_k^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.5}$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m} r_k a_k. \tag{1.6}$$

Similar results valid for semi-inner products may be found in [3] and [4].

For other inequalities related to the triangle inequality, see Chapter XVII of the book [5] and the references therein.

The main aim of this paper is to point out new reverses of the generalised triangle inequality which naturally complement the above results due to Diaz and Metcalf. New reverses for the Schwarz inequality are provided. Applications for vector-valued integrals in Hilbert spaces are pointed out as well.

2. SOME INEQUALITIES OF DIAZ-METCALF TYPE

The following result with a natural geometrical meaning holds:

Theorem 2.1. *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ and $\rho \in (0,1)$. If $x_i \in H$, $i \in \{1,\dots,n\}$ are such that*

$$\|x_i - a\| \leq \rho \ \text{for each}\ i \in \{1,\dots,n\}, \tag{2.1}$$

then we have the inequality

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|, \tag{2.2}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right) a. \tag{2.3}$$

Proof. From (2.1) we have

$$\|x_i\|^2 - 2\operatorname{Re}\langle x_i, a\rangle + 1 \le \rho^2,$$

giving

$$\|x_i\|^2 + 1 - \rho^2 \le 2\operatorname{Re}\langle x_i, a\rangle, \tag{2.4}$$

for each $i \in \{1,\dots,n\}$.

Dividing by $\sqrt{1-\rho^2} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \le \frac{2\operatorname{Re}\langle x_i, a\rangle}{\sqrt{1-\rho^2}}, \tag{2.5}$$

for each $i \in \{1,\dots,n\}$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha} + q\alpha \ge 2\sqrt{pq}, \quad p, q \ge 0,\ \alpha > 0 \tag{2.6}$$

we have

$$2\|x_i\| \le \frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \tag{2.7}$$

and thus, by (2.5) and (2.7), we deduce

$$\frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|} \ge \sqrt{1-\rho^2},$$

for each $i \in \{1,\dots,n\}$. Applying Theorem 1.1 for $r = \sqrt{1-\rho^2}$, we deduce the desired inequality (2.2). ∎

In a similar manner to the one used in the proof of Theorem 2.1 and by the use of the Diaz-Metcalf inequality incorporated in Theorem 1.2, we can also prove the following result:

Theorem 2.2. *Let $a_1,\dots,a_n$ be orthonormal vectors in H. Suppose the vectors $x_1,\dots,x_n \in H\backslash\{0\}$ satisfy*

$$\|x_i - a_k\| \le \rho_k \text{ for each } i \in \{1,\dots,n\},\ k \in \{1,\dots,m\}, \tag{2.8}$$

where $\rho_k \in (0,1)$. Then we have the following reverse of the triangle inequality

$$\left(m - \sum_{k=1}^{m}\rho_k^2\right)^{1/2}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.9}$$

The equality holds in (2.9) if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}\left(1-\rho_k^2\right)^{1/2} a_k. \tag{2.10}$$

The following result with a different geometrical meaning may be stated as well:

Theorem 2.3. *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ and $M\geq m>0$. If $x_i\in H$, $i\in\{1,\dots,n\}$ are such that either*

$$\operatorname{Re}\langle Ma-x_i,x_i-ma\rangle\geq 0 \tag{2.11}$$

or, equivalently,

$$\left\|x_i-\frac{M+m}{2}\cdot a\right\|\leq\frac{1}{2}(M-m) \tag{2.12}$$

holds for each $i\in\{1,\dots,n\}$, then we have the inequality

$$\frac{2\sqrt{mM}}{m+M}\sum_{i=1}^{n}\|x_i\|\leq\left\|\sum_{i=1}^{n}x_i\right\|, \tag{2.13}$$

or, equivalently,

$$(0\leq)\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}}\left\|\sum_{i=1}^{n}x_i\right\|. \tag{2.14}$$

The equality holds in (2.13) (or in (2.14)) if and only if

$$\sum_{i=1}^{n}x_i=\frac{2\sqrt{mM}}{m+M}\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{2.15}$$

Proof. Firstly, we remark that if $x,z,Z\in H$, then the following statements are equivalent

(i) $\operatorname{Re}\langle Z-x,x-z\rangle\geq 0$

and

(ii) $\left\|x-\frac{Z+z}{2}\right\|\leq\frac{1}{2}\|Z-z\|$.

Using this fact, one may simply realize that (2.11) and (2.12) are equivalent.

Now, from (2.11), we get

$$\|x_i\|^2+mM\leq(M+m)\operatorname{Re}\langle x_i,a\rangle,$$

for any $i\in\{1,\dots,n\}$. Dividing this inequality by $\sqrt{mM}>0$, we deduce the following inequality that will be used in the sequel

$$\frac{\|x_i\|^2}{\sqrt{mM}}+\sqrt{mM}\leq\frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x_i,a\rangle, \tag{2.16}$$

for each $i\in\{1,\dots,n\}$.

Using the inequality (2.6) from Theorem 2.1, we also have

$$2\|x_i\|\leq\frac{\|x_i\|^2}{\sqrt{mM}}+\sqrt{mM}, \tag{2.17}$$

for each $i\in\{1,\dots,n\}$.

Utilizing (2.16) and (2.17), we may conclude with the following inequality

$$\|x_i\|\leq\frac{M+m}{2\sqrt{mM}}\operatorname{Re}\langle x_i,a\rangle,$$

which is equivalent to

$$\frac{2\sqrt{mM}}{m+M}\leq\frac{\operatorname{Re}\langle x_i,a\rangle}{\|x_i\|} \tag{2.18}$$

for any $i\in\{1,\dots,n\}$.

Finally, on applying the Diaz-Metcalf result in Theorem 1.1 for $r = \frac{2\sqrt{mM}}{m+M}$, we deduce the desired conclusion.

The equivalence between (2.13) and (2.14) follows by simple calculation and we omit the details. ∎

Finally, by the use of Theorem 1.2 and a similar technique to that employed in the proof of Theorem 2.3, we may state the following result:

Theorem 2.4. *Let $a_1, \dots, a_n$ be orthonormal vectors in H. Suppose the vectors $x_1, \dots, x_n \in H \backslash \{0\}$ satisfy*

$$\operatorname{Re} \langle M_k a_k - x_i, x_i - \mu_k a_k \rangle \geq 0, \tag{2.19}$$

or, equivalently,

$$\left\| x_i - \frac{M_k + \mu_k}{2} a_k \right\| \leq \frac{1}{2} (M_k - \mu_k), \tag{2.20}$$

for any $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, where $M_k \geq \mu_k > 0$ for each $k \in \{1, \dots, m\}$.

Then we have the inequality

$$2 \left(\sum_{k=1}^{m} \frac{\mu_k M_k}{(\mu_k + M_k)^2} \right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.21}$$

The equality holds in (2.21) iff

$$\sum_{i=1}^{n} x_i = 2 \left(\sum_{i=1}^{n} \|x_i\| \right) \sum_{k=1}^{m} \frac{\sqrt{\mu_k M_k}}{\mu_k + M_k} a_k. \tag{2.22}$$

3. Some New Reverses of the Triangle Inequality

In this section we establish some additive reverses of the generalised triangle inequality in real or complex inner product spaces.

The following result holds:

Theorem 3.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e, x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $k_i \geq 0$, $i \in \{1, \dots, n\}$, are such that*

$$\|x_i\| - \operatorname{Re} \langle e, x_i \rangle \leq k_i \ \text{ for each } \ i \in \{1, \dots, n\}, \tag{3.1}$$

then we have the inequality

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} k_i. \tag{3.2}$$

The equality holds in (3.2) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \sum_{i=1}^{n} k_i \tag{3.3}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i \right) e. \tag{3.4}$$

Proof. If we sum in (3.1) over i from 1 to $n,$ then we get

$$\sum_{i=1}^{n}\|x_i\| \le \operatorname{Re}\left\langle e,\sum_{i=1}^{n}x_i\right\rangle + \sum_{i=1}^{n}k_i. \tag{3.5}$$

By Schwarz's inequality for e and $\sum_{i=1}^{n}x_i,$ we have

$$\begin{aligned}\operatorname{Re}\left\langle e,\sum_{i=1}^{n}x_i\right\rangle &\le \left|\operatorname{Re}\left\langle e,\sum_{i=1}^{n}x_i\right\rangle\right| \\ &\le \left|\left\langle e,\sum_{i=1}^{n}x_i\right\rangle\right| \le \|e\|\left\|\sum_{i=1}^{n}x_i\right\| = \left\|\sum_{i=1}^{n}x_i\right\|.\end{aligned} \tag{3.6}$$

Making use of (3.5) and (3.6), we deduce the desired inequality (3.1).

If (3.3) and (3.4) hold, then

$$\left\|\sum_{i=1}^{n}x_i\right\| = \left|\sum_{i=1}^{n}\|x_i\| - \sum_{i=1}^{n}k_i\right|\|e\| = \sum_{i=1}^{n}\|x_i\| - \sum_{i=1}^{n}k_i,$$

and the equality in the second part of (3.2) holds true.

Conversely, if the equality holds in (3.2), then, obviously (3.3) is valid and we need only to prove (3.4).

Now, if the equality holds in (3.2) then it must hold in (3.1) for each $i\in\{1,\dots,n\}$ and also must hold in any of the inequalities in (3.6).

It is well known that in Schwarz's inequality $|\langle u,v\rangle| \le \|u\|\|v\|$ $(u,v\in H)$ the case of equality holds iff there exists a $\lambda\in\mathbb{K}$ such that $u=\lambda v$. We note that in the weaker inequality $\operatorname{Re}\langle u,v\rangle \le \|u\|\|v\|$ the case of equality holds iff $\lambda\ge 0$ and $u=\lambda v.$

Consequently, the equality holds in all inequalities (3.6) simultaneously iff there exists a $\mu\ge 0$ with

$$\mu e = \sum_{i=1}^{n}x_i. \tag{3.7}$$

If we sum the equalities in (3.1) over i from 1 to $n,$ then we deduce

$$\sum_{i=1}^{n}\|x_i\| - \operatorname{Re}\left\langle e,\sum_{i=1}^{n}x_i\right\rangle = \sum_{i=1}^{n}k_i. \tag{3.8}$$

Replacing $\sum_{i=1}^{n}\|x_i\|$ from (3.7) into (3.8), we deduce

$$\sum_{i=1}^{n}\|x_i\| - \mu\|e\|^2 = \sum_{i=1}^{n}k_i,$$

from where we get $\mu = \sum_{i=1}^{n}\|x_i\| - \sum_{i=1}^{n}k_i$. Using (3.7), we deduce (3.4) and the theorem is proved. ∎

If we turn our attention to the case of orthogonal families, then we may state the following result as well.

Theorem 3.2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{e_k\}_{k\in\{1,\dots,m\}}$ a family of orthonormal vectors in H, $x_i\in H$, $M_{i,k}\ge 0$ for $i\in\{1,\dots,n\}$ and $k\in\{1,\dots,m\}$ such that*

$$\|x_i\| - \operatorname{Re}\langle e_k,x_i\rangle \le M_{ik} \ \textit{ for each } \ i\in\{1,\dots,n\},\ k\in\{1,\dots,m\}. \tag{3.9}$$

Then we have the inequality

$$\sum_{i=1}^{n}\|x_i\| \le \frac{1}{\sqrt{m}}\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m}M_{ik}. \tag{3.10}$$

The equality holds true in (3.10) if and only if

$$\sum_{i=1}^{n}\|x_i\| \ge \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m}M_{ik} \tag{3.11}$$

and

$$\sum_{i=1}^{n}x_i = \left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m}M_{ik}\right)\sum_{k=1}^{m}e_k. \tag{3.12}$$

Proof. If we sum over i from 1 to n in (3.9), then we obtain

$$\sum_{i=1}^{n}\|x_i\| \le \operatorname{Re}\left\langle e_k, \sum_{i=1}^{n}x_i\right\rangle + \sum_{i=1}^{n}M_{ik},$$

for each $k \in \{1,\dots,m\}$. Summing these inequalities over k from 1 to $m,$ we deduce

$$\sum_{i=1}^{n}\|x_i\| \le \frac{1}{m}\operatorname{Re}\left\langle \sum_{k=1}^{m}e_k, \sum_{i=1}^{n}x_i\right\rangle + \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m}M_{ik}. \tag{3.13}$$

By Schwarz's inequality for $\sum_{k=1}^{m}e_k$ and $\sum_{i=1}^{n}x_i$ we have

$$\begin{aligned}\operatorname{Re}\left\langle \sum_{k=1}^{m}e_k, \sum_{i=1}^{n}x_i\right\rangle &\le \left|\operatorname{Re}\left\langle \sum_{k=1}^{m}e_k, \sum_{i=1}^{n}x_i\right\rangle\right| \\ &\le \left|\left\langle \sum_{k=1}^{m}e_k, \sum_{i=1}^{n}x_i\right\rangle\right| \\ &\le \left\|\sum_{k=1}^{m}e_k\right\|\left\|\sum_{i=1}^{n}x_i\right\| \\ &= \sqrt{m}\left\|\sum_{i=1}^{n}x_i\right\|,\end{aligned} \tag{3.14}$$

since, obviously,

$$\left\|\sum_{k=1}^{m}e_k\right\| = \sqrt{\left\|\sum_{k=1}^{m}e_k\right\|^2} = \sqrt{\sum_{k=1}^{m}\|e_k\|^2} = \sqrt{m}.$$

Making use of (3.13) and (3.14), we deduce the desired inequality (3.10).

If (3.11) and (3.12) hold, then

$$\begin{aligned}\frac{1}{\sqrt{m}}\left\|\sum_{i=1}^{n}x_i\right\| &= \left|\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m}M_{ik}\right|\left\|\sum_{k=1}^{m}e_k\right\| \\ &= \frac{\sqrt{m}}{\sqrt{m}}\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m}M_{ik}\right) \\ &= \sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m}M_{ik},\end{aligned}$$

and the equality in (3.10) holds true.

Conversely, if the equality holds in (3.10), then, obviously (3.11) is valid.

Now if the equality holds in (3.10), then it must hold in (3.9) for each $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ and also must hold in any of the inequalities in (3.14).

It is well known that in Schwarz's inequality $\operatorname{Re}\langle u, v\rangle \leq \|u\| \|v\|$, the equality occurs iff $u = \lambda v$ with $\lambda \geq 0$, consequently, the equality holds in all inequalities (3.14) simultaneously iff there exists a $\mu \geq 0$ with

$$\mu \sum_{k=1}^{m} e_k = \sum_{i=1}^{n} x_i. \tag{3.15}$$

If we sum the equality in (3.9) over i from 1 to n and k from 1 to m, then we deduce

$$m \sum_{i=1}^{n} \|x_i\| - \operatorname{Re}\left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle = \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}. \tag{3.16}$$

Replacing $\sum_{i=1}^{n} x_i$ from (3.15) into (3.16), we deduce

$$m \sum_{i=1}^{n} \|x_i\| - \mu \sum_{k=1}^{m} \|e_k\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}$$

giving

$$\mu = \sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}.$$

Using (3.15), we deduce (3.12) and the theorem is proved. ∎

4. FURTHER REVERSES OF THE TRIANGLE INEQUALITY

In this section we point out different additive reverses of the generalised triangle inequality under simpler conditions for the vectors involved.

The following result holds:

Theorem 4.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e, x_i \in H$, $i \in \{1, \ldots, n\}$ with $\|e\| = 1$. If $\rho \in (0, 1)$ and x_i, $i \in \{1, \ldots, n\}$ are such that*

$$\|x_i - e\| \leq \rho \text{ for each } i \in \{1, \ldots, n\}, \tag{4.1}$$

then we have the inequality

$$\begin{aligned}(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| &\leq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1 + \sqrt{1-\rho^2}\right)} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \\ &\left(\leq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1 + \sqrt{1-\rho^2}\right)} \left\| \sum_{i=1}^{n} x_i \right\| \right).\end{aligned} \tag{4.2}$$

The equality holds in (4.2) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1 + \sqrt{1-\rho^2}\right)} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \tag{4.3}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right) e. \tag{4.4}$$

Proof. We know, from the proof of Theorem 3.1, that, if (4.1) is fulfilled, then we have the inequality

$$\|x_i\| \leq \frac{1}{\sqrt{1-\rho^2}} \operatorname{Re}\langle x_i, e\rangle$$

for each $i \in \{1, \dots, n\}$, implying

$$\begin{aligned} \|x_i\| - \operatorname{Re}\langle x_i, e\rangle &\leq \left(\frac{1}{\sqrt{1-\rho^2}} - 1 \right) \operatorname{Re}\langle x_i, e\rangle \\ &= \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\langle x_i, e\rangle \end{aligned} \tag{4.5}$$

for each $i \in \{1, \dots, n\}$.

Now, making use of Theorem 2.1, for

$$k_i := \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\langle x_i, e\rangle, \quad i \in \{1, \dots, n\},$$

we easily deduce the conclusion of the theorem.

We omit the details. ∎

We may state the following result as well:

Theorem 4.2. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space and* $e \in H$, $M \geq m > 0$. *If* $x_i \in H$, $i \in \{1, \dots, n\}$ *are such that either*

$$\operatorname{Re}\langle Me - x_i, x_i - me\rangle \geq 0, \tag{4.6}$$

or, equivalently,

$$\left\| x_i - \frac{M+m}{2} e \right\| \leq \frac{1}{2}(M-m) \tag{4.7}$$

holds for each $i \in \{1, \dots, n\}$, *then we have the inequality*

$$\begin{aligned} (0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| &\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \\ &\left(\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \left\| \sum_{i=1}^{n} x_i \right\| \right). \end{aligned} \tag{4.8}$$

The equality holds in (4.8) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \tag{4.9}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right) e. \tag{4.10}$$

Proof. We know, from the proof of Theorem 2.3, that if (4.6) is fulfilled, then we have the inequality

$$\|x_i\| \le \frac{M+m}{2\sqrt{mM}} \operatorname{Re} \langle x_i, e \rangle$$

for each $i \in \{1, \dots, n\}$. This is equivalent to

$$\|x_i\| - \operatorname{Re} \langle x_i, e \rangle \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \langle x_i, e \rangle$$

for each $i \in \{1, \dots, n\}$.

Now, making use of Theorem 3.1, we deduce the conclusion of the theorem. We omit the details. ∎

Remark 4.1. If one uses Theorem 3.2 instead of Theorem 3.1 above, then one can state the corresponding generalisation for families of orthonormal vectors of the inequalities (4.2) and (4.8) respectively. We do not provide them here.

Now, on utilising a slightly different approach, we may point out the following result:

Theorem 4.3. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $e, x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $r_i > 0$, $i \in \{1, \dots, n\}$ are such that*

$$\|x_i - e\| \le r_i \text{ for each } i \in \{1, \dots, n\}, \tag{4.11}$$

then we have the inequality

$$0 \le \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \frac{1}{2} \sum_{i=1}^{n} r_i^2. \tag{4.12}$$

The equality holds in (4.12) if and only if

$$\sum_{i=1}^{n} \|x_i\| \ge \frac{1}{2} \sum_{i=1}^{n} r_i^2 \tag{4.13}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{2} \sum_{i=1}^{n} r_i^2 \right) e. \tag{4.14}$$

Proof. The condition (4.11) is clearly equivalent to

$$\|x_i\|^2 + 1 \le \operatorname{Re} \langle x_i, e \rangle + r_i^2 \tag{4.15}$$

for each $i \in \{1, \dots, n\}$.

Using the elementary inequality

$$2\|x_i\| \le \|x_i\|^2 + 1, \tag{4.16}$$

for each $i \in \{1, \dots, n\}$, then, by (4.15) and (4.16), we deduce

$$2\|x_i\| \le 2 \operatorname{Re} \langle x_i, e \rangle + r_i^2,$$

giving

$$\|x_i\| - \operatorname{Re}\langle x_i, e\rangle \le \frac{1}{2} r_i^2 \tag{4.17}$$

for each $i \in \{1, \dots, n\}$.

Now, utilising Theorem 3.1 for $k_i = \frac{1}{2} r_i^2$, $i \in \{1, \dots, n\}$, we deduce the desired result. We omit the details. ∎

Finally, we may state and prove the following result as well.

Theorem 4.4. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e, x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $M_i \ge m_i > 0$, $i \in \{1, \dots, n\}$, are such that*

$$\left\| x_i - \frac{M_i + m_i}{2} e \right\| \le \frac{1}{2} (M_i - m_i), \tag{4.18}$$

or, equivalently,

$$\operatorname{Re}\langle M_i e - x, x - m_i e\rangle \ge 0 \tag{4.19}$$

for each $i \in \{1, \dots, n\}$, then we have the inequality

$$(0 \le) \sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\| \le \frac{1}{4} \sum_{i=1}^n \frac{(M_i - m_i)^2}{M_i + m_i}. \tag{4.20}$$

The equality holds in (4.20) if and only if

$$\sum_{i=1}^n \|x_i\| \ge \frac{1}{4} \sum_{i=1}^n \frac{(M_i - m_i)^2}{M_i + m_i} \tag{4.21}$$

and

$$\sum_{i=1}^n x_i = \left(\sum_{i=1}^n \|x_i\| - \frac{1}{4} \sum_{i=1}^n \frac{(M_i - m_i)^2}{M_i + m_i} \right) e. \tag{4.22}$$

Proof. The condition (4.18) is equivalent to:

$$\|x_i\|^2 + \left(\frac{M_i + m_i}{2}\right)^2 \le 2 \operatorname{Re}\left\langle x_i, \frac{M_i + m_i}{2} e \right\rangle + \frac{1}{4} (M_i - m_i)^2$$

and since

$$2\left(\frac{M_i + m_i}{2}\right) \|x_i\| \le \|x_i\|^2 + \left(\frac{M_i + m_i}{2}\right)^2,$$

then we get

$$2\left(\frac{M_i + m_i}{2}\right) \|x_i\| \le 2 \cdot \frac{M_i + m_i}{2} \operatorname{Re}\langle x_i, e\rangle + \frac{1}{4} (M_i - m_i)^2,$$

or, equivalently,

$$\|x_i\| - \operatorname{Re}\langle x_i, e\rangle \le \frac{1}{4} \cdot \frac{(M_i - m_i)^2}{M_i + m_i}$$

for each $i \in \{1, \dots, n\}$.

Now, making use of Theorem 3.1 for $k_i := \frac{1}{4} \cdot \frac{(M_i - m_i)^2}{M_i + m_i}$, $i \in \{1, \dots, n\}$, we deduce the desired result. ∎

Remark 4.2. If one uses Theorem 3.2 instead of Theorem 3.1 above, then one can state the corresponding generalisation for families of orthonormal vectors of the inequalities in (4.12) and (4.20) respectively. We omit the details.

5. REVERSES OF SCHWARZ INEQUALITY

In this section we outline a procedure showing how some of the above results for triangle inequality may be employed to obtain reverses for the celebrated Schwarz inequality.

For $a \in H$, $\|a\| = 1$ and $r \in (0,1)$ define the closed ball

$$\overline{D}(a,r) := \{x \in H, \|x - a\| \leq r\}.$$

The following reverse of the Schwarz inequality holds:

Proposition 5.1. *If $x, y \in \overline{D}(a,r)$ with $a \in H$, $\|a\| = 1$ and $r \in (0,1)$, then we have the inequality*

$$(0 \leq) \frac{\|x\|\|y\| - \operatorname{Re}\langle x,y\rangle}{(\|x\| + \|y\|)^2} \leq \frac{1}{2} r^2. \tag{5.1}$$

The constant $\frac{1}{2}$ in (5.1) is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. Using Theorem 2.1 for $x_1 = x, x_2 = y, \rho = r$, we have

$$\sqrt{1-r^2}(\|x\| + \|y\|) \leq \|x + y\|. \tag{5.2}$$

Taking the square in (5.2) we deduce

$$\left(1 - r^2\right)\left(\|x\|^2 + 2\|x\|\|y\| + \|y\|^2\right) \leq \|x\|^2 + 2\operatorname{Re}\langle x,y\rangle + \|y\|^2$$

which is clearly equivalent to (5.1).

Now, assume that (5.1) holds with a constant $C > 0$ instead of $\frac{1}{2}, i.e.$,

$$\frac{\|x\|\|y\| - \operatorname{Re}\langle x,y\rangle}{(\|x\| + \|y\|)^2} \leq Cr^2 \tag{5.3}$$

provided $x, y \in \overline{D}(a,r)$ with $a \in H$, $\|a\| = 1$ and $r \in (0,1)$.

Let $e \in H$ with $\|e\| = 1$ and $e \perp a$. Define $x = a + re, y = a - re$. Then

$$\|x\| = \sqrt{1 + r^2} = \|y\|, \ \operatorname{Re}\langle x,y\rangle = 1 - r^2$$

and thus, from (5.3), we have

$$\frac{1 + r^2 - (1 - r^2)}{\left(2\sqrt{1 + r^2}\right)^2} \leq Cr^2$$

giving

$$\frac{1}{2} \leq \left(1 + r^2\right) C$$

for any $r \in (0,1)$. If in this inequality we let $r \to 0+$, then we get $C \geq \frac{1}{2}$ and the proposition is proved. ∎

In a similar way, by the use of Theorem 2.3, we may prove the following reverse of the Schwarz inequality as well:

Proposition 5.2. *If $a \in H$, $\|a\| = 1$, $M \geq m > 0$ and $x, y \in H$ are so that either*

$$\operatorname{Re}\langle Ma - x, x - ma\rangle, \operatorname{Re}\langle Ma - y, y - ma\rangle \geq 0$$

or, equivalently,

$$\left\|x - \frac{m+M}{2}a\right\|, \left\|y - \frac{m+M}{2}a\right\| \leq \frac{1}{2}(M - m)$$

hold, then

$$(0 \leq) \frac{\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle}{(\|x\| + \|y\|)^2} \leq \frac{1}{2}\left(\frac{M-m}{M+m}\right)^2.$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity.

Remark 5.1. On utilising Theorem 2.2 and Theorem 2.4, we may deduce some similar reverses of Schwarz inequality provided $x, y \in \cap_{k=1}^{m} \overline{D}(a_k, \rho_k)$, assumed not to be empty, where $a_1, ..., a_n$ are orthonormal vectors in H and $\rho_k \in (0,1)$ for $k \in \{1, ..., m\}$. We omit the details.

Remark 5.2. For various different reverses of Schwarz inequality in inner product spaces, see the recent survey [2], that is available as a preprint in Mathematical Arχiv, where further references are given.

6. APPLICATIONS FOR VECTOR-VALUED INTEGRAL INEQUALITIES

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field, $[a,b]$ a compact interval in $\mathbb{R}$ and $\eta : [a,b] \to [0,\infty)$ a Lebesgue integrable function on $[a,b]$ with the property that $\int_a^b \eta(t)\,dt = 1$. If, by $L_\eta([a,b];H)$ we denote the Hilbert space of all Bochner measurable functions $f : [a,b] \to H$ with the property that $\int_a^b \eta(t)\|f(t)\|^2 dt < \infty$, then the norm $\|\cdot\|_\eta$ of this space is generated by the inner product $\langle\cdot,\cdot\rangle_\eta : H \times H \to \mathbb{K}$ defined by

$$\langle f, g\rangle_\eta := \int_a^b \eta(t)\langle f(t), g(t)\rangle\, dt.$$

The following proposition providing a reverse of the integral generalised triangle inequality may be stated.

Proposition 6.1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\eta : [a,b] \to [0,\infty)$ as above. If $g \in L_\eta([a,b];H)$ is so that $\int_a^b \eta(t)\|g(t)\|^2 dt = 1$ and $f_i \in L_\eta([a,b];H)$, $i \in \{1,\dots,n\}$, $\rho \in (0,1)$ are so that*

$$\|f_i(t) - g(t)\| \leq \rho \tag{6.1}$$

for a.e. $t \in [a,b]$ and each $i \in \{1,\dots,n\}$, then we have the inequality

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2} \leq \left(\int_a^b \eta(t)\left\|\sum_{i=1}^{n} f_i(t)\right\|^2 dt\right)^{1/2}. \tag{6.2}$$

The case of equality holds in (6.2) if and only if

$$\sum_{i=1}^{n} f_i(t) = \sqrt{1-\rho^2}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2} \cdot g(t)$$

for a.e. $t \in [a,b]$.

Proof. Observe, by (6.2), that

$$\begin{aligned}\|f_i - g\|_\eta &= \left(\int_a^b \eta(t)\|f_i(t) - g(t)\|^2 dt\right)^{1/2} \\ &\leq \left(\int_a^b \eta(t)\rho^2 dt\right)^{1/2} = \rho\end{aligned}$$

for each $i \in \{1,\dots,n\}$. Applying Theorem 2.1 for the Hilbert space $L_\eta([a,b];H)$, we deduce the desired result. ∎

The following result may be stated as well.

Proposition 6.2. *Let* H,η,g *be as in Proposition 6.1. If* $f_i \in L_\eta([a,b];H)$, $i\in\{1,\dots,n\}$ *and* $M \geq m > 0$ *are so that either*

$$\operatorname{Re}\langle Mg(t)-f_i(t), f_i(t)-mg(t)\rangle \geq 0$$

or, equivalently,

$$\left\| f_i(t) - \frac{m+M}{2} g(t) \right\| \leq \frac{1}{2}(M-m)$$

for a.e. $t\in[a,b]$ *and each* $i\in\{1,\dots,n\}$, *then we have the inequality*

$$\frac{2\sqrt{mM}}{m+M}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2} \leq \left(\int_a^b \eta(t)\left\|\sum_{i=1}^{n} f_i(t)\right\|^2 dt\right)^{1/2}. \tag{6.3}$$

The equality holds in (6.3) if and only if

$$\sum_{i=1}^{n} f_i(t) = \frac{2\sqrt{mM}}{m+M}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2}\cdot g(t),$$

for a.e. $t\in[a,b]$.

Remark 6.1. Similar integral inequalities may be stated on utilising the inequalities for inner products and norms obtained above, but we do not mention them here.

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