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VIABILITY THEORY AND DIFFERENTIAL LANCHESTER TYPE MODELS FOR COMBAT. DIFFERENTIAL SYSTEMS.

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ABSTRACT. In 1914, F.W. Lanchester proposed several mathematical models based on differential equations to describe combat situations [34]. Since then, his work has been extensively modified to represent a variety of competitions including entire wars. *Differential Lanchester type models* have been studied from many angles by many authors in hundreds of papers and reports. Lanchester type models are used in the planning of optimal strategies, supply and tactics. In this paper, we will show how these models can be studied from a *viability theory* stand point. We will introduce the notion of *winning cone* and show that it is a viable cone for these models. In the last part of our paper we will use the viability theory of differential equations to study Lanchester type models from the optimal theory point of view.

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1. INTRODUCTION

The *Lanchester theory of combat* owes its name and origin to F.W. Lanchester. During the first World War, in 1916, he published his book *Aircraft in Warfare: The Dawn of the Fourth Arm* [34] in which he first introduced the use of differential equations to the mathematical modelling of combat. For an excellent overview of the basis of the Lanchester theory of combat, the reader is referred to P.R. Wallis, *Recent Development in Lanchester Theory* [46].

Following the second World War, it became apparent that advances in technology was having a major impact on the way battles were fought. In an attempt to better understand the new dynamic of the battle field, what is now know as the Lanchester theory of combat gained rapidly in popularity and attracted the interest of many researchers generating hundreds of publications and unpublished technical reports. James G. Taylor published a very exhaustive and detailed synthesis of the work done prior to the early 1980's in his book *Lanchester Models of Warfare* [45].

Up until now, the research work carried out in this field of study has been done using the tools provided by classical analysis. As such, the theory has been explored to determine the existence of solutions along with their properties and behavior. Of particular interest, efforts have been made to develop some time-to-end-of-battle measures. See for example the results presented by [17], [20], [27], [31], [44], [47] to only name a few.

As the Lanchester type models were developed, numerous applications to historical battles have been attempted [8], [10], [11], [12], [16], [19], [18], [22], [25] for example. These verification of the validity and/or applicability of the various models to actual campaigns highlighted a major difficulty presented by such models. *This difficulty arises from the evaluation of the ever elusive* Lanchester coefficients. *These coefficients express, in some form or other depending on the specific model, the ability of a combatant to inflict damage to its opponent (see* The Lanchester Attrition-Rate Coefficient *by Seth Bonder* [6] *and followed up by C. Barfoot* [5]*). The problem is multi-fold, the nature of combat being as it is, the collection of data is at best incomplete and imprecise. Additionally, these coefficients vary through time and conditions in a manner hard to quantify.*

Throughout the evolution of the theory of combat, the analysis generated laws on the progression of combat such as the *Linear Law*, the *Square Law* [34] and the *Logarithmic Law* [39] among others (see Taylor [45] for further details). The validity of this analysis in respect to the coefficients as well as the laws of combat it introduced is often criticized [23], [1], [6], [24], [29], [32] to cite a few. Recently, Hembold [28] presented a very à propos paper on this issue highlighting what he called the *Constant Fallacy*. Validation of the models have repeatedly failed to prove the correctness of these laws of combat. *By our method presented in this paper, and based on viability theory, it is not necessary to to use such laws as linear, square or logarithmic.*

The interest for Lanchester type models is ongoing. They have found applications in numerous fields such as economy [38], biology and evolution theory [21]. There is some interesting work carried out using the Nash equilibrium strategies [43] in the context of armament race and control [35] [41] as well as some publications studying Lanchester models from a dynamical systems angle [13]. There is now an increased interest in the application of Lanchester type models not only to economy but also to problems dealing with competitive aspects.

In view of all that is presented above, in this article we are proposing some additions to the Lanchester theory of combat and bring to bear the tools of non-classical analysis. As a first step, we show how differential Lanchester type models can be studied from a viability theory *point of view. We base our work on the theory presented by J.P. Aubin* [4]. *To this end, we introduce the notion of* winning cone *and explore how this cone is viable for many models.*

As a last step, we take the analysis of the Lanchester type differential models to its natural progression into the domain of optimal control. In this last section, we will introduce the notion of *optimal control by viability* and explore its application to the Lanchester theory of combat.

2. PRELIMINARIES

In this article we will denote by $(H, \langle \cdot\, , \cdot \rangle)$ an arbitrary Hilbert space and by K a closed convex cone in H. Unless otherwise stated, the Hilbert space under consideration will be the Euclidean space $(R^n, \langle \cdot\, , \cdot \rangle)$. We recall that a convex set $\Omega \subseteq H$ is a subset of H such that

$$x, y \in \Omega \Longrightarrow (1-\lambda)x + \lambda y \in \Omega, \qquad \text{for any } \lambda \text{ such that } 0 < \lambda < 1.$$

A closed convex cone $K \subseteq H$ is a closed subset having the following properties:

(1) $K + K \subseteq K$,
(2) $\lambda K \subseteq K, \qquad \lambda \in \mathbf{R}_+$.

If in addition, K satisfies the property that $K \cap (-K) = 0$ then we say that K is a *pointed cone*.

The study of viability will require the application of the notion of contingent cone of which we now give a definition.

Definition 2.1 (Contingent Cone)**.** Let X be a Hilbert space, $K \subset X$ a non-empty subset and $B = \left\{ x \in X \,\middle|\, \|x\| \le 1 \right\}$. We say that the subset

$$T_K(x) = \bigcap_{\varepsilon > 0} \; \bigcap_{\alpha > 0} \bigcup_{0 < h < \alpha} \left(\frac{1}{h}\,(K - x) + \varepsilon B \right)$$

is the *contingent cone* (or the *Bouligand's contingent cone* [7]) to K at the point $x \in K$.

In the case when K is convex, we call $T_K(x)$ the *tangent* cone to K at x. Since this article studies the viability of dynamical systems using convex cones as the viable subset, we give the following characterization of $T_K(x)$.

Theorem 2.1. *Given a convex cone K, a subset of a Hilbert space, $x \in K$ and $T_K(x)$ the contingent cone at x satisfying definition (2.1) then*

$$K + \mathbf{R}x = K - \mathbf{R}_+ x \subset T_K(x)$$

furthermore

$$\overline{K - \mathbf{R}_+ x} = T_K(x). \tag{2.1}$$

Proof. Lets first prove that $K + \mathbf{R}x = K - \mathbf{R}_+ x$. As K is a cone, $\lambda x \in K$ for $\lambda \in \mathbf{R}_+$ and, therefore

$$\begin{aligned} \mathbf{R}_+ x &\subset K \\ K + \mathbf{R}_+ x \subset K + K &\subset K, \qquad \text{as } K \text{ is a cone} \end{aligned}$$

so

$$K + \mathbf{R}_+ x \subset K. \tag{2.2}$$

By decomposing $\mathbf{R}$ into its positive and negative parts $\mathbf{R}_+$ and $-\mathbf{R}_+$ respectively,

$$K + \mathbf{R}x = (K + \mathbf{R}_+ x) + (K - \mathbf{R}_+ x) \tag{2.3}$$

it is obvious that $K - \mathbf{R}_+ x \subset K + \mathbf{R}x$. Substituting equation (2.2) in the above:

$$K + \mathbf{R}x \subset K + (K - \mathbf{R}_+ x) \subset K - \mathbf{R}_+ x$$

which concludes the proof of the first equality.

To prove that $K - \mathbf{R}_+ x \subset T_K(x)$, lets consider $S_K(x) = \bigcup_{h>0} \left(\frac{K-x}{h}\right)$ and the fact that when K is convex, it is known that $\overline{S_K(x)} = T_K(x)$ (a proof can be found in Aubin and Frankowska [3] or Isac [30]). Starting with the definition of $S_K(x)$ and remarking that K is a cone implies $\frac{K}{h} = K$ for $h > 0$, we observe that

$$\begin{aligned}
S_K(x) &= \bigcup_{h>0} \left(\frac{K-x}{h}\right) \\
&= \bigcup_{h>0} \left(\frac{K}{h} - \frac{x}{h}\right) \\
&= \bigcup_{h>0} \left(K - \left(\frac{1}{h}\right) x\right) \\
&= \bigcup_{t\in\mathbf{R}_+} (K - tx), \qquad t \equiv \frac{1}{h},\ t \in \mathbf{R}_+ \\
&= K - \mathbf{R}_+ x.
\end{aligned}$$

Since $K - \mathbf{R}_+ x = S_K(x) \subset \overline{S_K(x)} = T_K(x)$, we can deduce that $K - \mathbf{R}_+ x \subset T_K(x)$ and $\overline{K - \mathbf{R}_+ x} = T_K(x)$. $\square$

3. DIFFERENTIAL LANCHESTER TYPE MODELS

Since the publication of Lanchester's book in 1916, numerous variations based on the original ideas have been proposed, see for instance the list presented by James Taylor [45]. All these models are based on a modeling that takes roots in defining the rates of variation in strength of opposing forces. These rates can be separated in three different categories as follows:

(1) **OLR**. Operational loss rate;
(2) **CLR**. Combat loss rate; and
(3) **RR**. Reinforcements rate.

The OLR represents the losses caused by non combat activities present in any conflict. The loss of a soldier to a driving accident is an example of such losses. The CLR are losses directly related to combat while the RR is the rate at which forces are added or removed from the theater of operations. The generic model of Courtney S. Coleman [15] is then represented by the differential system:

$$\begin{cases} \dfrac{dx_1}{dt} = OLR_{x_1} + CLR_{x_1} + RR_{x_1} \\ \dfrac{dx_2}{dt} = OLR_{x_2} + CLR_{x_2} + RR_{x_2}. \end{cases}$$

A conventional combat scenario is one where both forces are using conventional warfare tactics and could be modelled with:

$$(CONCOM) \begin{cases} \dfrac{dx_1}{dt} = -ax_1(t) - bx_2(t) + P(t) \\ \dfrac{dx_2}{dt} = -cx_1(t) - dx_2(t) + Q(t) \end{cases}$$

where a, b, c and d are non negative loss rate constants while $P(t)$ and $Q(t)$ are reinforcements rates typically in numbers of combatant per day. In this model, the OLR for x_i is $cste \times x_i$ while its CLR is $cste \times x_j$. That is to say that the larger a force is the bigger its operational losses and, the larger its enemy is, the bigger are its combat losses.

The above models assumes that both forces are visible to its enemy [34] [36]. If, on the other hand, both forces use guerilla type techniques, their elements are harder to detect and their combat loss rates (comparable to random hits of blind gun fire through a wooded area) become proportional to their own size as well as the enemy's. A guerilla warfare model is

$$(GUERCOM)\begin{cases} \dfrac{dx_1}{dt} = -ax_1(t) - bx_1(t)x_2(t) + P(t) \\ \dfrac{dx_2}{dt} = -dx_2(t) - cx_1(t)x_2(t) + Q(t) \end{cases}$$

where the terms in $x_1(t)x_2(t)$ are the CLR.

If we take for example the Vietnam conflict, one side was using conventional tactics while the other was taking its tactics out of guerilla warfare. Mixing both the above models where the conventional side is x_1, we obtain:

$$(VIETNAM)\begin{cases} \dfrac{dx_1}{dt} = -ax_1(t) - bx_2(t) + P(t) \\ \dfrac{dx_2}{dt} = -dx_2(t) - cx_1(t)x_2(t) + Q(t). \end{cases}$$

We will know review the various models introduced since, and including the work done in 1916 by Frederick W. Lanchester.

3.1. **Various Lanchester type models.**

3.1.1. *Aimed Fire.* When F.W. Lanchester introduced his model, he remarked that as a direct consequence of the greater range that recent advances in weaponry permitted, it was then possible for one force to *concentrate* its firepower in a battle. He considered two opponents, referred to as x_1 and x_2, and stipulated that each forces loss was directly proportional to the opponents numbers and their ability to destroy. In this model, the coefficients a and b represent each opponents *killing* ability. If the forces involved are homogeneous than the coefficients can be viewed as the destructive ability of one individual. The model introduced in 1914 [33] is expressed mathematically by

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 \end{cases}$$

where $a, b > 0$.

3.1.2. *Area Fire.* This model is also one of the first ones introduced by Lanchester [33]. As can be extrapolated from its title, it is representative of the case when the exact position of one opponent's is unknown but a general idea of his location is available. For example, forces hiding in a wooded area as seen from artillery. In such a case, the coefficients can be deemed to be related to the ratio of the area of effectiveness of each round fired and the density of enemy forces. As a result of the application of this model, the attrition rate of one force will be proportional to both the number of assailants as well as its own (the more there is the higher the density in the area). Mathematically,

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

where $a, b > 0$.

3.1.3. *Brackney.* In 1959, H. Brackney [9] introduced a model that is based on a combination of the *Aimed Fire* model in 3.1.1 and the *Area Fire* model of section 3.1.2. The intent is to consider the case where asymmetric forces are at play such as when a convention force x_1 meets with a guerilla force x_2. Typically, the guerilla forces are somewhat aware of the location and whereabouts of the conventional forces justifying the use of an aimed fire model for the attrition of the conventional forces. On the other hand, the guerilla forces are small in numbers and their location is often only known to be within a general region justifying in this case an attrition rates modeled on area fire. Brackney's model is expressed by

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \qquad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

where $a, b > 0$.

3.1.4. *Peterson.* At first glance, this model appears to be non applicable and only introduced for the sake of completeness as the attrition rate of one force is independent of the other. However, it has been noticed that at the onset of conflict, the casualty rates strangely appear to be more a function of one's size than of its opponent. In an attempt to explore this initial stage of battle, R. Peterson [39] introduced this model:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_2 \end{cases}$$

where $a, b > 0$.

3.1.5. *Morse and Kimball.* All the previous models attempted to provide a description of the casualty rates within the context of an engagement. However, military operations are not composed strictly of losses directly caused by the opponent. In fact, many losses are caused by the operations themselves (i.e. vehicle accidents, etc...). P. Morse and G. Kimball [36] put forth the hypothesis that losses from both combat and related operations contributed to the whole. The model is based on the operational loss ratio being proportional to one's numbers while the combat loss ratio is proportional to the size of its enemy.

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2 - \beta x_1, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 - \alpha x_2 \end{cases}$$

where $a, b, \alpha, \beta > 0$.

3.1.6. *Coleman.* This models introduces another aspect of battle: the reinforcements. In addition to losses related to combat and operations, the number of combatants involved in a battle also varies when reinforcements are brought to bear or when elements are withdrawn. As such, the model presented by Coleman [14] has an additional component to the expression of the variation of each forces.

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1 - bx_2 + R_{x_1}, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -cx_1 - dx_2 + R_{x_2} \end{cases}$$

where $a, b, c, d > 0$ and R_{x_1}, R_{x_2} can be either positive or negative and are generally considered to be step functions .

3.1.7. *Hembold.* In 1964, R. Hembold [26] put forward the idea that when two forces meet, if there is a sufficiently large difference in size, there is what he termed an *inefficiency of scale*. In other words, attempting to kill a fly with a sledge hammer introduces some inefficiencies. To compensate for this in the Aimed Fire model, he introduced two mappings that are a function of the ratio of the opposing forces numbers. The resulting model is

$$\begin{cases} \dfrac{dx_1}{dt} = -ag(\dfrac{x_1}{x_2})x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bh(\dfrac{x_2}{x_1})x_1 \end{cases}$$

where $a, b > 0$ while $g(.), h(.) \geq 0$ and $g(1) = h(1) = 1$.

3.1.8. *Weiss.* When H.K. Weiss introduced his models [48], he approached the issue of the effect that scale has on the rates from a vulnerability point of view. He suggested that the dominant element affecting losses when a unit's size grows in proportion to the opponent is its vulnerability. His model became

$$\begin{cases} \dfrac{dx_1}{dt} = -a(\dfrac{x_1}{x_2})^{1-W}x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b(\dfrac{x_2}{x_1})^{1-W}x_1 \end{cases}$$

where $a, b > 0$.

3.1.9. *Schreiber.* Col T.S. Schreiber [42] was interested in a model that looked at command and control. He thus put forth a model where the *efficiency of command* came to play through the introduction of command efficiency constants $e_{x_1}, e_{x_2} \in [0, 1]$. These constants represent the effectiveness of intelligence, command and control to bring to bear the power of their force directly on its enemey. A value of 1 signifies that fire power is never wasted and always aimed directly at its enemy while a value of 0 represents "blind" firing. With a command efficiency of 1 Schreiber's model becomes the same as the **Aimed Fire** model and with a command efficiency of 0 it becomes the **Area Fire** model. His model can be represented by

$$\begin{cases} \dfrac{dx_1}{dt} = -a\left\{\dfrac{x_1x_2}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\}, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b\left\{\dfrac{x_1x_2}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\} \end{cases}$$

where $a, b > 0$ and $e_{x_1}, e_{x_2} \in [0, 1]$.

3.1.10. *Richardson's model of Arms Race.* In the first part of the 20th century, L.F. Richardson [40] introduced the Lanchester type models to the arms race between nations where the variables $x_1(t)$ and $x_2(t)$ represent their respective armament levels (often expressed in dollar value). His model of the race between two nations is:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2 - \beta x_1 + g, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 - \alpha x_2 + h \end{cases}$$

where a and b are termed *defence coefficients* and are an expression of one nation's response to threat from the other. On the other hand, the *fatigue* or *restraint* coefficients, α and β, are an expression of the restraining influences, internal or external, exerted on a nations towards its disarmament. All four of these coefficients are considered to be positive. The remaining constants, g and h, express the agressive tendency of the nations. A posive value is significant of a nation that would tend to acquire weapons even when no threat is present.

All previous models are given with constant coefficients $a, b, \alpha, \beta, c, d, \cdots$. In all cases, related models are also considered by replacing these coefficients with variable ones $a(t), b(t), \alpha(t), \beta(t), \cdots$ where $a(t), b(t), \alpha(t), \beta(t), \cdots > 0, \forall t$.

4. Viable solutions for a differential system

In order to study the Lanchester models we must define what a *viable* solution consists of. In a nutshell, we will define a closed subset of state space, K, of the system to represent be the set of *acceptable* status of our combat equations. That closed subset will be considered *viable* under the differential system if for every initial $x_0 \in K$, there exists at least one solution to the system starting at that point and remaining in K for some time. More formely, using the definitions given by J.P. Aubin [4]:

Definition 4.1 (Viable function)**.** Let K be a subset of a finite dimensional vector space X. We shall say that a function $x(\cdot)$ from $[0, T]$ to X is viable in K on $[0, T]$ if $\forall t \in [0, T]$, $x(t) \in K$.

Consider the following differential equation for $f : \Omega \to X$, $\Omega \subset X$.

$$d(x(t))/dt = f(x(t)),\ x(0) = x_0 \in U. \tag{4.1}$$

Definition 4.2 (Viability and Invariance)**.** Let K be a subset of Ω. We shall say that K is *locally viable* under f if for any initial state x_0 of K, there exist $T > 0$ and a viable solution on $[0, T]$ to differential equation (4.1) starting at x_0. It is said to be (globally) *viable* under f if we can always take $T = \infty$.

The subset K is said to be *invariant* under f if for any initial state x_0 of K, all solutions to the differential equation (4.1) are viable in K.

Throughout the remainder of this article since we are concerned with Lanchester type combat models, given $K \in \mathbf{R}^n$ and a mapping $x(t) : \mathbf{R}_+ \to \mathbf{R}^n$, we will say that $x(t)$ is viable in K whenever $x(t) \in K, \forall t \in \mathbf{R}_+$.

The Nagumo theorem [37] provides an excellent tool to verify the existence of solutions that are viable within a subset U of a Hilbert space X based on $T_U(x)$, the contingent cone of U at x.

Theorem 4.1 (Nagumo)**.** *Let U be a closed subset of a Hilbert space H and f be a continuous map from U to H, $f : U \to H$, such that*

$$\forall x \in U, f(x) \in T_U(x). \tag{4.2}$$

Then for all $x_0 \in U$, there exists $T > 0$ such that equation (4.1) has a viable trajectory on $[0, T]$.

5. Winning cones

As presented in section 4, the Nagumo theorem gives viability conditions in relation to the contingent cone (or tangent cone in the case of a convex subset) at each viable point x.

As a usefull tool in viability analysis of Lanchester type models we will use the notion of *winning cone*. We give now the definition of *winning cone* in the general case since this notion can have other interesting applications in the theory of dynamical systems.

Let E be a real vector space and $K_\alpha, K_\beta \subset E$ be closed convex cones. Let "$\leq$" be the ordering defined by the convex cone $K_\leq \subset E$, i.e. $x \leq y \Leftrightarrow y - x \in K_\leq$.

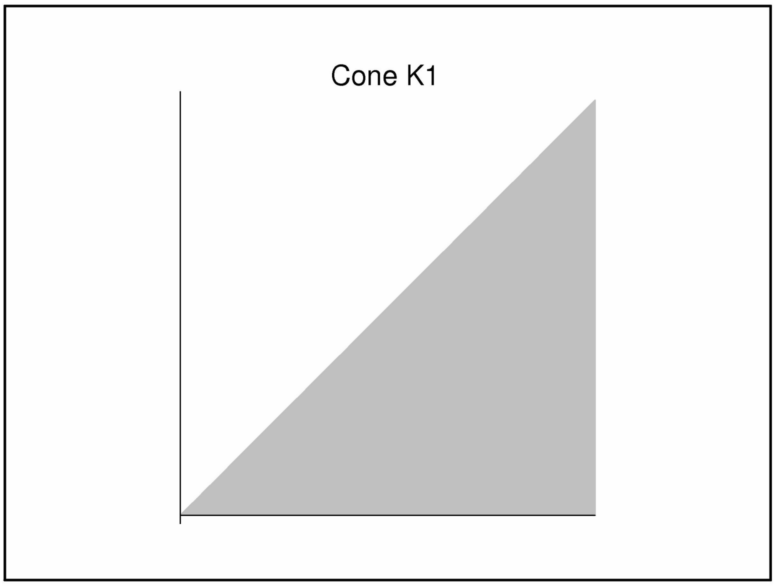

Figure 1: Viability defined by K_1

Definition 5.1 (Winning cone)**.** Consider the vector space $E \times E$. We say that the closed convex poined cone K_0 is a *winning cone* if it has the following properties:

(1) $K_0 \subset K_\alpha \times K_\beta$; and
(2) $(x, y) \in K_0 \Rightarrow x \geq y$, where "$\leq$" is the ordering defined by $K_\leq$.

The above notion was termed *winning cone* as for all points within such a cone, one component *dominates* the other. In many practical problems we have $E = \mathbf{R}^n$ and $K_\alpha, K_\beta, K_\leq = \mathbf{R}^n_+$. In our case, the analysis of Lanchester type models of combat, we will have $E = K_\beta = \mathbf{R}$, $K_\alpha = K_\leq = \mathbf{R}_+$, such that whenever $(x_1, x_2) \in K_0$ we have that x_1 dominates his oponent x_2.

As a first analytical step, we define the four *winning* cones where the first three are selected on the basis that they provide a set in which combat power of one opponent always dominates the other and a fourth cone that guarantees no loss of advantage. For each of these cones K we determine the tangent cone $T_K(x)$ associated with every $x = (x_1, x_2) \in K$, and the conditions it imposes in $\mathbf{R}^2$ on the dynamical system

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x), \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = f_2(x) \end{cases}$$

where $f(x) = \big(f_1(x), f_2(x)\big)$.

5.1. **Winning cone K_1.** The first cone selected, see figure 1, is the most intuitive one. The combat power (i.e. resources available to destroy the enemy) of those facing each other in a real armed conflict cannot be negative. Furthermore, as we wish to select the viable solutions where one, lets choose x_1, always dominates the other, x_2, we obtain the following definition for the first viable set:

$$K_1 := \left\{x \in \mathbf{R}^2 \mid (x_1 \geq x_2) \text{ and } (x_2 \geq 0)\right\}. \tag{5.1}$$

Prior to determining the tangent cone associated with each point in K_1 let us first note the well known fact that for $x \in int(K_1)$ the tangent cone $T_K(x) = X$, see for instance Aubin

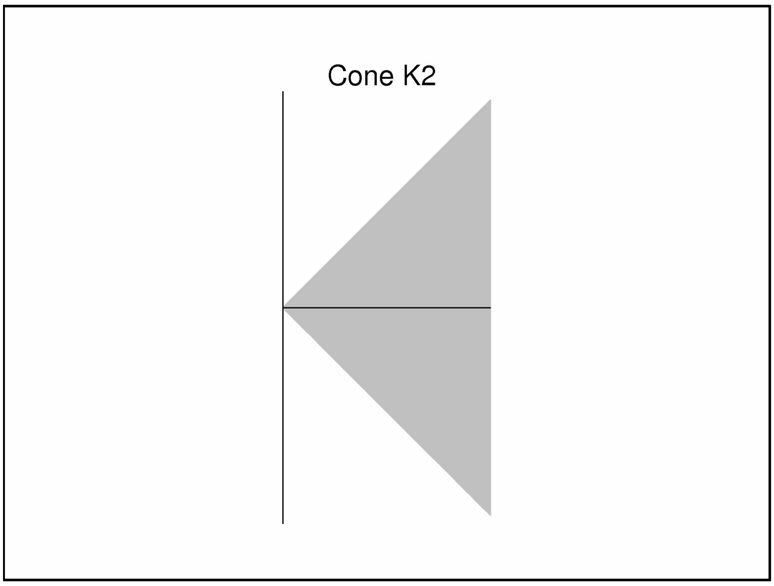

Figure 2: Viability defined by K_2

and Celina [2]. As a consequence, the problem is reduced to finding $T_K(x)$ when x is in the boundary of K_1.

5.1.1. *Case:* $x = 0$. This is the rather trivial case where $T_{K_1}(0) = K_1$ since

$$T_{K_1}(0) = \overline{K_1 - \mathbf{R}_+ 0} = \overline{K}_1 = K_1. \tag{5.2}$$

5.1.2. *Case: "Upper" boundary.* By "Upper" boundary, it is meant the half-line $x_1 = x_2$, where $x_1 > 0$. To determine the set $T_{K_1}(x)$ we consider

$$-\mathbf{R}_+ x = \left\{x \in \mathbf{R}^2 \mid x_1 = x_2,\ x_1 \leq 0\right\}$$
$$-\mathbf{R}_+ x + K = \left\{x \in \mathbf{R}^2 \mid x_1 \geq x_2\right\} = \overline{-\mathbf{R}_+ x + K_1}.$$

Therefore, in order for $f(x)$ to be an element of $T_{K_1}(x)$ it must respect the condition

$$f_1(x) \geq f_2(x), \qquad x_1 = x_2,\ x_1 > 0. \tag{5.3}$$

5.1.3. *Case: "Lower" boundary.* By "Lower" boundary, it is meant the half-line $x_2 = 0$, where $x_1 > 0$. In a manner similar to the previous boundary,

$$-\mathbf{R}_+ x = \left\{x \in \mathbf{R}^2 \mid x_2 = 0,\ x_1 \leq 0\right\}$$
$$-\mathbf{R}_+ x + K_1 = \left\{x \in \mathbf{R}^2 \mid x_2 \geq 0\right\} = \overline{-\mathbf{R}_+ x + K_1}.$$

Therefore, in order for $f(x)$ to be an element of $T_{K_1}(x)$ it must respect the condition

$$f_2(x) \geq 0, \qquad x_2 = 0,\ x_1 > 0. \tag{5.4}$$

5.2. **Winning cone K_2.** The second cone considered, see figure 2, can be viewed a natural extension to the first cone presented in section 5.1. Remember that these cones are meant to be used with Lanchester type models and, for these models, solutions to the system that cross the x_1 axis on the positive side are not only acceptable but desirable. To relax the conditions and allow these desired solutions as part of the viable set, consider the following cone:

$$K_2 := \left\{x \in \mathbf{R}^2 \mid x_1 \geq |x_2|\right\}. \tag{5.5}$$

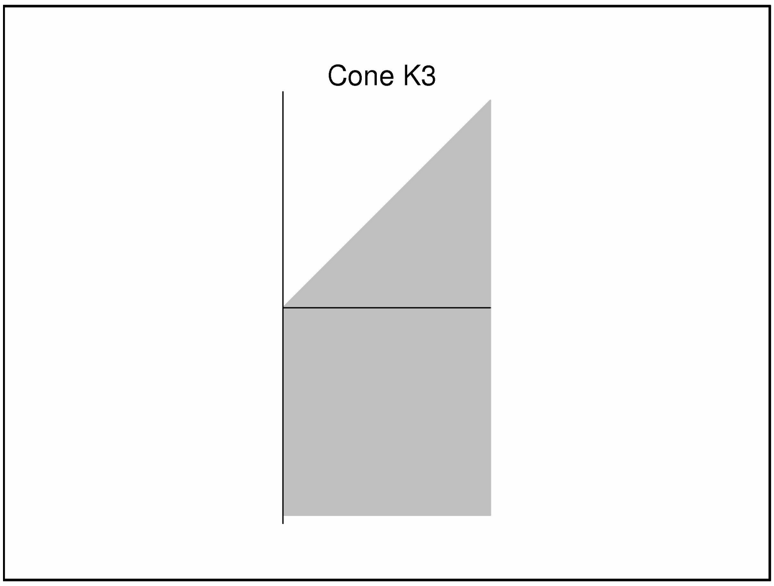

Figure 3: Viability defined by K_3

As for K_1, the tangent cone at $x \in int(K_2)$ is $T_{K_2}(x) = \mathbf{R}^2$. All that then remains to examine is the tangent cone at points on the boundaries of K_2.

5.2.1. *Case:* $x = 0$. Again, this is trivial and $T_{K_2}(0) = K_2$ since

$$T_{K_2}(0) = \overline{K_2 - \mathbf{R}_+ 0} = \overline{K}_2 = K_2. \tag{5.6}$$

5.2.2. *Case: "Upper" boundary.* Here the "Upper" boundary is the same half-line used for the previous cone: $x_1 = x_2$, where $x_1 > 0$. To determine the set $T_{K_2}(x)$ we consider

$$-\mathbf{R}_+ x = \left\{x \in \mathbf{R}^2 \mid x_1 = x_2,\ x_1 \leq 0\right\}$$
$$-\mathbf{R}_+ x + K_2 = \left\{x \in \mathbf{R}^2 \mid x_1 \geq x_2\right\} = \overline{-\mathbf{R}_+ x + K_2}.$$

Therefore, in order for $f(x)$ to be an element of $T_{K_2}(x)$ it must respect the condition

$$f_1(x) \geq f_2(x), \qquad x_1 = x_2,\ x_1 > 0. \tag{5.7}$$

5.2.3. *Case: "Lower" boundary.* Here the "Lower" boundary differs from the previous cone. It is the half-line $x_1 = -x_2$, where $x_1 > 0$ and

$$-\mathbf{R}_+ x = \left\{x \in \mathbf{R}^2 \mid x_1 = -x_2,\ x_1 \leq 0\right\}$$
$$-\mathbf{R}_+ x + K_2 = \left\{x \in \mathbf{R}^2 \mid x_1 \geq -x_2\right\} = \overline{-\mathbf{R}_+ x + K_2}.$$

Therefore, in order for $f(x)$ to be an element of $T_{K_2}(x)$ it must respect the condition

$$f_1(x) \geq -f_2(x), \qquad x_1 = -x_2,\ x_1 > 0. \tag{5.8}$$

5.3. **Winning cone K_3.** This third cone, see figure 3, is the least restrictive one where x_1 exceeds x_2 without allowing x_1 to fall below 0, effectively disallowing solutions that intersect the axis x_2. Furthermore, this is the *largest* cone satisfying Definition 5.1 with $E = K_\beta = \mathbf{R}$, $K_\alpha = K_\leq = \mathbf{R}_+$ and we can write $K_2 = \left\{(x_1, x_2) \in K_\alpha \times K_\beta \,\middle|\, x \geq y\right\}$. The subset of $\mathbf{R}^2$ defining K_3 is

$$K_3 := \left\{x \in \mathbf{R}^2 \mid x_1 \geq x_2 \text{ and } x_1 \geq 0\right\}. \tag{5.9}$$

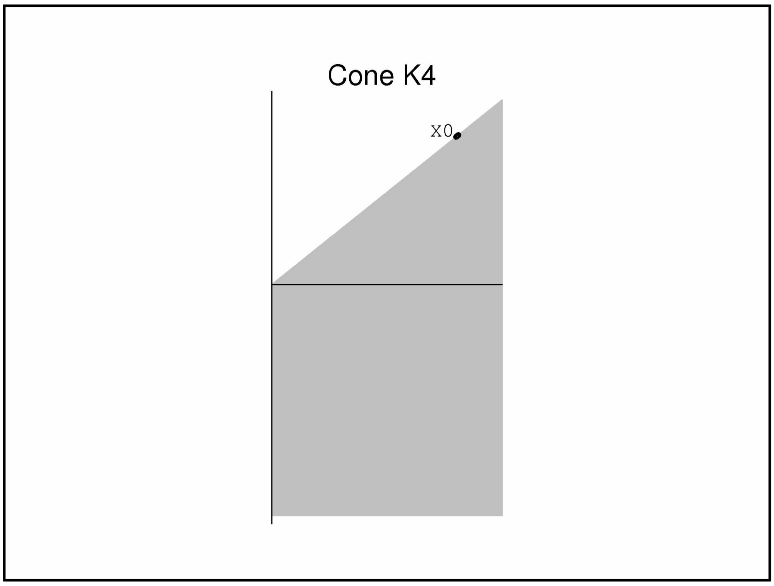

Figure 4: Viability defined by K_4

As for both previous cones, the tangent cone at $x \in int(K_3)$ is $\mathbf{R}^2$. Again, all that then remains to examine is the tangent cone at points on the boundaries of K_3.

5.3.1. *Case:* $x = 0$. It is clear that $T_{K_3}(0) = K_3$ since

$$T_{K_3}(0) = \overline{K_3 - \mathbf{R}_+ 0} = \overline{K}_3 = K_3. \tag{5.10}$$

5.3.2. *Case: "Upper" boundary.* Once more, the "Upper" boundary, is the same half-line used in previous cones: $x_1 = x_2$, where $x_1 > 0$. To determine the set $T_{K_3}(x)$ we consider

$$-\mathbf{R}_+ x = \left\{x \in \mathbf{R}^2 \mid x_1 = x_2,\ x_1 \leq 0\right\}$$
$$-\mathbf{R}_+ x + K_3 = \left\{x \in \mathbf{R}^2 \mid x_1 \geq x_2\right\} = \overline{-\mathbf{R}_+ x + K_3}.$$

In order for $f(x)$ to be an element of $T_{K_3}(x)$ it must respect the condition

$$f_1(x) \geq f_2(x), \qquad x_1 = x_2,\ x_1 > 0. \tag{5.11}$$

5.3.3. *Case: "Lower" boundary.* This "Lower" boundary is simply the half-line $x_1 = 0$, where $x_2 < 0$ and

$$-\mathbf{R}_+ x = \left\{x \in \mathbf{R}^2 \mid x_1 = 0,\ x_2 \geq 0\right\}$$
$$-\mathbf{R}_+ x + K_3 = \left\{x \in \mathbf{R}^2 \mid x_1 \geq 0\right\} = \overline{-\mathbf{R}_+ x + K_3}.$$

Therefore, in order for $f(x)$ to be an element of $T_{K_3}(x)$ it must respect the condition

$$f_1(x) \geq 0, \qquad x_1 = 0,\ x_2 < 0. \tag{5.12}$$

5.4. **Winning cone K_4.** This fourth and last cone, see figure 4, is somewhat different then the previous ones as its definition is dependant on the starting point x_0. This *gaining* cone expresses the desire to improve one's position in reference to its opponent. The subset of $\mathbf{R}^2$ defining K_4 is

$$K_4 := \left\{x \in \mathbf{R}^2 \mid x_1 \geq \frac{x_1(0)}{x_2(0)} x_2 \text{ and } x_1 \geq 0\right\}. \tag{5.13}$$

Again, the tangent cone at $x \in int(K_3)$ is $\mathbf{R}^2$ while that at its boundaries is as follows.

Winning cone(K)	**Points in K**	**Conditions on $f(x)$**
K_1	$x \in int(K_1)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_1$
	$x_1 = x_2,\ x_1 > 0$	$f_1(x) \geq f_2(x)$
	$x_2 = 0,\ x_1 > 0$	$f_2(x) \geq 0$
K_2	$x \in int(K_2)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_2$
	$x_1 = x_2,\ x_1 > 0$	$f_1(x) \geq f_2(x)$
	$x_1 = -x_2,\ x_1 > 0$	$f_1(x) \geq -f_2(x)$
K_3	$x \in int(K_3)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_3$
	$x_1 = x_2,\ x_1 > 0$	$f_1(x) \geq f_2(x)$
	$x_1 = 0,\ x_2 < 0$	$f_1(x) \geq 0$
K_4	$x \in int(K_4)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_4$
	$x_1 = \frac{x_1(0)}{x_2(0)}x_2,\ x_1 > 0$	$f_1(x) \geq \frac{x_1(0)}{x_2(0)}f_2(x)$
	$x_1 = 0,\ x_2 < 0$	$f_1(x) \geq 0$

Table 5.1: Tangent cones at $x \in K$

5.4.1. *Case:* $x = 0$. As for the previous *Winning Cones*, $T_{K_4}(0) = K_4$ since

$$T_{K_4}(0) = \overline{K_4 - \mathbf{R}_+ 0} = \overline{K}_4 = K_4. \tag{5.14}$$

5.4.2. *Case: "Upper" boundary.* In this case, the "Upper" boundary varies for each Cauchy problem as it depends on x_0. Points on this boundary are those meeting the two conditions: $x_1 = \frac{x_1(0)}{x_2(0)}x_2$ and $x_1 > 0$. To determine the set $T_{K_4}(x)$ we consider

$$-\mathbf{R}_+ x = \left\{ x \in \mathbf{R}^2 \mid x_1 = \frac{x_1(0)}{x_2(0)}x_2,\ x_1 \leq 0 \right\}$$
$$-\mathbf{R}_+ x + K_4 = \left\{ x \in \mathbf{R}^2 \mid x_1 \geq \frac{x_1(0)}{x_2(0)}x_2 \right\} = \overline{-\mathbf{R}_+ x + K_4}.$$

Therefore, in order for $f(x)$ to be an element of $T_{K_4}(x)$ it must respect the condition

$$f_1(x) \geq \frac{x_1(0)}{x_2(0)}f_2(x), \qquad x_1 = \frac{x_1(0)}{x_2(0)}x_2,\ x_1 > 0. \tag{5.15}$$

5.4.3. *Case: "Lower" boundary.* This "Lower" boundary is simply the half-line $x_1 = 0$, where $x_2 < 0$ and the resulting tangent cone is the same as for K_3.

$$-\mathbf{R}_+ x = \left\{ x \in \mathbf{R}^2 \mid x_1 = 0,\ x_2 \geq 0 \right\}$$
$$-\mathbf{R}_+ x + K_4 = \left\{ x \in \mathbf{R}^2 \mid x_1 \geq 0 \right\} = \overline{-\mathbf{R}_+ x + K_4}.$$

Therefore, in order for $f(x)$ to be an element of $T_{K_4}(x)$ it must respect the condition

$$f_1(x) \geq 0, \qquad x_1 = 0,\ x_2 < 0. \tag{5.16}$$

5.5. **Compilation of conditions – Winning cones.** The results of this section are important in the further study of viable solutions. For easy reference, the various conditions generated on all cones are presented in table 5.1.

6. WINNING CONES FOR DIFFERENTIAL LANCHESTER TYPE MODELS

In the past, one of the biggest difficulty in the analysis of combat through the use of Lanchester type mathematical models has been the determination of the coefficients in each of them. The problem is further amplified by the lack of data available and its lack of precision that is caused by the unclear collection or recollection of information during a conflict. This lack of clear perception of the ongoing battle is often referred to as the *fog of war*. To avoid this slippery ground, our approach is based on the application of viability theory where we establish conditions on the Lanchester coefficients that ensure viable solutions. Through careful definition, the viable sets selected in section 5 guarantee one's victory against his opponent.

For each model presented, we will derive a set of conditions on the coefficients to satisfy the viability requirements listed in table 5.1 such that once a trajectory of a solution to the differential equation enters the viable set, it never leaves it again .

6.1. **Aimed Fire.** As we recall, from section 3.1.1 the model introduced in 1914 is expressed mathematically by

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 \end{cases}$$

where $a, b > 0$. To link the models with the differential equation given earlier, the right hand side of equation (4.1) is defined by

$$f(x) = \begin{cases} f_1(x) = -ax_2 \\ f_2(x) = -bx_1 \end{cases}$$

where $x = (x_1, x_2)$.

6.1.1. *Winning cone is defined by* K_1. From the tangent cones defined in table 5.1, it is clear that for points belonging to the interior of the cone, $x \in int(K_1)$, any values for a and b are viable. Similarly, since $f(0) = 0$ for any $a, b > 0$ we have $f(x) \in K_1$ for any choice of coefficients. It remains to examine the coefficients required to meet the conditions of the Nagumo theorem at the cone's boundaries. For the *upper* boundary, we must have

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -ax_1 &\geq -bx_1 \\ -ax_1 &\geq -bx_1, && x_1 = x_2 \\ -a &\geq -b, && x_1 > 0 \\ a &\leq b. \end{aligned} \tag{6.1}$$

While the restrictions on the coefficients for the *lower* boundary become

$$\begin{aligned} f_2(x) &\geq 0 \\ -bx_1 &\geq 0 \\ -b(1) &\geq 0, && \text{choose } x = (1,0) \text{ on boundary} \\ -b &\geq 0 \\ b &\leq 0 \end{aligned} \tag{6.2}$$

but since the model requires $b > 0$ it implies that there is no viable solution to equation (4.1) that remain viable once it reaches the lower boundary of cone K_1.

6.1.2. *Winning cone is defined by* K_2. As for cone K_1 in section 6.1.1, there are no restrictions for the choice of a or b for $x \in int(K_2)$ or $x = 0$. As the tangent cone for interior points is the totality of the space independently of the model chosen, further analysis in this article will be confined to the study of the behavior at the boundaries of cones. As the *upper* boundary of K_2 yields the same condition on $f(x)$ as did K_1, the conditions on the coefficients is given by equation (6.1). For the last region of K_2, the *lower* boundary, from table 5.1 we have

$$\begin{aligned} f_1(x) &\geq -f_2(x) \\ -ax_2 &\geq bx_1 \\ -a(-x_1) &\geq bx_1, & x_2 = -x_1 \\ ax_1 &\geq bx_1 \\ a &\geq b, & x_1 > 0. \end{aligned} \tag{6.3}$$

To combine all the conditions on the regions of cone K_2, we are required to meet the inequalities of equations (6.1) and (6.3). For a solution to the *aimed fire* model to remain viable once it enters K_2, its coefficients must be such that $a = b$. We can see why using K_2 as a viable set for this particular model is too restrictive when we compare it to the results obtained in section 6.1.3.

6.1.3. *Winning cone is defined by* K_3. It is obvious through the similarities with the previous cones that the only restrictions additional to those imposed by the model are those generated by the *upper* and *lower* boundaries. In the case of the former, the results are also identical and given by equation (6.1) while for the latter, using table 5.1 we have

$$\begin{aligned} f_1(x) &\geq 0 \\ -ax_2 &\geq 0 \\ -a &\leq 0, & x_2 < 0 \\ a &\geq 0. \end{aligned} \tag{6.4}$$

The restrictions imposed by equation (6.4) are more relaxed then that imposed by the model. As a consequence only the additional inequality of equation (6.1) is required to consider. For a solution to the *aimed fire* model to remain viable once it enters K_3, its coefficients must be such that $a \leq b$.

6.1.4. *Winning cone is defined by* K_4. From the inspection of table 5.1 the only differences between this cone and K_3 are at the “Upper” boundary. From the corresponding entry in the table we have

$$\begin{aligned} f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\ -ax_2 &\geq \frac{x_1(0)}{x_2(0)}(-b)x_1 \\ -ax_2 &\geq \frac{x_1(0)}{x_2(0)}(-b)\frac{x_1(0)}{x_2(0)}x_2, & x_1 = \frac{x_1(0)}{x_2(0)}x_2 \\ -a &\geq \left(\frac{x_1(0)}{x_2(0)}\right)^2 (-b), & x_2 > 0 \\ a &\leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 b. \end{aligned} \tag{6.5}$$

The restrictions on the coefficients imposed by the model are more restrictive then the boundary conditions with the exception of (6.5). As such, to make K_4 viable the coefficients must meet $\frac{a}{b} \leq \left(\frac{x_1(0)}{x_2(0)}\right)^2$ and $a, b > 0$.

6.2. **Area Fire.** In section 3.1.2 we introduced the model

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

where $a, b > 0$. Rewriting this model in the form of the differential equation given earlier, the right hand side of equation (4.1) is defined by

$$f(x) = \begin{cases} f_1(x) = -ax_1x_2 \\ f_2(x) = -bx_1x_2 \end{cases}$$

where $x = (x_1, x_2)$.

6.2.1. *Winning cone is defined by* K_1. Let us consider the tangent cones defined in table 5.1 in conjunction with the *Area Fire* model. At the origin, since $f(0) = 0$ for any $a, b > 0$ we have $f(x) \in K_1$ for any choice of coefficients. To meet the requirements of the Nagumo theorem, the coefficients at the *upper* boundary must be such that

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -ax_1x_2 &\geq -bx_1x_2 \\ -a &\geq -b, \qquad\qquad x_1, x_2 > 0 \\ a &\leq b. \end{aligned} \tag{6.6}$$

While the restrictions on the coefficients for the *lower* boundary become

$$\begin{aligned} f_2(x) &\geq 0 \\ -bx_1x_2 &\geq 0 \\ b &\in \mathbf{R}, \qquad\qquad x_1, x_2 > 0 \end{aligned} \tag{6.7}$$

and of course, there is no further restrictions on a. As a result, combining the results of (6.6) and (6.7) with the initial conditions imposed by the model, for a solution entering K_2 to remain viable, the coefficients must be such that $0 < a \leq b$.

6.2.2. *Winning cone is defined by* K_2. As for K_1, the origin does not impose restrictions on the coefficients and the behavior at the *upper* boundary is identical. Focusing our attention to the remaining frontier,

$$\begin{aligned} f_1(x) &\geq -f_2(x) \\ -ax_1x_2 &\geq bx_1x_2 \\ -a &\leq b, \qquad\qquad x_2 = -x_1,\ x_1x_2 < 0. \end{aligned} \tag{6.8}$$

Since $a,\ b > 0$ from the model's initial conditions, this last inequality does not further restrict the viable values for the coefficients. Consequently the conditions of the Nagumo theorem are met for K_2 when $0 < a \leq b$.

6.2.3. *Winning cone is defined by* K_3. Again, the only analysis remaining to be carried out is that of the behavior at the *lower* boundary. From table 5.1

$$\begin{aligned} f_1(x) &\geq 0 \\ -ax_1x_2 &\geq 0 \\ 0 &\geq 0, & x_1 = 0 \\ a, b &\in \mathbf{R}. \end{aligned} \tag{6.9}$$

Combining this result with the conditions of equation (6.6) we are again left with the viability conditions $0 < a \leq b$ for K_3.

6.2.4. *Winning cone is defined by* K_4. As above, we only need to inspect the behavior of the system at the "Upper" boundary since this is the only difference with K_3. From the corresponding entry in table 5.1 we have

$$\begin{aligned} f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\ -ax_1x_2 &\geq \frac{x_1(0)}{x_2(0)} (-b)x_1x_2 \\ -a &\geq \frac{x_1(0)}{x_2(0)} (-b), & x_1, x_2 > 0 \\ a &\leq \frac{x_1(0)}{x_2(0)} b. \end{aligned} \tag{6.10}$$

When examining the conditions at all the boundaries and including the limitations on the coefficients imposed by the model, we obtain that for K_4 to be viable, it is required that $0 < a \leq \frac{x_1(0)}{x_2(0)} b$. In this case, the limiting factor is from equation (6.10).

6.3. **Brackney.** To model the case where asymmetric forces are at play, such as when a convention force x_1 meets with a guerilla force x_2, we introduced the following in section 3.1.3.

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, & x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

where $a, b > 0$. To fit the original differential equation form as before, the right hand side of equation (4.1) becomes

$$f(x) = \begin{cases} f_1(x) = -ax_2 \\ f_2(x) = -bx_1x_2 \end{cases}$$

where $x = (x_1, x_2)$.

6.3.1. *Winning cone is defined by* K_1. A quick analysis of the system at the origin finds that $f(x) \in K_1$ for any choice of coefficients ($f(0) = 0$). Applying the conditions of table 5.1 to explore the first boundary:

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -ax_2 &\geq -bx_1x_2 \\ -a &\geq -bx_1, & x_2 > 0 \\ a &\leq bx_1 \\ \frac{a}{b} &\leq x_1, & b > 0. \end{aligned} \tag{6.11}$$

Unfortunately, the model allows (for this boundary) $x_1 \in (0, \infty)$ which results for $a = \inf_{x_1 \in (0,\infty)} bx_1 = 0$ which is impossible as $a > 0$ from the initial coefficients definition. However, the reality being modelled does not typically allow for $x_1, x_2 < 1$ so we could modify the model to become

$$f(x) = \begin{cases} f_1(x) = -ax_2, & x_1 \geq 1; \quad 0 \; otherwise \\ f_2(x) = -bx_1x_2, & x_2 \geq 1; \quad 0 \; otherwise \end{cases} \tag{6.12}$$

resulting in the restriction $\frac{a}{b} \leq 1$ to obtain $f(x)$ viable in K_1. At the *lower* boundary

$$\begin{aligned} f_2(x) &\geq 0 \\ -bx_1x_2 &\geq 0 \\ 0 &\geq 0, \qquad\qquad x_2 = 0 \\ a, b &\in \mathbf{R}, \end{aligned} \tag{6.13}$$

which does not further restrict the coefficients. If the model being considered is the modified version provided at equation (6.12) then obviously $f_2(x) \geq 0$ since $f_2(x) = 0$ along this boundary. As a consequence, no conditions on the coefficients allow this model (in it's original form) to remain viable in K_1. However the modified version expressed at equation (6.12) remains viable if $0 < a \leq b$.

6.3.2. *Winning cone is defined by* K_2. The problems identified in section 6.3.1 are again present for this cone at the *upper* boundary. Exploring the behavior at the *lower* end based on table 5.1,

$$\begin{aligned} f_1(x) &\geq -f_2(x) \\ -ax_2 &\geq bx_1x_2 \\ -a &\leq bx_1, \qquad\qquad x_2 < 0 \\ -\frac{a}{b} &\leq x_1 \end{aligned} \tag{6.14}$$

and since $a, b, x_1 > 0$ at this boundary, it does not further restrict the values of the coefficients. However, as the *upper* boundary required the introduction of a modified version, the behavior at this side of the cone is affected and becomes

$$\begin{aligned} f_1(x) &\geq -f_2(x) \\ -ax_2 &\geq 0 \\ -a &\leq 0, \qquad\qquad x_2 < 0 \\ a &\geq 0 \end{aligned} \tag{6.15}$$

which is less restrictive than the original conditions on a and does not impose any constraints on b. To discuss a possible viability of Brackney's model in K_2 the modified version presented at equation (6.12) should be considered resulting on the constraint $0 < a \leq b$.

6.3.3. *Winning cone is defined by* K_3. If we were to analyse the original model, the *lower* boundary would not yield any additional restrictions on the coefficients however, the problems cited in section 6.3.1 are still present. The modified model yields $f(x) = 0$ for all x on the *lower* boundary which guarantees $f(x) \in T_{K_3}(x)$. Again, as with the other cones, the Nagumo conditions are satisfied whenever $0 < a \leq b$.

6.3.4. *Winning cone is defined by* K_4. The inspection of the behavior of the system at the "Upper" boundary yields:

$$\begin{aligned} f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\ -ax_2 &\geq \frac{x_1(0)}{x_2(0)}(-b)x_1x_2 \\ -a &\geq \frac{x_1(0)}{x_2(0)}(-b)x_1, && x_2 > 0 \\ \frac{a}{b} &\leq \frac{x_1(0)}{x_2(0)}x_1, && -b < 0. \end{aligned} \tag{6.16}$$

Again, as for equation (6.11) of section 6.3.1 this is impossible for $a > 0$. However, in a similar fashion, modifying the model to become

$$f(x) = \begin{cases} f_1(x) = -ax_2, \quad x_1 \geq 1; & 0 \; otherwise \\ f_2(x) = -bx_1x_2, \quad x_2 \geq \dfrac{x_2(0)}{x_1(0)}; & 0 \; otherwise \end{cases} \tag{6.17}$$

allows K_4 to be viable for $0 < a \leq \frac{x_1(0)}{x_2(0)} b$ (from equation (6.16) with $x_1 = 1$).

6.4. **Peterson.** We recall that R. Peterson [39] introduced this model:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_2 \end{cases}$$

where $a, b > 0$. This model yields for the right hand side of equation (4.1)

$$f(x) = \begin{cases} f_1(x) = -ax_1 \\ f_2(x) = -bx_2 \end{cases}$$

where $x = (x_1, x_2)$.

6.4.1. *Winning cone is defined by* K_1. Concentrating our observations to the boundaries, let us first remark that $f(0) = 0 \in T_{K_1}(0)$. At the boundary $x_1 = x_2$ Nagumo's conditions require

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -ax_1 &\geq -bx_2 \\ ax_1 &\leq bx_2 \\ ax_1 &\leq bx_1, && x_1 = x + 2 \\ a &\leq b, && x_1 > 0 \end{aligned} \tag{6.18}$$

which is the same type of conditions as that obtained on the previous models. Examining the frontier $x_2 = 0$ yields

$$\begin{aligned} f_2(x) &\geq 0 \\ -bx_2 &\geq 0 \\ 0 &\geq 0, && x_2 = 0 \\ a, b &\in \mathbf{R}. \end{aligned} \tag{6.19}$$

Since the initial restrictions on the coefficients is more restrictive, we remain with $a, b > 0$ at this boundary. As such, the conditions of Nagumo's theorem are met throughout K_1 when $0 < a \leq b$.

6.4.2. *Winning cone is defined by* K_2. The only subset remaining to investigate for this cone and this model is the *lower* boundary. From table 5.1 we have

$$\begin{aligned} f_1(x) &\geq -f_2(x) \\ -ax_1 &\geq bx_2 \\ -ax_1 &\geq b(-x_1), && x_2 = -x_1 \\ a &\leq b, && -x_1 < 0 \end{aligned} \tag{6.20}$$

which together with equation (6.18) once more sets the Nagumo condition to $0 < a \leq b$.

6.4.3. *Winning cone is defined by* K_3. Analysis of the boundary at $x_1 = 0$ can easily be seen to always satisfy $f_1(x) \geq 0$ and therefore provides no addition conditions to equation (6.18). The restrictions on the coefficients are therefore the same as that of K_1 and K_2.

6.4.4. *Winning cone is defined by* K_4. The inspection of the behavior of the system at the "Upper" boundary yields:

$$\begin{aligned} f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\ -ax_1 &\geq \frac{x_1(0)}{x_2(0)}(-b)x_2 \\ ax_1 &\leq \frac{x_1(0)}{x_2(0)} bx_2 \\ a &\leq \frac{x_1(0)}{x_2(0)} b, && x_1 = x_2 > 0. \end{aligned} \tag{6.21}$$

Here again $f_1(0) = 0$ at the lower boundary and (6.21) is the only additional restriction for the viability of the model.

6.5. **Morse and Kimball.** With the introduction of operational losses, section 3.1.5 presented the following model by Morse and Kimball.

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2 - \beta x_1, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 - \alpha x_2 \end{cases}$$

where $a, b, \alpha, \beta > 0$. This model yields for the right hand side of equation (4.1)

$$f(x) = \begin{cases} f_1(x) = -ax_2 - \beta x_1 \\ f_2(x) = -bx_1 - \alpha x_2 \end{cases}$$

where $x = (x_1, x_2)$.

6.5.1. *Winning cone is defined by* K_1. The first point to consider is the behavior at the origin. As $f(0) = 0 \in T_K(0)$, the condition's of Nagumo's theorem are met for all values of coefficients (a, b, α, β). Again, using the results compiled in section 5.5 exploration of the *upper* boundary reveals

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -ax_2 - \beta x_1 &\geq -bx_1 - \alpha x_2 \\ -(a+\beta)x_1 &\geq -(b+\alpha)x_1, & x_1 = x_2 \\ a + \beta &\leq b + \alpha. \end{aligned} \tag{6.22}$$

At the remaining frontier of K_1, the requirement is

$$\begin{aligned} f_2(x) &\geq 0 \\ -bx_1 - \alpha x_2 &\geq 0 \\ -bx_1 &\geq 0, & x_2 = 0 \\ -b &\geq 0, & x_1 > 0 \\ b &\leq 0 \end{aligned} \tag{6.23}$$

which is impossible since the model stipulates, $b > 0$. There is therefore no conditions on the coefficients that would make $f(x)$ viable on K_1 for the model of Morse and Kimball.

6.5.2. *Winning cone is defined by* K_2. As for the previous models, the only variation between cones is at the *lower* boundary. Starting with the general condition

$$\begin{aligned} f_1(x) &\geq -f_2(x) \\ -ax_2 - \beta x_1 &\geq bx_1 + \alpha x_2 \\ -a(-x_1) - \beta x_1 &\geq bx_1 + \alpha(-x_1), & x_2 = -x_1 \\ (a-\beta)x_1 &\geq (b-\alpha)x_1 \\ a - \beta &\geq b - \alpha, & x_1 > 0. \end{aligned} \tag{6.24}$$

Combining the conditions imposed by all the boundaries,

$$\begin{aligned} a - b &\leq \alpha - \beta, & \text{from (6.22)} \\ a - b &\geq \beta - \alpha, & \text{from (6.24)} \\ |a - b| &\leq \alpha - \beta. \end{aligned} \tag{6.25}$$

As a direct consequence, it is clear that $\alpha - \beta \geq 0$. As a conclusion, $f(x)$ will be viable in K_2 if $|a - b| \leq \alpha - \beta$ which is equivalent to saying $(\alpha - \beta, a - b) \in K_2$.

6.5.3. *Winning cone is defined by* K_3. By increasing the viable set to K_3, the viability condition becomes

$$\begin{aligned} f_1(x) &\geq 0 \\ -ax_2 - \beta x_1 &\geq 0 \\ -ax_2 &\geq 0, & x_1 = 0 \\ -a &\leq 0, & x_2 < 0 \\ a &\geq 0 \end{aligned} \tag{6.26}$$

which does not further restrain the range of viable coefficients. The only remaining requirement to Nagumo's theorem is therefore stated by equation 6.22.

6.5.4. *Winning cone is defined by* K_4. The inspection of the behavior of the system at the "Upper" boundary yields:

$$\begin{aligned}
f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\
(-ax_2 - \beta x_1) &\geq \frac{x_1(0)}{x_2(0)}(-bx_1 - \alpha x_2) \\
-(a+\beta)x_1 &\geq -\frac{x_1(0)}{x_2(0)}(b+\alpha)x_1, \qquad x_1 = x_2 \\
a + \beta &\leq \frac{x_1(0)}{x_2(0)}(b+\alpha), \qquad x_1 > 0.
\end{aligned} \tag{6.27}$$

The lower boundary yields the same result as for K_3 and as such, the viability conditions are those expressed by (6.27).

6.6. **Coleman.** This model, from section 3.1.6, introduced another aspect of battle: the reinforcements.

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1 - bx_2 + R_{x_1}, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -cx_1 - dx_2 + R_{x_2} \end{cases}$$

where $a, b, c, d > 0$ and R_{x_1}, R_{x_2} can be either positive or negative and are generally considered to be step functions . This model yields for the right hand side of equation (4.1)

$$f(x) = \begin{cases} f_1(x) = -ax_1 - bx_2 + R_{x_1} \\ f_2(x) = -cx_1 - dx_2 + R_{x_2} \end{cases}$$

where $x = (x_1, x_2)$.

6.6.1. *Winning cone is defined by* K_1. To begin the analysis consider $f(0)$. Since $T_{K_1}(0) = K_1$ this implies that viability requires that reinforcements when both forces are at the brink of annihilation be such that $0 \leq R_{x_2} \leq R_{x_1}$. Let us turn our attention to the *upper* boundary of K_1.

$$\begin{aligned}
f_1(x) &\geq f_2(x) \\
-ax_1 - bx_2 + R_{x_1} &\geq -cx_1 - dx_2 + R_{x_2} \\
-(a+b)x_1 + R_{x_1} &\geq -(c+d)x_1 + R_{x_2}, \qquad x_1 = x_2 \\
(a+b)x_1 &\leq (c+d)x_1 + R_{x_1} - R_{x_2} \\
(a+b-c-d)x_1 &\leq R_{x_1} - R_{x_2} \\
(a+b-c-d) &\leq \frac{R_{x_1} - R_{x_2}}{x_1}, \qquad x_1 > 0.
\end{aligned} \tag{6.28}$$

To continue the analysis, first consider the situation where $R_{x_1} < R_{x_2}$. Since $x_1 \in (0, \infty)$ it would be required that $a + b - c - d \leq -\infty$ which means that no value of a, b, c, d could guarantee Nagumo's condition in this case. The model must than be restricted to $R_{x_1} \geq R_{x_2}$ which has for consequence that $\lim_{x_1 \to \infty} \frac{R_{x_1} - R_{x_2}}{x_1} = 0$. Applying this to equation (6.28) results in

$$\begin{aligned}
(a+b-c-d) &\leq 0 \\
a + b &\leq c + d.
\end{aligned} \tag{6.29}$$

We have yet to examine the behavior of the system at the *lower* boundary where we require

$$\begin{aligned} f_2(x) &\geq 0 \\ -cx_1 - dx_2 + R_{x_2} &\geq 0 \\ -cx_1 + R_{x_2} &\geq 0, \qquad\qquad x_2 = 0 \\ R_{x_2} &\geq cx_1 \end{aligned} \tag{6.30}$$

although mathematically possible, in practice, such a condition could require the opponent to commit and have an astronomical amount of forces in reserve.

6.6.2. *Winning cone is defined by* K_2. The results stated at equation (6.29) are obviously still valid. Turning our attention to the boundary where $x_1 = -x_2$:

$$\begin{aligned} f_1(x) &\geq -f_2(x) \\ -ax_1 - bx_2 + R_{x_1} &\geq cx_1 + dx_2 - R_{x_2} \\ -ax_1 + bx_1 + R_{x_1} &\geq cx_1 - dx_1 - R_{x_2}, \qquad\qquad x_2 = -x_1 \\ (b - a + d - c)x_1 &\geq -R_{x_1} - R_{x_2} \\ (a - b + c - d)x_1 &\leq R_{x_1} + R_{x_2} \\ a - b + c - d &\leq \frac{R_{x_1} + R_{x_2}}{x_1}, \qquad\qquad x_1 > 0. \end{aligned} \tag{6.31}$$

We must again consider two cases. If $R_{x_1} + R_{x_2} < 0$ then, using the same reasoning as that of section 6.6.1, it is impossible to determine satisfactory coefficients. We must therefore impose the restriction $R_{x_1} + R_{x_2} \geq 0$ resulting in $a - b + c - d \leq 0$.

Putting all this together for K_2 we conclude that for $f(x)$ to be viable, it is also required that $R_{x_1} \geq R_{x_2}$, $R_{x_1} + R_{x_2} \geq 0$ which is equivalent to $|R_{x_2}| \leq R_{x_1}$ (i.e. $(R_{x_1}, R_{x_2}) \in K_2$). In addition, the coefficients must satisfy $|b - c| \leq d - a$ (i.e. $(d - a, b - c) \in K_2$).

6.6.3. *Winning cone is defined by* K_3. As for the previous models, it is only required to examine the behavior of the system at the boundary where $x_1 = 0$. As per table 5.1

$$\begin{aligned} f_1(x) &\geq 0 \\ -ax_1 - bx_2 + R_{x_1} &\geq 0 \\ -bx_2 + R_{x_1} &\geq 0, \qquad\qquad x_1 = 0 \\ bx_2 &\leq R_{x_1} \\ b &\geq \frac{R_{x_1}}{x_2}, \qquad\qquad x_2 < 0. \end{aligned} \tag{6.32}$$

To remain viable in K_3 the system must be such that whenever $x_1 = 0$ then $b \geq \frac{R_{x_1}}{x_2}$. Additionally from equation (6.29) it is required that $R_{x_1} \geq R_{x_2}$.

6.6.4. *Winning cone is defined by* K_4. The inspection of the behavior of the system at the "Upper" boundary yields:

$$\begin{aligned} f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\ -ax_1 - bx_2 + R_{x_1} &\geq \frac{x_1(0)}{x_2(0)}(-cx_1 - dx_2 + R_{x_2}) \\ -(a+b)x_1 + R_{x_1} &\geq -\frac{x_1(0)}{x_2(0)}(c+d)x_1 + \frac{x_1(0)}{x_2(0)} R_{x_2}, \qquad x_1 = x_2 \\ ((a+b)x_1 - \frac{x_1(0)}{x_2(0)}(c+d))x_1 &\leq R_{x_1} - \frac{x_1(0)}{x_2(0)} R_{x_2} \end{aligned}$$

$$(a+b) - \frac{x_1(0)}{x_2(0)}(c+d) \leq \frac{R_{x_1} - \frac{x_1(0)}{x_2(0)} R_{x_2}}{x_1}. \tag{6.33}$$

Again, in this case, the denominator of the right hand side must be positive. Considering the $\lim_{x_1 \to \infty}$ we obtain:

$$(a+b) \leq \frac{x_1(0)}{x_2(0)}(c+d) \tag{6.34}$$

to guarantee viability.

6.7. **Hembold.** To deal with inefficiency of scale, Hembold (see section 3.1.7) introduced the model:

$$\begin{cases} \dfrac{dx_1}{dt} = -ag(\dfrac{x_1}{x_2})x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bh(\dfrac{x_2}{x_1})x_1 \end{cases}$$

where $a, b > 0$ while $g(.), h(.) \geq 0$ and $g(1) = h(1) = 1$. Accordingly, the right hand side of equation (4.1) is defined by

$$f(x) = \begin{cases} f_1(x) = -ag(\dfrac{x_1}{x_2})x_2 \\ f_2(x) = -bh(\dfrac{x_2}{x_1})x_1 \end{cases}$$

where $x = (x_1, x_2)$.

Although in general the model does not require so, most attempt at applying it consider $g \equiv h$. Hembold himself focused his interest on $g(\frac{x_1}{x_2}) = h(\frac{x_1}{x_2}) = (\frac{x_1}{x_2})^c$.

6.7.1. *Winning cone is defined by* K_1. In a first step, consider $f(0)$. As it currently stands, the value is undefined. To overcome this, we will consider a modification to $f(x)$ such that $f(0) = 0$ and consequently $f(0) \in T_K(0)$. At the *upper* boundary, the conditions are such that we must have

$$\begin{aligned}
f_1(x) &\geq f_2(x) \\
-ag(\frac{x_1}{x_2})x_2 &\geq -bh(\frac{x_2}{x_1})x_1 \\
ag(\frac{x_1}{x_2})x_2 &\leq bh(\frac{x_2}{x_1})x_1 \\
ag(\frac{x_1}{x_1})x_1 &\leq bh(\frac{x_1}{x_1})x_1, && x_1 = x_2 \\
ag(1)x_1 &\leq bh(1)x_1 \\
ax_1 &\leq bx_1, && g(1) = h(1) = 1 \\
a &\leq b, && x_1 > 0 \qquad (6.35)
\end{aligned}$$

which, not surprisingly, is similar to the results in section 6.1. At the remaining boundary set,

$$\begin{aligned}
f_2(x) &\geq 0 \\
-bh(\frac{x_2}{x_1})x_1 &\geq 0 \\
bh(\frac{x_2}{x_1})x_1 &\leq 0 \\
bh(\frac{x_2}{x_1}) &\leq 0, && x_1 > 0. \qquad (6.36)
\end{aligned}$$

We must therefore have that either $b = 0$ or $h(\frac{x_2}{x_1}) = 0$ and since $b > 0$ we conclude that we must have $h(\frac{x_2}{x_1}) = 0$ whenever $x_2 = 0$ (i.e $h(0) = 0$). Combining this with the results expressed at equation (6.35) we have that the system is viable in K_1 when $a \leq b$ and $h(0) = 0$.

6.7.2. *Winning cone is defined by* K_2. To verify viability in K_2, we need to verify the behavior of $f(x)$ at the cone's *lower* boundary where we need

$$\begin{aligned}
f_1(x) &\geq -f_2(x) \\
-ag(\frac{x_1}{x_2})x_2 &\geq bh(\frac{x_2}{x_1})x_1 \\
-ag(\frac{x_1}{-x_1})(-x_1) &\geq bh(\frac{-x_1}{x_1})x_1, && x_2 = -x_1 \\
ag(-1) &\geq bh(-1), && x_1 > 0 \\
\frac{a}{b} &\geq \frac{h(-1)}{g(-1)}, && b > 0, g(-1) \neq 0. \qquad (6.37)
\end{aligned}$$

However, in most application $g \equiv h$ and we have $\frac{a}{b} \geq 1$ which, when combined with equation (6.35) yields $1 \leq \frac{a}{b} \leq 1$ which restricts viability to the case $a = b$.

6.7.3. *Winning cone is defined by* K_3. The interest in this cone is that its lower boundary rarely generates additional constraints for viability. This the case for this model as we can see by verifying that the last condition in table 5.1 is met:

$$\begin{aligned} f_1(x) &\geq 0 \\ -ag(\frac{x_1}{x_2})x_2 &\geq 0 \\ ag(\frac{x_1}{x_2})x_2 &\leq 0 \\ ag(\frac{x_1}{x_2}) &\geq 0, \qquad\qquad x_2 < 0 \end{aligned} \tag{6.38}$$

and since we have from the model definition $a > 0$ and $g(.) \geq 0$ we can remark that no additional restrictions are generated. Therefore for the system to remain viable in K_3 it suffices to have $0 < a \leq b$.

6.7.4. *Winning cone is defined by* K_4. The inspection of the behavior of the system at the "Upper" boundary yields:

$$\begin{aligned} f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\ -ag(\frac{x_1}{x_2})x_2 &\geq -\frac{x_1(0)}{x_2(0)} bh(\frac{x_2}{x_1})x_1 \\ -a &\geq \frac{x_1(0)}{x_2(0)}(-b), \qquad\qquad x_1 = x_2,\ g(1) = h(1) = 1 \\ a &\leq \frac{x_1(0)}{x_2(0)} b. \end{aligned} \tag{6.39}$$

As the lower boundary does not bring about any additional conditions, viability is dependant on:

$$0 < a \leq \frac{x_1(0)}{x_2(0)} b. \tag{6.40}$$

6.8. **Weiss.** This section's model is expressed by:

$$\begin{cases} \dfrac{dx_1}{dt} = -a(\dfrac{x_1}{x_2})^{1-W} x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b(\dfrac{x_2}{x_1})^{1-W} x_1 \end{cases}$$

where $a, b > 0$. Accordingly, the right hand side of equation (4.1) is defined by

$$f(x) = \begin{cases} f_1(x) = -a(\dfrac{x_1}{x_2})^{1-W} x_2 \\ f_2(x) = -b(\dfrac{x_2}{x_1})^{1-W} x_1 \end{cases}$$

where $x = (x_1, x_2)$.

We will not carry any further analysis of this model except to say that if we let $c = 1 - W$, Weiss' model can be viewed as a specialization of Hembold's presented in section 6.7.

6.9. **Schreiber.** The Schreiber model deals with the efficiency of command and is expressed mathematically by:

$$\begin{cases} \dfrac{dx_1}{dt} = -a\left\{\dfrac{x_1x_2}{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}\right\}, & x(0)=x_0 \\ \dfrac{dx_2}{dt} = -b\left\{\dfrac{x_1x_2}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\right\} \end{cases}$$

where $a, b > 0$ and $e_{x_1}, e_{x_2} \in [0,1]$. Accordingly, the right hand side of equation (4.1) is defined by

$$f(x) = \begin{cases} f_1(x) = -a\left\{\dfrac{x_1x_2}{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}\right\} \\ f_2(x) = -b\left\{\dfrac{x_1x_2}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\right\} \end{cases}$$

where $x = (x_1, x_2)$.

6.9.1. *Winning cone is defined by* K_1. It is clear that $f(0) = 0$ and therefore $f(0) \in T_K(0)$. Applying again the results presented in table 5.1, at the *upper* boundary

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -a\left\{\frac{x_1x_2}{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}\right\} &\geq -b\left\{\frac{x_1x_2}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\right\} \\ -a(x_{2,0}-e_{x_1}(x_{2,0}-x_2)) &\geq -b(x_{1,0}-e_{x_2}(x_{1,0}-x_1)), && x_1 = x_2 > 0 \\ a(x_{2,0}-e_{x_1}(x_{2,0}-x_2)) &\leq b(x_{1,0}-e_{x_2}(x_{1,0}-x_1)) \\ \frac{a}{b} &\leq \frac{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}{x_{2,0}-e_{x_1}(x_{2,0}-x_1)}, && b > 0, x_1 = x_2 \\ \frac{a}{b} &\leq \frac{(1-e_{x_2})x_{1,0}+e_{x_2}x_1}{(1-e_{x_1})x_{2,0}+e_{x_1}x_1}. \end{aligned} \tag{6.41}$$

It can be verified that the right hand side of the last inequality in a monotone function of x_1. Since $\lim_{x_1\to\infty} = \frac{e_{x_2}}{e_{x_1}}$ and that $\lim_{x_1\to 0} = \frac{(1-e_{x_2})x_{1,0}}{(1-e_{x_1})x_{2,0}}$ we obtain for a restriction on the coefficients:

$$\frac{a}{b} \leq \inf_{x_1>0}\left(\frac{e_{x_2}}{e_{x_1}}, \frac{(1-e_{x_2})x_{1,0}}{(1-e_{x_1})x_{2,0}}\right). \tag{6.42}$$

It is interesting to note that the conditions on the coefficients are dependant on the value of x_0. Let us now consider what is happening at the boundary where $x_2 = 0$:

$$\begin{aligned} f_2(x) &\geq 0 \\ -b\left\{\frac{x_1x_2}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\right\} &\geq 0 \\ 0 &\geq 0, && x_2 = 0 \end{aligned} \tag{6.43}$$

which of course is always true. A consequence of this and considering the symmetrical problem by interchanging x_1 and x_2, we can conclude that any solution to the Cauchy problem where x_0 is in the first quadrant never leaves the quadrant. Since all applications of combat models start within the first quadrant (reality has it that negative forces are hard to come by) it is unnecessary to pursue any cones extending outside for purpose of viability and are only left with the analysis at K_4 "Upper" boundary.

6.9.2. *Winning cone is defined by* K_4. The inspection of the behavior of the system at the "Upper" boundary yields:

$$\begin{aligned} f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\ -a\left\{\frac{x_1x_2}{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}\right\} &\geq -\frac{x_1(0)}{x_2(0)} b\left\{\frac{x_1x_2}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\right\} \\ -a(x_{2,0}-e_{x_1}(x_{2,0}-x_2)) &\geq -\frac{x_1(0)}{x_2(0)} b(x_{1,0}-e_{x_2}(x_{1,0}-x_1)), \qquad x_1 = x_2 > 0 \\ a(x_{2,0}-e_{x_1}(x_{2,0}-x_2)) &\leq \frac{x_1(0)}{x_2(0)} b(x_{1,0}-e_{x_2}(x_{1,0}-x_1)) \\ \frac{a}{b} &\leq \frac{x_1(0)}{x_2(0)} \frac{(1-e_{x_2})x_{1,0}+e_{x_2}x_1}{(1-e_{x_1})x_{2,0}+e_{x_1}x_1}. \end{aligned} \tag{6.44}$$

As for K_1, considering the limits we obtain as a viability condition:

$$\frac{a}{b} \leq \frac{x_1(0)}{x_2(0)} \inf_{x_1>0}\left(\frac{e_{x_2}}{e_{x_1}}, \frac{(1-e_{x_2})x_{1,0}}{(1-e_{x_1})x_{2,0}}\right). \tag{6.45}$$

6.10. **Variable coefficients.** The previous analysis provided some insights on the use of viability with Lanchester type models. We now consider the "less restrictive" case where the coefficients involved are variable in time. These systems are considered less restrictive as the conditions imposed on the coefficients need only be met when the boundaries are reached. As these models represent war scenarios, we will only inspect their behavior at the "Upper" boundary. The work is presented for a few examples and the compiled results are then given in table format.

6.10.1. *Aimed fire.* Considering the model presented in section 3.1.1 and replacing the coefficients by variable ones we get

$$\begin{cases} \dfrac{dx_1}{dt} = -a(t)x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b(t)x_1. \end{cases}$$

In the case of K_3, the conditions at the "Upper" boundary become

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -a(t)x_2 &\geq -b(t)x_1 \\ -a(t) &\geq -b(t), \qquad x_1 = x_2 > 0 \\ a(t) &\leq b(t). \end{aligned} \tag{6.46}$$

Here, as we are not dealing with constant coefficients, there is no reason to require that this condition be met for all $t \in \mathbf{R}_+$. It is therefore only necessary to have $a(t) \leq b(t)$ when

Model	K_1 **to** K_3	**Gaining cone** K_4
Aimed Fire	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Area Fire	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Brackney (Modified)	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Peterson	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Morse and Kimball	$a(t) + \beta(t) \leq b(t) + \alpha(t)$	$a(t) + \beta(t) \leq \frac{x_1(0)}{x_2(0)} (b(t) + \alpha(t)$
Coleman	$(a(t) + b(t)) - (c(t) + d(t)) \leq \frac{R_{x_1}(t) + R_{x_2}(t)}{x_1(t)}$	$(a(t) + b(t)) - (c(t) + d(t)) \leq \frac{x_1(0)(R_{x_1}(t) + R_{x_2}(t))}{x_2(0) x_1(t)}$
Hembold	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Schreiber	$\frac{a(t)}{b(t)} \leq \frac{(1-e_{x_2})x_1(0) + e_{x_2} x_1(t)}{(1-e_{x_1})x_2(0) + e_{x_1} x_1(t)}$	$\frac{a(t)}{b(t)} \leq \frac{x_1(0)}{x_2(0)} \frac{(1-e_{x_2})x_1(0) + e_{x_2} x_1(t)}{(1-e_{x_1})x_2(0) + e_{x_1} x_1(t)}$

Table 6.1: Viability conditions for variable coefficients models

$x_1(t) = x_2(t) > 0$. In the case of K_4, we obtain

$$
\begin{aligned}
f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\
-a(t)x_2 &\geq -\frac{x_1(0)}{x_2(0)} b(t) x_1 \\
-a(t) &\geq -\frac{x_1(0)}{x_2(0)} b(t), \qquad\qquad x_1 = x_2 > 0 \\
a(t) &\leq \frac{x_1(0)}{x_2(0)} b(t). \qquad (6.47)
\end{aligned}
$$

And once more, this condition only has to be met when $x_1(t) = x_2(t) > 0$.

6.10.2. *Hembold.* Modifying the model of section 3.1.7 with variable coefficients, it becomes

$$
\begin{cases}
\dfrac{dx_1}{dt} = -a(t) g(\dfrac{x_1}{x_2}) x_2, \quad x(0) = x_0 \\
\dfrac{dx_2}{dt} = -b(t) h(\dfrac{x_2}{x_1}) x_1
\end{cases}
$$

and considering K_3, viability conditions become

$$
\begin{aligned}
f_1(x) &\geq f_2(x) \\
-a(t) g(\frac{x_1}{x_2}) x_2 &\geq -b(t) h(\frac{x_2}{x_1}) x_1 \\
-a(t) g(1) &\geq -b(t) h(1), \qquad\qquad x_1 = x_2 > 0 \\
-a(t) &\geq -b(t), \qquad\qquad g(1) = h(1) = 1 \\
a(t) &\leq b(t). \qquad (6.48)
\end{aligned}
$$

It is interesting to note that again, with this choice of *winning cone* the system's viability conditions are exactly those of the Aimed fire model at this boundary.

6.10.3. *Compiled results for variable coefficients.* The work for the determination of the viability conditions for the remaining variable coefficients models and cones in omitted but the results for each of them is given in table 6.1.

6.11. **Comments.** In this section, we have looked at various Lanchester type models to examine the restrictions on the coefficients that *guarantee* victory for one side. This *guarantee* was established through the verification of viability using the notion of *winning cone*. This analysis method provided us with conditions to meet on the Lanchester coefficients to lead one side to victory. We will now proceed to the study of those models through optimal control by viability.

7. OPTIMAL CONTROL BY VIABILITY

In combat situations, as for many others, the control over variables is of a discrete nature. For example, the decision as to whether or not artillery should be brought to bear, reinforcements committed to battle or simply to rest the troops in order for them to recuperate.

These decisions have an impact either on their effectiveness in combat or in their number, resulting in changes in the variables at play in the models used. Optimal Control in such cases shouldn't be about minimizing/optimizing a function but should rather reflect the reality being modelled. That is where *Optimal Control by Viability* provides us with interesting tools.

With the application of viability concepts, the set of desired states can be defined, such as our *winning cones*, and the conditions for viability determined. Once these conditions are known, the evolution of the dynamical system can be studied and the *moments* when they will become unsatisfied can be established. This identifies the situations/states at which control must be applied to ensure continued viability. This process is what we call *Optimal Control by Viability*.

Given the reality to which models such as Lanchester types are applied, it is more appropriate to consider this type of optimal control then that commonly defined.

7.1. **Optimal control by viability of Lanchester type models.** In the previous work presented in this article, conditions have been developed to make the systems viable for the respective cones. However, for optimal control by viability, we are interested in using these conditions to determine the moment T at which the system will cease to be viable.

Let us consider, for example, the Aimed Fire model. In the case of fixed coefficients, it is a somewhat simple process. If $a \leq b$, the model is forever viable while if $a > b$, then we need to find T such that $x_1(T) = x_2(T)$.

The variable coefficients case is more interesting. As the coefficients vary over time, we need to identify the time T such that $a(T) > b(T)$ *and* $x_1(T) = x_2(T)$. It no longer suffices to look at the coefficients to determine T.

From an optimal control by viability point of view we are interested in

$$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, a(t) > b(t)\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}. \tag{7.1}$$

This represents the moment when a new control scheme must be employed. For the Lanchester type models, it may represent a moment when engagement should be broken to allow troops to rest, or a time to change the type of forces employed, or a time to apply other tactical tools available to the commander. In either case, this represents a moment of decision taking. For the various models presented herein, the decisions times T can be determined from table 6.1 and are given in table 7.1 for the "Upper" boundaries of the winning cones.

It is clear that *Optimal Control by Viability* closer models the real system and how effects are generated by the participants. Models using such a concept provide and excellent decision making assistance tool as well as a mean of studying the various effects of command decisions.

Model	Decision time
Aimed Fire	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, a(t) > b(t)\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$
Area Fire	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, a(t) > b(t)\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$
Brackney (Modified)	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, a(t) > b(t)\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$
Peterson	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, a(t) > b(t)\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$
Morse and Kimball	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, a(t) + \beta(t) > b(t) + \alpha(t)\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$
Coleman	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, (a(t)+b(t)) - (c(t)+d(t)) > \frac{R_{x_1}(t)+R_{x_2}(t)}{x_1(t)}\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$
Hembold	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, a(t) > b(t)\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$
Schreiber	$inf\left\{\left\{t \in \mathbf{R}_+ \,\middle|\, \frac{a(t)}{b(t)} > \frac{(1-e_{x_2})x_1(0)+e_{x_2}x_1(t)}{(1-e_{x_1})x_2(0)+e_{x_1}x_1(t)}\right\} \cap \left\{t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t)\right\}\right\}$

Table 7.1: Viability conditions for variable coefficients models

8. CONLUSION

In this article we have introduced the new concept of *Winning Cone* and have seen how it can be applied to dynamical systems such as the Lanchester type models for combat. Noting that these models are used in other areas (such as economy, biology ...) and that other dynamical systems can also employ the notion of winning cone, this concept has a wide range of application.

Following the above analysis, we have introduced the idea of *Optimal Control by Viability*. We have seen how combining this notion with that of *Winning Cone* generated a very useful tool in Command and Control. We will develop this idea in a next paper.

To this point, we have not addressed the issue of the difficulty in determining the Lanchester coefficients raised in the introduction (see [6],[5]). This is the subject of our next article. In this second part we will show how *Interval Analysis*, *Differential Inclusions*, *Viability* and *Optimal Control* can be used together to resolve this issue.

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