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FEKETE-SZEGÖ INEQUALITY FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. In this present investigation, the authors obtain Fekete-Szegö inequality for a certain class of analytic functions $f(z)$ for which $\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}$ $(\alpha, \beta \geq 0)$ lies in a region starlike with respect to 1 and symmetric with respect to the real axis. Also certain application of our main result for a class of functions defined by Hadamard product (convolution) is given. As a special case of our result we obtain Fekete-Szegö inequality for a class of functions defined through fractional derivatives. Also we obtain Fekete-Szegö inequality for the inverse functions.

Key words and phrases: Analytic functions, starlike functions, convex functions, inverse functions, subordination, coefficient problem, Fekete-Szegö inequality, fractional derivatives, Hadamard product (or convolution).

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ defined on

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and $\mathcal{A}_0$ be the family of functions $f(z) \in \mathcal{A}$ normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Such functions $f \in \mathcal{A}_0$ have the Taylor series expansion given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta). \tag{1.1}$$

Let $\mathcal{S}$ be the family of functions $f \in \mathcal{A}_0$ which are univalent. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \Delta$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad z \in \Delta.$$

These classes were introduced and studied by Ma and Minda [5]. The familiar class $S^*(\alpha)$ of *starlike functions* of order α and the class $C(\alpha)$ of *convex functions* of order α, $0 \leq \alpha < 1$ are the special case of $S^*(\phi)$ and $C(\phi)$ respectively when $\phi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$.

We now define a class of functions which unifies the class $S^*(\phi)$ and $C(\phi)$ in the following:

Definition 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the class $M_{\alpha,\beta}(\phi)$ if

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} \prec \phi(z) \quad (0 \leq \alpha \leq 1).$$

It follows that

$$M_{0,1}(\phi) \equiv C(\phi) \quad \text{and} \quad M_{1,0}(\phi) \equiv S^*(\phi).$$

Ma and Minda [5] obtained the Fekete-Szegö inequality for functions in the class $C(\phi)$ and in view of the Alexander result between the class $S^*(\phi)$ and $C(\phi)$, the Fekete-Szegö inequality for functions in $S^*(\phi)$. Similar problem for a class of Bazilevič functions was considered recently by Ravichandran *et al.* [8].

In the present paper, we prove the Fekete-Szegö inequality in Theorem 2.1 for a more general class of analytic functions which we have defined above in Definiton1.1. Also we give applications of our results to certain functions defined through Hadamard product and in particular we consider a class defined by fractional derivatives. Also we discuss the Fekete-Szegö inequality for the inverse functions.

To prove our main result, we need the following:

Lemma 1.1. [5] *If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is a function with positive real part in* Δ*, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0, \\ 2 & \text{if } 0 \leq v \leq 1, \\ 4v - 2 & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ *or* $v > 1$*, the equality holds if and only if* $p_1(z)$ *is* $(1+z)/(1-z)$ *or one of its rotations. If* $0 < v < 1$*, then equality holds if and only if* $p_1(z)$ *is* $(1+z^2)/(1-z^2)$ *or one of its rotations. If* $v = 0$*, the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$*, the equality holds if and only if* p_1 *is the reciprocal of one of the functions such that the equality holds in the case of* $v = 0$*.*

Also the above upper bound is sharp, it can be improved as follows when $0 < v < 1$*:*

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).$$

We also need the following:

Lemma 1.2. (cf. [13]) *If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is a function with positive real part in* Δ*, then*

$$|c_2 - vc_1^2| \leq 2\max\{1; |2v - 1|\}.$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

2. FEKETE-SZEGÖ PROBLEM

By making use of the Lemma 1.1, we prove the following:

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$*. Let*

$$\begin{aligned} \sigma_1 &:= \frac{2(\alpha+2\beta)^2(B_2 - B_1) - [(\alpha+2\beta)^2 - 3(\alpha+4\beta)B_1^2}{4(\alpha+3\beta)]B_1^2}, \\ \sigma_2 &:= \frac{2(\alpha+2\beta)^2(B_2 + B_1) - [(\alpha+2\beta)^2 - 3(\alpha+4\beta)B_1^2}{4(\alpha+3\beta)]B_1^2}, \\ \sigma_3 &:= \frac{2(\alpha+2\beta)^2 B_2 - [(\alpha+2\beta)^2 - 3(\alpha+4\beta)B_1^2}{4(\alpha+3\beta)]B_1^2}, \\ \gamma &:= (\alpha+2\beta)^2 - 3(\alpha+4\beta) + 4\mu(\alpha+3\beta). \end{aligned}$$

If $f(z)$ given by (1.1) belongs to $M_{\alpha,\beta}(\phi)$, then

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{1}{4(\alpha+3\beta)}\left[2B_2-\dfrac{B_1^2}{(\alpha+2\beta)^2}\gamma\right] & \text{if} \quad \mu\leq\sigma_1, \\ \dfrac{1}{2(\alpha+3\beta)}B_1 & \text{if} \quad \sigma_1\leq\mu\leq\sigma_2, \\ \dfrac{1}{4(\alpha+3\beta)}\left[-2B_2+\dfrac{B_1^2}{(\alpha+2\beta)^2}\gamma\right] & \text{if} \quad \mu\geq\sigma_2. \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3-\mu a_2^2| + \frac{(\alpha+2\beta)^2}{2(\alpha+3\beta)B_1}\left[1-\frac{B_2}{B_1}+\frac{\gamma B_1}{2(\alpha+2\beta)^2}\right]|a_2|^2 \leq \frac{B_1}{2(\alpha+3\beta)}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3-\mu a_2^2| + \frac{(\alpha+2\beta)^2}{2(\alpha+3\beta)B_1}\left[1+\frac{B_2}{B_1}-\frac{\gamma B_1}{2(\alpha+2\beta)^2}\right]|a_2|^2 \leq \frac{B_1}{2(\alpha+3\beta)}.$$

These results are sharp.

Proof. If $f(z)\in M_{\alpha,\beta}(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0)=0$ and $|w(z)|<1$ in Δ such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} = \phi(w(z)). \tag{2.1}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1+w(z)}{1-w(z)} = 1+c_1z+c_2z^2+\cdots. \tag{2.2}$$

Since $w(z)$ is a Schwarz function, we see that $\Re p_1(z)>0$ and $p_1(0)=1$. Define the function $p(z)$ by

$$p(z) := \left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} = 1+b_1z+b_2z^2+\cdots. \tag{2.3}$$

In view of the equations (2.1), (2.2), (2.3), we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{2.4}$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1z+(c_2-\frac{c_1^2}{2})z^2+(c_3+\frac{c_1^3}{4}-c_1c_2)z^3+\cdots\right]$$

therefore

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1+\frac{1}{2}B_1c_1z+\left[\frac{1}{2}B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{4}B_2c_1^2\right]z^2+\cdots.$$

From (2.4), we obtain

$$b_1 = \frac{1}{2}B_1c_1$$

and

$$b_2 = \frac{1}{2}B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{4}B_2c_1^2.$$

A computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3a_2)z^3 + \cdots$$

and therefore we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = 1 + \alpha a_2 z + (2\alpha a_3 + \frac{\alpha^2 - 3\alpha}{2} a_2^2)z^2 + \cdots .$$

Similarly we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + \cdots$$

and therefore

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} = 1 + 2\beta a_2 z + \left(6\beta a_3 + 2(\beta^2 - 3\beta)a_2^2\right) z^2 + \cdots .$$

Thus we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta}$$
$$= 1 + (\alpha + 2\beta)a_2 z + \left[2(\alpha + 3\beta)a_3 + \frac{(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)}{2} a_2^2\right] z^2 + \ldots$$

and in view of the equation (2.3), we see that

$$b_1 = (\alpha + 2\beta)a_2 \tag{2.5}$$

$$b_2 = 2(\alpha + 3\beta)a_3 + \frac{(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)}{2} a_2^2 \tag{2.6}$$

or, equivalently, we have

$$a_2 = \frac{B_1 c_1}{2(\alpha + 2\beta)},$$
$$a_3 = \frac{1}{2(\alpha + 3\beta)} \left\{\frac{B_1 c_2}{2} - \frac{1}{4}\left[B_1 - B_2 + \frac{[(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)]}{2(\alpha + 2\beta)^2} B_1^2\right] c_1^2\right\}.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{4(\alpha + 3\beta)} \left\{c_2 - v c_1^2\right\} \tag{2.7}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{[(\alpha + 2\beta)^2 + 4\mu(\alpha + 3\beta) - 3(\alpha + 4\beta)]}{2(\alpha + 2\beta)^2} B_1\right].$$

Our result now follows by an application of Lemma 1.1.

To show that the bounds are sharp, we define the functions $K_{\phi n}$ $(n = 2, 3, \ldots)$ by

$$\left(\frac{zK'_{\phi n}(z)}{K_{\phi n}(z)}\right)^{\alpha} \left(1 + \frac{zK''_{\phi n}(z)}{K'_{\phi n}(z)}\right)^{\beta} = \phi(z^{n-1}), \quad K_{\phi n}(0) = 0 = [K_{\phi n}]'(0) - 1$$

and the function F_λ and G_λ $(0 \leq \lambda \leq 1)$ by

$$\left(\frac{zF'_{\lambda}(z)}{F_{\lambda}(z)}\right)^{\alpha} \left(1 + \frac{zF''_{\lambda}(z)}{F'_{\lambda}(z)}\right)^{\beta} = \phi\left(\frac{z(z + \lambda)}{1 + \lambda z}\right), \quad F_{\lambda}(0) = 0 = F'_{\lambda}(0) - 1$$

and

$$\left(\frac{zG'_\lambda(z)}{G_\lambda(z)}\right)^\alpha \left(1+\frac{zG''_\lambda(z)}{G'_\lambda(z)}\right)^\beta = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G_\lambda(0)=0=G'_\lambda(0)-1.$$

Clearly the functions $K_{\phi n}, F_\lambda, G_\lambda \in M_{\alpha,\beta}(\phi)$. Also we write $K_\phi := K_{\phi 2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations.

∎

By making use of Lemma 1.2, we immediately obtain the following:

Theorem 2.2. *Let $f(z)$, $\phi(z)$ be as in Theorem 2.1. For complex μ, we have*

$$|a_3-\mu a_2^2| = \frac{B_1}{\alpha+3\beta}\max\left\{1, \left|-\frac{B_2}{B_1}+\frac{[(\alpha+2\beta)^2+4\mu(\alpha+3\beta)-3(\alpha+4\beta)]}{2(\alpha+2\beta)^2}B_1\right|\right\}.$$

The result is sharp.

Remark 2.1. The coefficient bounds for $|a_2|$ and $|a_3|$ are special cases of our Theorem 2.1.

3. APPLICATION TO FUNCTIONS DEFINED BY CONVOLUTION AND TO THE INVERSE FUNCTIONS

Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$. Define $M^g_{\alpha,\beta}(\phi)$ to be the class of all functions $f(z)$ such that $(f * g)(z) \in M_{\alpha,\beta}(\phi)$. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M^g_{\alpha,\beta}(\phi)$ if and only if $(f*g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_{\alpha,\beta}(\phi)$, we obtain the coefficient estimate for functions in the class $M^g_{\alpha,\beta}(\phi)$ from the corresponding estimate for functions in the class $M_{\alpha,\beta}(\phi)$. See [8] for more details.

Define the inverse function f^{-1} by

$$f^{-1}(f(z)) = z = f(f^{-1}(z)).$$

Then

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \quad (|w| < r_0), \tag{3.1}$$

where r_0 is greater than the radius of the Koebe domain for the class $M_{\alpha,\beta}(\phi)$.

Since $f^{-1}(f(z)) = z$, we have

$$\begin{aligned} a_2 + d_2 &= 0 \\ a_3 + 2a_2d_2 + d_3 &= 0 \end{aligned}$$

and therefore we have

$$\begin{aligned} d_2 &= -a_2 \\ d_3 &= -a_3 + 2a_2^2. \end{aligned}$$

A computation now shows that

$$|d_3 - \mu d_2^2| = |a_3 - (2 - \mu)a_2^2|$$

and hence the Fekete-Szegö inequality for the inverse function follows from that of the function $f(z)$. We omit the details.

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