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A MAJORIZATION PROBLEM FOR THE SUBCLASS OF p-VALENTLY ANALYTIC FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. The main purpose of this paper is to investigate a majorization problem for the class $C_{p,q}^{m}(\gamma)$. Relevant connections of the main result obtained in this paper with those given by earlier workers on the subject are also pointed out.

Key words and phrases: Analytic functions, Majorization problems, p-valent functions, Salagean operator.

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1. INTRODUCTION

Let the functions $f(z)$ and $g(z)$ be analytic in the open unit disk

$$U = \{z : z \in \mathbf{C} \text{ and } |z| < 1\}.$$

It is called that $f(z)$ is majorized by $g(z)$ in U and write

$$f(z) \ll g(z) \ \ (z \in U)$$

if there exists a function $\varphi(z)$, analytic in U, such that

$$|\varphi(z)| \leq 1 \text{ and } f(z) = \varphi(z)g(z) \ \ (z \in U).$$

Let $A(p)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \ \ (p \in N = \{1, 2, 3, \ldots\}),$$

which are analytic and p-valent the unit disk U. Note that $A = A(1)$.

Salagean [8] has introduced the following operator called the Salagean operator:

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = zf'(z)$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \ \ (n \in N).$$

Note that if $f(z) \in A(p)$ then

$$D^n f(z) = p^n z^p + \sum_{j=p+1}^{\infty} j^n a_j z^j.$$

A function $f(z) \in A(p)$ is said to be in the class $C^n_{p,q}(\gamma)$ of p-valently analytic functions of complex order $\gamma \neq 0$ in U if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{\left(zD^{n+1}f^{(q)}(z)\right)' - D^{n+1}f^{(q)}(z)}{D^{n+1}f^{(q)}(z)} - p + q + n + 1\right)\right\} > 0$$

$$(z \in U, \ \ p \in N, \ \ n, q \in N_0 = N \cup \{0\}, \ \gamma \in \mathbf{C} - \{0\}, |2\gamma - p + q + n| \leq p - q - n)$$

where $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order $q \in N_0$. We have the following relationship:

$$C^0_{1,0}(\gamma) = C(\gamma) \ \ (\gamma \in \mathbf{C} - \{0\})$$

and

$$C^0_{1,0}(1 - \alpha) = C(1 - \alpha) = K(\alpha) \ \ (0 \leq \alpha < 1),$$

where $C(\gamma)$ denotes the class of convex functions of complex order $\gamma \neq 0$ in U which were considered by Nasr and Aouf [6] and Wiatrowski [9], and $K(\alpha)$ denote the class of convex functions of order α in U which were introduced by Robertson [7].

A function $f(z) \in A(p)$ is said to be in the class $S^n_{p,q}(\gamma)$ of p-valently analytic functions of complex order $\gamma \neq 0$ in U if and only if

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{D^{n+1}f^{(q)}(z)}{D^{n}f^{(q)}(z)}-p+q+n\right)\right\}>0$$

$$(z\in U,\ \ p\in N,\ \ n,q\in N_0=N\cup\{0\}\,,\ \gamma\in \mathbf{C}-\{0\},|2\gamma-p+q+n|\leq p-q-n)$$

where $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order $q\in N_0$. Clearly, we have the following relationship:

$$S^0_{1,0}(\gamma)=S(\gamma)\ \ (\gamma\in\mathbf{C}-\{0\})$$

and

$$S^0_{1,0}(1-\alpha)=S(1-\alpha)=S^*(\alpha)\ \ (0\leq\alpha<1),$$

where $S(\gamma)$ denotes the class of starlike functions of complex order $\gamma\neq 0$ in U which were considered by Nasr and Aouf [6] and Wiatrowski [9], and $S^*(\alpha)$ denote the class of starlike functions of order α in U which were introduced by Robertson [7].

A majorization problem for the classes $S(\gamma)$ and $C(\gamma)$ have been investigated by Altintaş, Özkan and Srivastava [1], p. 211, Theorem 1, p. 214, Theorem 2. Also, a majorization problem for the classes $S^*=S^*(0)$ and $K(0)=K$ have been investigated by MacGregor [5], p. 96, Theorem 1B, p. 96 Theorem 1C. Altintaş and Srivastava [2], p. 177, Theorem1 worked an majorization problem for the classes $S^0_{p,q}(\gamma)=S_{p,q}(\gamma)$ and $C^0_{p,q}(\gamma)=C_{p,q}(\gamma)$ $(\gamma\in\mathbf{C}-\{0\})$. Then, Kadıoğlu [4], Theorem 1 worked a majorization problem for the class $S^n_{p,q}(\gamma)$ $(\gamma\in\mathbf{C}-\{0\})$.

2. MAJORIZATION PROBLEMS FOR THE CLASS $C^n_{p,q}(\gamma)$

The results for the class $C^n_{p,q}(\gamma)$ is based on following theorem.

Theorem 2.1. *If* $f\in C^n_{p,q}(\gamma)$ $(\gamma\in\mathbf{C}-\{0\})$*, then* $f\in S^n_{p,q}(\frac{1}{2}\gamma)$*, that is,*

$$C^n_{p,q}(\gamma)\subset S^n_{p,q}(\frac{1}{2}\gamma).$$

Proof. Altintaş and Srivastava [1], p. 180, Lemma shows that, if $f\in C_{p,q}(\gamma)$,

$$\operatorname{Re}\left\{1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}-p+q+1\right\}>0\Rightarrow\operatorname{Re}\left\{1+\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}-p+q\right\}>\frac{1}{2}\ .$$

We can write

$$\operatorname{Re}\left\{1+\frac{Df^{(q+1)}(z)}{D^0f^{(q+1)}(z)}-p+q+1\right\}>0\Rightarrow\operatorname{Re}\left\{1+\frac{Df^{(q)}(z)}{D^0f^{(q)}(z)}-p+q\right\}>\frac{1}{2}$$

or

$$\operatorname{Re}\left\{1+\frac{D\left(\frac{1}{z}Df^{(q)}(z)\right)}{D^0\left(\frac{1}{z}Df^{(q)}(z)\right)}-p+q+1\right\}>0\Rightarrow\operatorname{Re}\left\{1+\frac{Df^{(q)}(z)}{D^0f^{(q)}(z)}-p+q\right\}>\frac{1}{2}$$

by using the operator D. We have

$$\operatorname{Re}\left\{1+\frac{D\left(\frac{1}{z}D^{n+1}f^{(q)}(z)\right)}{D^0\left(\frac{1}{z}D^{n+1}f^{(q)}(z)\right)}-p+q+n+1\right\}>0$$
$$\Rightarrow \operatorname{Re}\left\{1+\frac{D^{n+1}f^{(q)}(z)}{D^nf^{(q)}(z)}-p+q+n\right\}>\frac{1}{2}$$

or

$$\operatorname{Re}\left\{1+\frac{\left(zD^{n+1}f^{(q)}(z)\right)'-D^{n+1}f^{(q)}(z)}{D^{n+1}f^{(q)}(z)}-p+q+n+1\right\}>0$$
$$\Rightarrow \operatorname{Re}\left\{1+\frac{D^{n+1}f^{(q)}(z)}{D^nf^{(q)}(z)}-p+q+n\right\}>\frac{1}{2}$$

for $f^{(q)}(z)\to D^nf^{(q)}(z)$. This yields

$$1+\frac{\left(zD^{n+1}f^{(q)}(z)\right)'-D^{n+1}f^{(q)}(z)}{D^{n+1}f^{(q)}(z)}-p+q+n+1=\frac{1-w(z)}{1+w(z)}$$
$$\Rightarrow 1+\frac{D^{n+1}f^{(q)}(z)}{D^nf^{(q)}(z)}-p+q+n=\frac{1}{1+w(z)}.$$

Using these equalities we obtain

$$1+\frac{1}{\gamma}\left(\frac{\left(zD^{n+1}f^{(q)}(z)\right)'-D^{n+1}f^{(q)}(z)}{D^{n+1}f^{(q)}(z)}-p+q+n+1\right)=\frac{\gamma+(\gamma-2)w(z)}{\gamma(1+w(z))}$$
$$\Rightarrow 1+\frac{2}{\gamma}\left(\frac{D^{n+1}f^{(q)}(z)}{D^nf^{(q)}(z)}-p+q+n\right)=\frac{\gamma+(\gamma-2)w(z)}{\gamma(1+w(z))}.$$

Thus we can write

$$C^n_{p,q}(\gamma)\subset S^n_{p,q}(\frac{1}{2}\gamma).$$

∎

Theorem 2.2. *Let the function* $f(z)$ *be in the* $A(p)$ *and suppose that* $g\in C^n_{p,q}(\gamma)$. *If* $D^nf^{(q)}(z)$ *is majorized by* $D^ng^{(q)}(z)$ *in* U *for* $q\in N_0$ *then*

$$|D^{n+1}f^{(q)}(z)|\le|D^{n+1}g^{(q)}(z)|\ \ (|z|\le r),$$

where

$$r=r(p,q,n;\gamma)=\frac{k-\sqrt{k^2-4(p-q-n)|\gamma-p+q+n|}}{2|\gamma-p+q+n|}$$
$$(k=2+p-q-n+|\gamma-p+q+n|;\ p\in N,\ n,q\in N_0,\ \gamma\in\mathbf{C}-\{0\}).$$

Proof. Replacing γ in Theorem 1, proved by Kadıoğlu [4] , by $\frac{1}{2}\gamma$, if we apply the above Theorem 2.1, the proof is completed. ∎

If we set $n=0$ in Theorem 2.2, we obtain

Corollary 2.3. (Altintas and Srivastava [2], p. 181, Theorem 2). *Let the function* $f(z)$ *be in the class* $A(p)$ *and suppose that* $g \in C_{p,q}(\gamma)$*. If* $f^{(q)}(z)$ *is majorized by* $g^{(q)}(z)$ *in* U *for* $q \in N_0$ *then*

$$\left|f^{(q+1)}(z)\right| \leq \left|g^{(q+1)}(z)\right| \quad (|z| \leq r),$$

where

$$r = r(p,q;\gamma) = \frac{k - \sqrt{k^2 - 4(p-q)|\gamma - p + q|}}{2|\gamma - p + q|},$$
$$(k = 2 + p - q + |\gamma - p + q|,\ p \in N;\ q \in N_0;\ \gamma \in \mathbf{C} - \{0\}).$$

A special case of Theorem 2.2 when $n = 0,\ p = 1$ and $q = 0$ yields.

Corollary 2.4. (Altintas et al. [1], p. 214, Theorem 2). *Let the function* $f(z)$ *be analytic in* U *and suppose that* $g \in C(\gamma)$*. If* $f(z)$ *is majorized by* $g(z)$ *in* U*, then*

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$r = r(\gamma) = \frac{3 + |\gamma - 1| - \sqrt{9 + 2|\gamma - 1| + |\gamma - 1|^2}}{2|\gamma - 1|}.$$

If we set $n = 0,\ p = 1,\ q = 0$ and in its limit case when $\gamma \to 1$ in Theorem 2.2, we obtain

Corollary 2.5. (MacGregor [5], p. 96, Theorem 1C). *Let the function* $f(z)$ *be analytic in* U *and suppose that* $g \in K = K(0)$*. If* $f(z)$ *is majorized by* $g(z)$ *in* U*, then*

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq \frac{1}{3}).$$

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