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FIXED POINT THEOREMS FOR A FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, K be a nonempty bounded closed convex subset of E, $T_i : K \to K, i = 1, 2, \dots, r$ be a finite family of asymptotically nonexpansive mappings such that for each i, $\{k_{n_i}\} \subset [1, \infty)$. Let $\cap_{i=1}^r F(T_i)$ be a nonempty set of common fixed points of $\{T_i\}_{i=1}^r$ and define

$$S^n := \alpha_0 I + \alpha_1 T_1^n + \alpha_2 T_2^n + \cdots + \alpha_r T_r^n,$$

$n \geq 1$. Let $u \in K$ be fixed and let $\{t_n\} \subset (0, 1)$ be such that $t_n \to 1$ as $n \to \infty$. We prove that the sequence $\{x_n\}_n$ satisfying the relation

$$x_n = (1 - \frac{t_n}{p_n})u + \frac{t_n}{p_n} S^n x_n,$$

$p_n \in [1, \infty)$ associated with S^n, converges strongly to a fixed point of S provided E possesses uniform normal structure. Futhermore we prove that the iterative process: $z_1 \in K$,

$$z_{n+1} := (1 - \frac{t_n}{p_n})u + \frac{t_n}{p_n} S^n z_n,$$

$n \geq 1$, converges strongly to a fixed point of S.

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1. INTRODUCTION

Let E be a Banach space and K a nonempty subset of E. The mapping $T : K \to K$ is said to be *Lipschitzian* if for any integer $n \geq 1$, there exists a constant $k_n > 0$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$. A Lipschitzian mapping T is called *uniformly k-Lipschtzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* if $k_n \in [1, \infty)$ and $\lim_n k_n = 1$.

In [1] Kirk introduced an iterative process given by

$$x_{n+1} := \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \cdots + \alpha_r T^r x_n, \tag{1.1}$$

where $\alpha_i \geq 0, \alpha_0 > 0$ and $\sum_{i=0}^{r} \alpha_i = 1$, for approximating fixed points of nonexpansive mappings on convex subsets of uniformly convex Banach spaces. Maiti and Saha [2] extended the results of Kirk as follows:

Let K be a nonempty closed convex and *bounded* subset of a real Banach space E. Let $T_i : K \to K$, for $i = 1, 2, \ldots, r$ be nonexpansive mappings and let

$$S := \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_r T_r, \tag{1.2}$$

where $\alpha_i \geq 0,\ \alpha_0 > 0$ and $\sum_{i=0}^{r} \alpha_i = 1$. Mappings $T_i,\ i = 1, 2, \ldots, r$ with nonempty common fixed points set $D := \cap_{i=1}^{r} F(T_i)$ where $F(T_i) := \{x \in K : T_i(x) = x\}$ in K are said to satisfy *condition A*: (see, e.g., [2], [3], [4]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0,\ f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Sx\| \geq f(d(x, D))$ for all $x \in D$, where $d(x, D) := \inf\{\|x - z\| : z \in D\}$.

In [3], Liu, Lei and Li introduced an iteration process

$$x_{n+1} := S x_n \tag{1.3}$$

where $x_1 \in K$, and showed that $\{x_n\}_n$ defined by (1.3) converges to a common fixed point of $\{T_i,\ i = 1, 2, \ldots, r\}$ in Banach spaces provided that $T_i,\ i = 1, 2, \ldots, r$ satisfy condition A.

In [5] Jung removed the *strong* condition A and proved the following (though not applicable to L_p spaces $1 < p < \infty, p \neq 2$):

Let E be a reflexive and strictly convex Banach space with uniformly Gâteaux differentiable norm. Let $T_i : E \to E,\ i = 1, 2, \ldots, r$ be nonexpansive mappings and $\{x_n\}_n$ be a sequence in E defined by the recursion relation (1.3) and suppose that $J^{-1} : E^* \to E$ is weakly sequentially continuous at 0 if $\cap_{i=1}^{r} F(T_i)$ is nonempty, then $\{x_n\}_n$ converges *weakly* to a common fixed point of $\{T_1, T_2, \ldots, T_r\}$.

Chidume, Zegeye and Prempeh [6] proved the following theorem which does not require *condition A*, and is applicable to Banach spaces including L_p spaces, $1 < p < \infty$, and the convergence is *strong*:

Theorem CZP. [6]. *Let K be a nonempty closed convex subset of a strictly convex real Banach space E which has uniformly Gâteaux differentiable norm. Assume K is a sunny nonexpansive retract of E with Q as sunny nonexpansive retraction. Let $T_i : K \to E, i = 1, 2, \ldots, r$ be a family of nonexpansive weakly inward mappings. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. For given $u, x_1 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Q S x_n,\ n \geq 1, \tag{1.4}$$

where $S := a_0 I + a_1 T_1 + \cdots + a_r T_r$, *for* $0 < a_i < 1,\ i = 0, 1, 2, \ldots, r$, $\sum_{i=0}^{r} a_i = 1$, *and* $\{\alpha_n\}$ *is a real sequence which satisfies the following conditions: (i)* $\lim \alpha_n = 0$*; (ii)* $\sum \alpha_n = \infty$ *and (iii)* $\lim \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$*. Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, \ldots, T_r\}$.

Most recently Chidume and Udomene [7] have proved the following:

Let E be a real reflexive Banach space with a uniformly G$\hat{a}$teaux differentiable norm, K be a nonempty bounded closed convex subset of E, $T : K \to K$ be asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed and let $\{t_n\} \subset (0, 1)$ be such that $t_n \to 1$ as $n \to \infty$, then the sequence $\{x_n\}_n$ satisfying the relation

$$x_n := (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_n,\ n \geq 1 \tag{1.5}$$

converges strongly to a fixed point of T provided $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ and E possesses uniform normal structure. Furthermore the iterative process $z_1 \in K$,

$$z_{n+1} := (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n z_n,\ n \geq 1 \tag{1.6}$$

converges strongly to a fixed point of T.
Given that $T_i : K \to E$ for $i = 1, 2, \cdots, r$, T_i asymptotically nonexpansive and

$$S := \alpha_0 I + \alpha_1 T_1 + \cdots + \alpha_r T_r,$$

then we are especially informed by the modified Mann iteration method, Schu [8] (See Theorem 1.5) to define for all $n \geq 1$,

$$S^n := \alpha_0 I + \alpha_1 T_1^n + \cdots + \alpha_r T_r^n.$$

Our main purpose in this paper to extend the Theorem CZP to a family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^{r}$ with common fixed point set $D := \cap_{i=1}^{r} F(T_i)$.

2. PRELIMINARIES

Let E with dual E^* be a Banach space. E is said to be *smooth* if for each $x \in S(0, 1)$ the unit sphere of E, the limit

$$\vartheta = \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2.1}$$

exists for all $y \in S(0, 1)$. If this limit exists and is attained uniformly in $x, y \in S(0, 1)$,then E is said to be *uniformly smooth*. The norm is said to be *uniformly G$\hat{a}$teaux differentiable* if for each $y \in S(0, 1)$, the limit exists uniformly for $x \in S(0, 1)$.

The *normalised duality mapping* $J : E \to E^*$, defined by

$$J(x) := \{x^* \in E^* :< x, x^* >= \|x\|^2 = \|x^*\|^2\}, x \in E.$$

If E is smooth the duality mapping is single-valued, and if E has a uniformly G$\hat{a}$teaux differentiable norm then the duality mapping is $norm{-}to{-}weak^*$ uniformly continuous on bounded sets. (See [6], [7], [9])

Let K be a nonempty closed convex and bounded subset of the Banach space E. Let the *diameter* of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let

$r(x, K) := \sup\{\|x - y\| : y \in K\}$ and denote the $\check{C}eby\check{s}ev$ radius of K by
$r(K) := \inf\{r(x, K) : x \in K\}$. The *normal structure coefficient* of K, denoted by

$$N(K) := \inf\{\frac{d(K)}{r(K)} : K \text{is closed convex bounded subset of E with } d(K) > 0\}$$

(See [10]). A space E such that $N(E) > 1$ is said to have *uniform normal structure*. It is known that a space with uniform structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure. (See [9]).

Let LIM be a Banach limit. $LIM \in (l^\infty)^*$ such that $\|LIM\| = 1$,
$\liminf a_n \leq LIM_n a_n \leq \limsup a_n$ and $LIM_n a_n = LIM_n a_{n+1}$ for all $\{a_n\} \in l^\infty$. Furthermore if $\{a_n\}_n$, $\{b_n\}_n \in l^\infty$ then

$$\limsup a_n + LIM_n b_n \leq \limsup(a_n + b_n). \tag{2.2}$$

(See [7]).

A Banach space E is said to be *strictly convex* if $\|\frac{x_1+x_2+\cdots+x_r}{r}\| < 1$ for $x_i \in E,\ i = 1, 2, \ldots, r$ with $\|x_i\| = 1,\ i = 1, 2, \ldots, r$ and $x_i \neq x_j$, for some $i \neq j$. In a strictly convex Banach space E, we have that if $\|x_1\| = \|x_2\| = \cdots = \|x_r\| = \|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r\|$, for $x_i \in E,\ \alpha_i \in (0, 1),\ i = 1, 2, \ldots, r$ and such that $\sum_{i=1}^r \alpha_i = 1$ then $x_1 = x_2 = \cdots = x_r$. (See [6]).

We shall require the following technical lemmas in the sequel.

Lemma 2.1. ([11]). *Let E be an arbitrary normed space. For each $x, y \in E$ and $j \in J(x + y)$ we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j > \tag{2.3}$$

Lemma 2.2. ([9]). *Suppose E is a Banach space with uniform normal structure, K is a nonempty bounded subset of E and $T : K \to K$ is a uniformly k-Lipschtzian mapping with $k < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset C of K with the following property (P):*
(P): $x \in C$ implies $\omega_w(x) \subset C$
where ω_w is the weak $\omega - limit$ set of T at x, that is, the set

$$\{y \in E : y = weak - lim_j T^{n_j} x \text{ for some } n_j \uparrow \infty\}.$$

Then T has a fixed point in C.

Lemma 2.3. ([12]). *Let $\{a_n\}_n$ be a sequence of nonnegative real numbers satisfying the following relation: $a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n,\ n \geq 1$ where (i) $0 < \alpha_n < 1$; (ii) $\sum_{n=1}^\infty \alpha_n = \infty$. Suppose, either (a) $\sigma_n = o(\alpha_n)$, or (b) $\limsup \sigma_n \leq 0$. Then $a_n \to 0$ as $n \to \infty$.*

3. MAIN RESULTS

We now prove the following lemma and theorems. In the sequel we denote k_{n_i} as the Lipschitz's constant of T_i^n.

Lemma 3.1. *Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_i : K \to E,\ i = 1, 2, \ldots, r$ be a family of asymptotically nonexpansive mappings such that $\cap_{i=1}^r F(T_i) \neq \phi$. Let $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_r$ be a finite number of real numbers in $(0, 1)$ such that $\sum_{i=0}^r \alpha_i = 1$ and for $n \geq 1$ let $S^n := \alpha_0 I + \alpha_1 T_1^n + \cdots + \alpha_r T_r^n$. Then (i) S is asymptotically nonexpansive and (ii) $F(S) = \cap_{i=1}^r F(T_i)$.*

Proof. (i) Let $x, y \in K$ and for each $n \geq 1$, for $1 \leq i \leq r$, $\|T_i^n x - T_i^n y\| \leq k_{n_i}\|x-y\|$, $k_{n_i} \geq 1$ and $\lim_n k_{n_i} = 1$.

$$\begin{aligned}
\|S^n x - S^n y\| &= \|(\alpha_0 x + \alpha_1 T_1^n x + \cdots + \alpha_r T_r^n x) - (\alpha_0 y + \alpha_1 T_1^n y + \cdots + \alpha_r T_r^n y)\| \\
&\leq \alpha_0\|x-y\| + \sum_{i=1}^{r} \alpha_i k_{n_i}\|x-y\| \\
&\leq \alpha_0\|x-y\| + p_n\|x-y\| \sum_{i=1}^{1} \alpha_i, p_n := \max\{k_{n_i},\ i = 1, 2, \ldots, r\} \\
&= [(1 - \sum_{i=1}^{r} \alpha_i) + p_n \sum_{i=1}^{r} \alpha_i]\|x-y\| \\
&= [1 + (p_n - 1)\sum_{i=1}^{r} \alpha_i]\|x-y\| \\
&\leq p_n\|x-y\|
\end{aligned}$$

where $p_n \geq 1$ for all n and $\lim_n p_n = 1$. Therefore S is asymptotically nonexpansive.
(ii) Let $w \in \cap_{i=1}^r F(T_i)$, then $\forall i,\ T_i w = w$. Then:

$$\begin{aligned}
Sw &= \alpha_0 w + \alpha_1 T_1 w + \cdots + \alpha_r T_r w \\
&= w \sum_{0}^{r} \alpha_i \\
&= w.
\end{aligned}$$

Thus $\cap_{i=1}^r F(T_i) \subset F(S)$.

Next we show that $F(S) \subset \cap_{i=1}^r F(T_i)$. Suppose $v \in F(S)$, then $v \in F(S^n)$ and for $w \in \cap_{i=1}^r F(T_i)$, we have

$$\begin{aligned}
\|v - w\| &= \|\alpha_0 v + \alpha_1 T_1^n v + \cdots + \alpha_r T_r^n v - w\| \\
&= \|\alpha_0(v-w) + \alpha_1(T_1^n v - w) + \cdots + \alpha_r(T_r^n v - w)\| \\
&\leq \alpha_0\|v-w\| + \sum_{i=1}^{r} \alpha_i\|T_i^n v - w\|.
\end{aligned}$$

Then taking $\lim_n$ throughout we have:

$$\begin{aligned}
\|v - w\| &\leq \lim\{\alpha_0\|v-w\| + \sum_{i=1}^{r} \alpha_i\|T_i^n v - w\|\} \\
&= \alpha_0\|v-w\| + \sum_{i=1}^{r-1} \alpha_i \lim\|T_i^n v - w\| + \alpha_r \lim\|T_r^n v - w\| \\
&\leq \alpha_0\|v-w\| + \sum_{i=1}^{r-1} \alpha_i \lim k_{n_i}\|v-w\| + \lim \alpha_r\|T_r^n v - w\| \\
&\leq \alpha_0\|v-w\| + \sum_{i=1}^{r} \alpha_i \lim k_{n_i}\|v-w\|
\end{aligned}$$

$$\begin{aligned} &\leq \alpha_0\|v-w\| + \sum_{i=1}^{r}\alpha_i \lim p_n\|v-w\| \\ &= \|v-w\|\sum_{j=0}^{r}\alpha_j \\ &= \|v-w\|. \end{aligned}$$

From the foregoing we have

$$\|v-w\| = \sum_{j=0}^{r-1}\alpha_j\|v-w\| + \alpha_r \lim\|T_r^n v - w\|$$

therefore

$$\alpha_r\|v-w\| = \alpha_r \lim\|T_r^n v - w\|.$$

Hence,

$$\|v-w\| = \lim\|T_r^n v - w\|.$$

Similarly we have

$$\|v-w\| = \lim\|T_i^n v - w\|,\ i = 1, 2, \ldots, r-1$$

but also

$$\|v-w\| = \|\alpha_0(v-w) + \sum_{i=1}^{r}\alpha_r(T_i^n v - w)\|.$$

By strict convexity of E,

$$v - w = \lim(T_i^n v - w),\ i = 1, 2, \ldots, r.$$

Hence

(3.1) $$v = \lim T_i^n v,\ i = 1, 2, \ldots, r.$$

Then, $T_i v = \lim T_i(T_i^n v) = \lim T_i^{n+1} v = v$, by (3.1). Therefore $T_i v = v,\ i = 1, 2, \ldots, r$. Hence $v \in F(T_i),\ i = 1, 2, \ldots, r$, and so $v \in \cap_{i=1}^r F(T_i)$ implying that $F(S) \subset \cap_{i=1}^r F(T_i)$. Hence $F(S) = \cap_i^r F(T_i)$, for $n \in \mathbb{N}$. □

Theorem 3.2. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K a nonempty closed convex and bounded subset of E, $S : K \to K$ an asymptotically nonexpansive mapping with $\{p_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed, $\{t_n\}_n \subset (0, 1)$ be such that $\lim t_n = 1$, and $\lim \frac{p_n - 1}{p_n - t_n} = 0$. Then*
(i) for each integer $n \geq 1$, there exists a unique $x_n \in K$ such that

(3.2) $$x_n = (1 - \frac{t_n}{p_n})u + \frac{t_n}{p_n}S^n x_n$$

and, if in addition, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$, then
(ii) the sequence $\{x_n\}_n$ converges strongly to a fixed point of S.

Proof. First we observe that since $N(E) > 1$ and $p_n \to 1$ as $n \to \infty$ there exists an integer $N > 0$ and a constant $L > 0$ such that

$$p_n \leq L \leq N(E)^{\frac{1}{2}}, \forall n \geq N.$$

For each integer $n \geq 1$, the mapping $f_n : K \to K$ defined for each $u \in K$ by

$$f_n x := (1 - \frac{t_n}{p_n})u + \frac{t_n}{p_n}S^n x$$

is a contraction. It follows that there exists a unique $x_n \in K$ such that $f_n x_n = x_n$. Define the mapping $\phi : K \to \mathbb{R}$ by

$$\phi(y) = LIM_n \|x_n - y\|^2, \ \forall y \in K.$$

Since E is reflexive, $\phi(y) \to \infty$ as $\|y\| \to \infty$, ϕ is continuous and convex, there is some $x \in K$ such that $\phi(x) = \inf_{y\in K} \phi(y)$. Thus the set $K_{\min} := \{x \in K : \phi(x) = \inf_{y\in K} \phi(y)\} \neq \phi$. It is also convex and closed. Further, $K_{\min}$ has property (P). This follows as in [7], [9]. Hence $K_{\min} \cap F(S) \neq \phi$.

Let $x^* \in K_{\min} \cap F(S)$ and let $t \in (0,1)$. Then $(1-t)x^* + tu \in K$. It follows that $\phi(x^*) \leq \phi((1-t)x^* + tu)$. Using inequality (2.3) we have

$$\begin{aligned} 0 &\leq \frac{\phi((1-t)x^* + tu) - \phi(x^*)}{t} \\ &\leq -2LIM_n < u - x^*, j(x_n - x^* - t(u - x^*)) > . \end{aligned}$$

This implies that $LIM_n < u - x^*, j(x_n - x^* - t(u-x^*)) > \leq 0$. In the limit as $t \to 0$, since j is $norm - to - weak^*$ continuous, we have that

$$LIM_n < u - x^*, j(x_n - x^*) > \leq 0. \tag{3.3}$$

Since S is asymptotically nonexpansive with $\{p_n\}_n \subset [1, \infty)$ we have

$$\begin{aligned} < x_n - S^n x_n, j(x_n - x^*) > &= < x_n - x^*, j(x_n - x^*) > - < S^n x_n - x^*, j(x_n - x^*) > \\ &\geq \|x_n - x^*\|^2 - \|S^n x_n - x^*\| \|x_n - x^*\| \\ &\geq \|x_n - x^*\|^2 - p_n \|x_n - x^*\|^2 \\ &= -(p_n - 1)\|x_n - x^*\|^2 \\ &\geq -(p_n - 1)d^2 \end{aligned}$$

where $d = diamK$. Since x_n is a fixed point of f_n, it follows that

$$x_n - S^n x_n = \frac{p_n - t_n}{t_n}(u - x_n)$$

and from the last inequality,

$$\frac{p_n - t_n}{t_n} < u - x_n, j(x_n - x^*) > \geq -(p_n - 1)d^2$$

or

$$< x_n - u, j(x_n - x^*) > \leq \frac{t_n(p_n - 1)}{p_n - t_n} d^2$$

where $\frac{t_n(p_n-1)}{p_n - t_n} \to 0$ as $n \to \infty$. Therefore

$$\limsup < x_n - u, j(x_n - x^*) > \leq 0. \tag{3.4}$$

We also have using (2.1)

$$\limsup \|x_n - x^*\|^2 + LIM_n < x^* - u, j(x_n - x^*) > \leq \limsup < x_n - u, j(x_n - x^*) > .$$

From inequalities (3.3) and (3.4) we deduce that

$$\begin{aligned} \limsup \|x_n - x^*\|^2 &\leq \limsup < x_n - u, j(x_n - x^*) > \\ &\leq 0. \end{aligned}$$

Hence $\{x_n\}_n$ converges strongly to $x^* \in F(S)$. □

Theorem 3.3. *Let E be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, $S : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{p_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed, $\{t_n\}_n \subset (0,1)$ be such that $\lim t_n = 1,\ t_n p_n < 1$, and $\lim \frac{p_n - 1}{p_n - t_n} = 0$. Define the sequence $\{z_n\}_n$ iteratively by $z_1 \in K$,*

$$z_{n+1} = (1 - \frac{t_n}{p_n})u + \frac{t_n}{p_n} S^n z_n,\ n = 1, 2, \dots. \tag{3.5}$$

Then
(i) for each integer $n \geq 1$, there is a unique $x_n \in K$ such that

$$x_n = (1 - \frac{t_n}{p_n})u + \frac{t_n}{p_n} S^n x_n;$$

and if in addition $\lim \|x_n - Sx_n\| = 0$, $\|z_n - S^n z_n\| = o(1 - \frac{t_n}{p_n})$, then
(ii) $\{z_n\}_n$ converges strongly to a fixed point of S.

Proof. From (3.2)

$$x_n - z_n = (1 - \frac{t_n}{p_n})(u - z_n) + \frac{t_n}{p_n}(S^n x_n - z_n).$$

Applying inequality (2.3), we have

$$\begin{aligned}
\|x_n - z_n\|^2 &\leq \frac{t_n^2}{p_n^2}\|S^n x_n - z_n\|^2 + \\
&\quad + 2(1 - \frac{t_n}{p_n}) < u - z_n, j(x_n - z_n) > \\
&\leq \frac{t_n^2}{p_n^2}\|S^n x_n - z_n\|^2 + \\
&\quad + 2(1 - \frac{t_n}{p_n}) < u - x_n, j(x_n - z_n) > + p_n^2\|x_n - z_n\|^2 \\
&\leq \frac{t_n^2}{p_n^2}\{p_n^2\|x_n - z_n\|^2 + 2p_n\|x_n - z_n\|\|S^n z_n - z_n\| + \\
&\quad + \|S^n z_n - z_n\|^2\} + 2(1 - \frac{t_n}{p_n})\{< u - x_n, j(x_n - z_n) > + p_n^2\|x_n - z_n\|^2\} \\
&= \{1 - (1 - \frac{t_n}{p_n})\}^2 p_n^2\|x_n - z_n\|^2 + \\
&\quad + \|S^n z_n - z_n\|\{2p_n\|x_n - z_n\| + \|S^n z_n - z_n\|\} + \\
&\quad + 2(1 - \frac{t_n}{p_n})\{< u - x_n, j(x_n - z_n) > + p_n^2\|x_n - z_n\|^2\} \\
&\leq \{1 + (1 - \frac{t_n}{p_n})^2\} p_n^2\|x_n - z_n\|^2 + \|S^n z_n - z_n\| M + \\
&\quad + 2(1 - \frac{t_n}{p_n}) < u - x_n, j(x_n - z_n) >
\end{aligned}$$

for some constant M. It follows that

$$\begin{aligned}
\limsup < u - x_n, j(z_n - x_n) > &\leq \frac{[p_n^2 - 1 + p_n^2(1 - \frac{t_n}{p_n})^2]}{(1 - \frac{t_n}{p_n})} \limsup \|x_n - z_n\|^2 + \\
&\quad + \limsup \frac{M\|z_n - S^n z_n\|}{1 - \frac{t_n}{p_n}}
\end{aligned}$$

Since $\{z_n\}$ and $\{x_n\}$ are bounded, $\{S^n z_n\}$ is bounded, and also since $\|z_n - S^n z_n\| = o(1 - \frac{t_n}{p_n})$, it follows that

$$\limsup < u - x_n, j(z_n - x_n) > \leq 0. \tag{3.6}$$

Moreover by Theorem (3.2), $x_n \to x^* \in F(S)$ as $n \to \infty$. But

$$\begin{aligned} < u - x_n, j(z_n - x_n) > &= < u - x^*, j(z_n - x^*) > + < u - x^*, j(z_n - x_n) - j(z_n - x^*) > + \\ &\quad + < x^* - x_n, j(z_n - x_n) > \end{aligned} \tag{3.7}$$

Now $< x^* - x_n, j(z_n - x_n) > \to 0$ as $n \to \infty$, $< u - x^*, j(z_n - x_n) - j(z_n - x^*) > \to 0$ as $n \to \infty$. Therefore from (3.6) and (3.7) we obtain

$$\limsup < u - x^*, j(z_n - x^*) > \leq 0.$$

From the iterative process (3.1) and inequality (3.2) we have

$$z_{n+1} - x^* = (1 - \frac{t_n}{p_n})(u - x^*) + \frac{t_n}{p_n}(S^n z_n - x^*) \tag{3.8}$$

and the following estimates:

$$\begin{aligned} \|z_{n+1} - x^*\|^2 &\leq \frac{t_n^2}{p_n^2}\|S^n z_n - x^*\|^2 + 2(1 - \frac{t_n}{p_n}) < u - x^*, j(z_{n+1} - x^*) > \\ &\leq \frac{t_n}{p_n}\|z_n - x^*\|^2 + 2(1 - \frac{t_n}{p_n}) < u - x^*, j(z_{n+1} - x^*) > \\ &= (1 - \alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\beta_n \end{aligned}$$

where $\alpha_n := (1 - \frac{t_n}{p_n})$ and $\beta_n :=< u - x^*, j(z_{n+1} - x^*) >$, and that $\limsup \alpha_n\beta_n \leq 0$. It therefore follows from Lemma 2.3 that $z_n \to x^*$, as $n \to \infty$, completing the proof. □

Remark 3.1. Lim and Xu [9] have shown that a sequence $\{t_n\}_n \subset (0, 1)$ satisfying the conditions of the theorems above always exists. The example given by them is: $t_n := \min\{1 - (p_n - 1)^{\frac{1}{2}}, 1 - n^{-1}\}$, $n = 1, 2, \ldots$.

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