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MONOTONICITY PROPERTIES FOR GENERALIZED LOGARITHMIC MEANS

CHAO-PING CHEN AND FENG QI

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DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS, RESEARCH INSTITUTE OF APPLIED MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN 454000, CHINA
chenchaoping@hpu.edu.cn

DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS, RESEARCH INSTITUTE OF APPLIED MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN 454000, CHINA
qifeng@hpu.edu.cn, fengqi618@member.ams.org
URL: http://rgmia.vu.edu.au/qi.html, http://dami.hpu.edu.cn/qifeng.html

ABSTRACT. In this paper, we consider the monotonicity properties for ratio of two generalized logarithmic means, and then use it to extend and complement a recently published result of F. Qi and B.-N. Guo.

Key words and phrases: Monotonicity, inequality, generalized logarithmic mean, one-parameter mean.

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The generalized logarithmic mean $L_r(a,b)$ of two positive numbers a and b is introduced in [2, 5, 6] for $a = b$ by $L_r(a,b) = a$ and for $a \neq b$ by

$$L_r(a,b) = \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r \neq -1, 0; \tag{1.1}$$

$$L_{-1}(a,b) = \frac{b-a}{\ln b - \ln a} = L(a,b); \tag{1.2}$$

$$L_0(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} = I(a,b), \tag{1.3}$$

where $L(a,b)$ and $I(a,b)$ are respectively called the logarithmic or exponential mean of two positive numbers a and b. When $a \neq b$, $L_r(a,b)$ is a strictly increasing function of r. In particular,

$$\lim_{r\to-\infty} L_r(a,b) = \min\{a,b\}, \quad \lim_{r\to\infty} L_r(a,b) = \max\{a,b\}.$$

The logarithmic mean $L(a,b)$ can be generalized to the one-parameter mean [1, 7] for $a = b$ by $J_r(a,b) = a$ and for $a \neq b$ by

$$J_r(a,b) = \frac{r(b^{r+1} - a^{r+1})}{(r+1)(b^r - a^r)}, \quad r \neq 0, -1; \tag{1.4}$$

$$J_0(a,b) = L(a,b); \tag{1.5}$$

$$J_{-1}(a,b) = \frac{[G(a,b)]^2}{L(a,b)}. \tag{1.6}$$

When $a \neq b$, $J_r(a,b)$ is a strictly increasing function of r. In particular,

$$\lim_{r\to-\infty} J_r(a,b) = \min\{a,b\}, \quad \lim_{r\to\infty} J_r(a,b) = \max\{a,b\}.$$

This work is motivated by two papers of F. Qi and B.-N. Guo [3, 4], who proved that let $b > a > 0$ and $\delta > 0$, then for $r > 0$,

$$\frac{b}{b+\delta} < \left(\frac{\frac{1}{b-a}\int_a^b x^r \, \mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \, \mathrm{d}x}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{1.7}$$

Both bounds in (1.7) are the best possible because of

$$\lim_{r\to\infty}\left(\frac{\frac{1}{b-a}\int_a^b x^r \, \mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \, \mathrm{d}x}\right)^{1/r} = \frac{b}{b+\delta}, \tag{1.8}$$

$$\lim_{r\to0+}\left(\frac{\frac{1}{b-a}\int_a^b x^r \, \mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \, \mathrm{d}x}\right)^{1/r} = \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{1.9}$$

It is easy to see that the inequality (1.7) can be written for $r > 0$ as

$$\frac{b}{b+\delta} < \frac{L_r(a,b)}{L_r(a,b+\delta)} < \frac{I(a,b)}{I(a,b+\delta)}. \tag{1.10}$$

The following theorem can conclude that $g(r) = L_r(a,b)/L_r(a,b+\delta)$ is a strictly decreasing function of $r \in (-\infty, \infty)$, and then, (1.7) can be obtained as a consequence.

Theorem 1. *Let* $a > 0$, $r, s \in \mathbb{R}$ *and* $r \neq s$, *define for* $x > 0$,

$$f(x) = \begin{cases} \dfrac{L_r(a,x)}{L_s(a,x)}, & x \neq a; \\ 1, & x = a. \end{cases} \tag{1.11}$$

(1) *If* $r > s$, *then the function* f *is strictly decreasing on* $(0, a)$ *and strictly increasing on* (a, ∞);
(2) *If* $r < s$, *then the function* f *is strictly increasing on* $(0, a)$ *and strictly decreasing on* (a, ∞).

Proof. Taking logarithm and differentiating yields

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{1}{x-a}\left(\frac{rx^{r+1}-(r+1)ax^r+a^{r+1}}{r(x^{r+1}-a^{r+1})} - \frac{sx^{s+1}-(s+1)ax^s+a^{s+1}}{s(x^{s+1}-a^{s+1})}\right) \\ &= \frac{1}{x-a}\left(\frac{rx^{r+1}-(r+1)ax^r+a^{r+1}}{r(x^{r+1}-a^{r+1})} - 1\right) \\ &\quad - \frac{1}{x-a}\left(\frac{sx^{s+1}-(s+1)ax^s+a^{s+1}}{s(x^{s+1}-a^{s+1})} - 1\right) \\ &= \frac{a}{x-a}\left(-\frac{(r+1)(x^r-a^r)}{r(x^{r+1}-a^{r+1})} + \frac{(s+1)(x^s-a^s)}{s(x^{s+1}-a^{s+1})}\right) \\ &= \frac{a}{x-a}\left(-\frac{1}{J_r(a,x)} + \frac{1}{J_s(a,x)}\right) \\ &= \frac{a(J_r(a,x)-J_s(a,x))}{(x-a)J_r(a,x)J_s(a,x)}. \end{aligned} \tag{1.12}$$

Since $J_r(a,b)$ is a strictly increasing function of r when $a \neq b$, it is easy to see that
(1) If $r > s$, then $f'(x) < 0$ for $0 < x < a$ and $f'(x) > 0$ for $x > a$;
(2) If $r < s$, then $f'(x) > 0$ for $0 < x < a$ and $f'(x) < 0$ for $x > a$.

The proof is complete. □

Remark 1. If $r > s$, then, for $c > b > a > 0$,

$$\frac{L_r(a,b)}{L_s(a,b)} < \frac{L_r(a,c)}{L_s(a,c)}.$$

This shows that for any fixed $c > b > a > 0$, the function $g(r) = L_r(a,b)/L_r(a,c)$ is strictly decreasing on $(-\infty, \infty)$.

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