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GENERALIZATIONS OF TWO THEOREMS ON ABSOLUTE SUMMABILITY METHODS

H. S. ÖZARSLAN AND H. N. ÖĞDÜK

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DEPARTMENT OF MATHEMATICS, ERCIYES UNIVERSITY, 38039 KAYSERI, TURKEY
seyhan@erciyes.edu.tr
nogduk@erciyes.edu.tr
URL: http://fef.erciyes.edu.tr/math/hikmet.htm

ABSTRACT. In this paper two theorems on $|A, p_n; \delta|_k$ summability methods, which generalize two theorems of Bor [2] on $\left|\bar{N}, p_n\right|_k$ summability methods, have been proved.

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1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the partial sums (s_n), and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v,\ n = 0, 1, ... \tag{1.1}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} |\overline{\Delta} A_n(s)|^k < \infty \tag{1.2}$$

where

$$\overline{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.3}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $|\ \bar{N}, p_n\ |_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\ t_n - t_{n-1}\ |^k < \infty, \tag{1.5}$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \overline{\Delta} A_n(s) \right|^k < \infty. \tag{1.6}$$

We say that the series $\sum a_n$ is summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{\delta k + k - 1} \left| \overline{\Delta} A_n(s) \right|^k < \infty. \tag{1.7}$$

In the special case when $\delta = 0$, $|A, p_n; \delta|_k$ summability is the same as $|A, p_n|_k$ summability. Also if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ summability.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of f is zero, so that

$$\int_{-\pi}^{\pi} f(t) dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t).$$

It is well known that the convergence of the Fourier series at $t = x$ is a local property of f (i.e., depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and so the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f.

Bor [2] has proved the following theorems for $\left|\bar{N}, p_n\right|_k$ summability methods.

Theorem 1.1. *Let $k \geq 1$. If the sequence (s_n) is bounded and (λ_n) is a sequence such that*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |\lambda_n|^k = O(1) \ \ as \ \ m \to \infty \tag{1.8}$$

and

$$\sum_{n=1}^{m} |\Delta\lambda_n| = O(1) \ \ as \ \ m \to \infty, \tag{1.9}$$

then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N}, p_n\right|_k$.

Theorem 1.2. *Let $k \geq 1$. The summability $\left|\overline{N}, p_n\right|_k$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (1.8) and (1.9) are satisfied.*

2. THE MAIN RESULTS.

The aim of this paper is to generalize above theorems for $|A, p_n; \delta|_k$ summability methods, where $k \geq 1$ and $\delta \geq 0$. Before stating the main theorem we must first introduce some further notation.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \ n, v = 0, 1, ... \tag{2.1}$$

and

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}, \ \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \ n = 1, 2, ... \tag{2.2}$$

It may be noted that $\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \overline{a}_{nv}a_v \tag{2.3}$$

and

$$\overline{\Delta}A_n(s) = \sum_{v=0}^{n} \widehat{a}_{nv}a_v. \tag{2.4}$$

Now, we shall prove the following theorems.

Theorem 2.1. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (s_n) be a bounded sequence and suppose that $A = (a_{nv})$ is a positive normal matrix such that*

$$a_{n-1,v} \geq a_{nv},\ for\ n \geq v+1, \tag{2.5}$$

$$\overline{a}_{no} = 1,\ n = 0, 1, ..., \tag{2.6}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{2.7}$$

$$\sum_{n=v+1}^{\infty} (\frac{P_n}{p_n})^{\delta k} |\Delta_v(\widehat{a}_{nv})| = O\{(\frac{P_v}{p_v})^{\delta k-1}\} \tag{2.8}$$

and

$$\sum_{n=v+1}^{\infty} (\frac{P_n}{p_n})^{\delta k} |\widehat{a}_{n,v+1}| = O\{(\frac{P_v}{p_v})^{\delta k}\}. \tag{2.9}$$

If a sequence (λ_n) holds the following conditions,

$$\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{\delta k-1} |\lambda_n|^k < \infty \tag{2.10}$$

and

$$\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{\delta k} |\Delta\lambda_n| < \infty, \tag{2.11}$$

then the series $\sum a_n\lambda_n$ is summable $|A, p_n; \delta|_k$.

Theorem 2.2. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. The summability $|A, p_n; \delta|_k$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (2.5)-(2.11) are satisfied.*

It may be remarked that, if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 2.1 and Theorem 2.2, then we get Theorem 1.1 and Theorem 1.2, respectively.

3. Proof of Theorem 2.1

Let (T_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then we have, by (2.3) and (2.4),

$$\overline{\Delta}T_n = \sum_{v=0}^{n} \widehat{a}_{nv} a_v \lambda_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\overline{\Delta}T_n &= \sum_{v=0}^{n-1} \Delta_v(\widehat{a}_{nv})\lambda_v s_v + \sum_{v=0}^{n-1} \widehat{a}_{n,v+1}\Delta\lambda_v s_v + a_{nn}\lambda_n s_n \\
&= T_n(1) + T_n(2) + T_n(3),\ say.
\end{aligned}$$

Since

$$|T_n(1) + T_n(2) + T_n(3)|^k \leq 3^k(|T_n(1)|^k + |T_n(2)|^k + |T_n(3)|^k),$$

to complete the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{\delta k+k-1} |T_n(r)|^k < \infty\ for\ r = 1, 2, 3.$$

Since (s_n) is bounded, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}\sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} |T_n(1)|^k &\leq \sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} \{\sum_{v=0}^{n-1} |\Delta_v(\widehat{a}_{nv})||\lambda_v||s_v|\}^k \\ &= O(1) \sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} \{\sum_{v=0}^{n-1} |\Delta_v(\widehat{a}_{nv})||\lambda_v|^k\} \times \{\sum_{v=0}^{n-1} |\Delta_v(\widehat{a}_{nv})|\}^{k-1}.\end{aligned}$$

Since

$$\begin{aligned}\Delta_v(\widehat{a}_{nv}) &= \widehat{a}_{nv} - \widehat{a}_{n,v+1} \\ &= \overline{a}_{nv} - \overline{a}_{n-1,v} - \overline{a}_{n,v+1} + \overline{a}_{n-1,v+1} \\ &= a_{nv} - a_{n-1,v},\end{aligned}$$

by using (2.5) and (2.6)

$$\sum_{v=0}^{n-1} |\Delta_v(\widehat{a}_{nv})| = \sum_{v=0}^{n-1} (a_{n-1,v} - a_{nv}) = 1 - 1 + a_{nn} = a_{nn},$$

we get

$$\begin{aligned}\sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} |T_n(1)|^k &= O(1) \sum_{v=0}^{m} |\lambda_v|^k \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k} |\Delta_v(\widehat{a}_{nv})| \\ &= O(1) \sum_{v=0}^{m} (\frac{P_v}{p_v})^{\delta k-1} |\lambda_v|^k \\ &= O(1) \; as \; m \to \infty,\end{aligned}$$

by virtue of the hypothesis of Theorem 2.1.

Again using Hölder's inequality,

$$\begin{aligned}\sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} |T_n(2)|^k &\leq \sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} \{\sum_{v=0}^{n-1} |\widehat{a}_{n,v+1}||\Delta\lambda_v||s_v|\}^k \\ &= O(1) \sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} \{\sum_{v=0}^{n-1} |\widehat{a}_{n,v+1}||\Delta\lambda_v|\} \times \{\sum_{v=0}^{n-1} |\widehat{a}_{n,v+1}||\Delta\lambda_v|\}^{k-1}.\end{aligned}$$

Taking account of (2.5) and (2.6) we have, for $1 \leq v \leq n-1$,

$$\begin{aligned}
\widehat{a}_{n,v+1} &= \overline{a}_{n,v+1} - \overline{a}_{n-1,v+1} \\
&= \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i} \\
&= 1 - \sum_{i=0}^{v} a_{ni} - 1 + \sum_{i=0}^{v} a_{n-1,i} \\
&= \sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) \\
&\leq \sum_{i=0}^{n-1} (a_{n-1,i} - a_{ni}) \\
&= 1 - 1 + a_{nn} = a_{nn},
\end{aligned}$$

where

$$\sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) \geq 0.$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} |T_n(2)|^k &= O(1) \sum_{v=0}^{m} |\Delta\lambda_v| \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k} |\widehat{a}_{n,v+1}| \\
&= O(1) \sum_{v=0}^{m} (\frac{P_v}{p_v})^{\delta k} |\Delta\lambda_v| = O(1) \; as \; m \to \infty,
\end{aligned}$$

by virtue of the hypothesis of Theorem 2.1.

Finally, we have that

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k+k-1} |T_n(3)|^k = O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k-1} |\lambda_n|^k = O(1) \; as \; m \to \infty,$$

by virtue of the hypothesis of Theorem 2.1.

Therefore, we get that

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k+k-1} |T_n(r)|^k = O(1) \; as \; m \to \infty, \; for \; r = 1, 2, 3.$$

This completes the proof of Theorem 2.1.

4. PROOF OF THEOREM 2.2

Since the behaviour of the Fourier series for a particular value of x, as far as convergence is concerned, depends on the behaviour of the function in the immediate neighbourhood of this point only, Theorem 2.2 is a necessary consequence of Theorem 2.1.

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