GENERALIZATIONS OF TWO THEOREMS ON ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. In this paper two theorems on $|A, p_n; \delta|_k$ summability methods, which generalize two theorems of Bor [2] on $|N, p_n|_k$ summability methods, have been proved.

Key words and phrases: Absolute summability, summability factors, infinite series.

2000 Mathematics Subject Classification 40D15, 40F05, 40G99.

ISSN (electronic): 1449-5910
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1. INTRODUCTION

Let \( \sum a_n \) be a given infinite series with the partial sums \( (s_n) \), and let \( A = (a_{nv}) \) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then \( A \) defines the sequence-to-sequence transformation, mapping the sequence \( s = (s_n) \) to \( As = (A_n(s)) \), where

\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots
\]

The series \( \sum a_n \) is said to be summable \( |A|_k, k \geq 1 \), if (see [5])

\[
\sum_{n=1}^{\infty} n^{k-1} |\Delta A_n(s)|^k < \infty
\]

where

\[
\Delta A_n(s) = A_n(s) - A_{n-1}(s).
\]

Let \( (p_n) \) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).
\]

The sequence-to-sequence transformation

\[
t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v
\]

defines the sequence \( (t_n) \) of the \( (\bar{N}, p_n) \) mean of the sequence \( (s_n) \), generated by the sequence of coefficients \( (p_n) \) (see [3]). The series \( \sum a_n \) is said to be summable \( |\bar{N}, p_n|_k, k \geq 1 \), if (see [1])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty,
\]

and it is said to be summable \( |A, p_n|_k, k \geq 1 \), if (see [14])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |\Delta A_n(s)|^k < \infty.
\]

We say that the series \( \sum a_n \) is summable \( |A, p_n; \delta|_k, k \geq 1 \) and \( \delta \geq 0 \), if

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |\Delta A_n(s)|^k < \infty.
\]

In the special case when \( \delta = 0 \), \( |A, p_n; \delta|_k \) summability is the same as \( |A, p_n|_k \) summability. Also if we take \( \delta = 0 \) and \( a_{nv} = \frac{P_v}{P_n} \), then \( |A, p_n; \delta|_k \) summability is the same as \( |\bar{N}, p_n|_k \) summability.

Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable \( (L) \) over \((-\pi, \pi)\). Without any loss of generality we may assume that the constant term in the Fourier series of \( f \) is zero, so that

\[
\int_{-\pi}^{\pi} f(t) dt = 0
\]
and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t).$$

It is well known that the convergence of the Fourier series at \( t = x \) is a local property of \( f \)
(i.e., depends only on the behaviour of \( f \) in an arbitrarily small neighbourhood of \( x \)), and so the
summability of the Fourier series at \( t = x \) by any regular linear summability method is also a
local property of \( f \).

Bor [2] has proved the following theorems for \( |\bar{N}, p_n|_k \) summability methods.

**Theorem 1.1.** Let \( k \geq 1 \). If the sequence \( (s_n) \) is bounded and \( (\lambda_n) \) is a sequence such that

\[
(1.8) \quad \sum_{n=1}^{m} \frac{p_n}{P_n} |\lambda_n|^k = O(1) \quad \text{as} \quad m \to \infty
\]

and

\[
(1.9) \quad \sum_{n=1}^{m} |\Delta \lambda_n| = O(1) \quad \text{as} \quad m \to \infty,
\]

then the series \( \sum a_n \lambda_n \) is summable \( |\bar{N}, p_n|_k \).

**Theorem 1.2.** Let \( k \geq 1 \). The summability \( |\bar{N}, p_n|_k \) of the series \( \sum A_n(t) \lambda_n \) at a point is a
local property of the generating function if the conditions \( (1.8) \) and \( (1.9) \) are satisfied.

### 2. The main results.

The aim of this paper is to generalize above theorems for \( |A, p_n; \delta|_k \) summability methods,
where \( k \geq 1 \) and \( \delta \geq 0 \). Before stating the main theorem we must first introduce some further
notation.

Given a normal matrix \( A = (a_{nv}) \), we associate two lower semimatrices \( \bar{A} = (\bar{a}_{nv}) \) and
\( \hat{A} = (\hat{a}_{nv}) \) as follows:

\[
(2.1) \quad \bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots
\]

and

\[
(2.2) \quad \hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots
\]

It may be noted that \( \bar{A} \) and \( \hat{A} \) are the well-known matrices of series-to-sequence and series-to-
series transformations, respectively. Then, we have

\[
(2.3) \quad A_n(s) = \sum_{v=0}^{n} a_{nv}s^v = \sum_{v=0}^{n} \bar{a}_{nv}\bar{a}_{v}
\]

and

\[
(2.4) \quad \Delta A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}\bar{a}_{v}.
\]

Now, we shall prove the following theorems.
Theorem 2.1. Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let $(s_n)$ be a bounded sequence and suppose that $A = (a_{nv})$ is a positive normal matrix such that

\begin{align*}
(2.5) & \quad a_{n-1,v} \geq a_{nv}, \text{ for } n \geq v + 1, \\
(2.6) & \quad \bar{a}_{no} = 1, \text{ for } n = 0, 1, \\
(2.7) & \quad a_{nn} = O\left(\frac{p_n}{p_v}\right),
\end{align*}

and

\begin{align*}
(2.8) & \quad \sum_{n=v+1}^{\infty} \frac{P_n}{p_n} \delta^k |\Delta_v(\hat{a}_{nv})| = O\left\{ \frac{(P_v)}{p_v} \delta^{k-1} \right\} \\
(2.9) & \quad \sum_{n=v+1}^{\infty} \frac{P_n}{p_n} \delta^k |\hat{a}_{n,v+1}| = O\left\{ \frac{(P_v)}{p_v} \delta^k \right\}.
\end{align*}

If a sequence $(\lambda_n)$ holds the following conditions,

\begin{align*}
(2.10) & \quad \sum_{n=1}^{\infty} \frac{P_n}{p_n} \delta^{k-1} |\lambda_n|^k < \infty \\
(2.11) & \quad \sum_{n=1}^{\infty} \frac{P_n}{p_n} \delta^k |\Delta \lambda_n| < \infty,
\end{align*}

then the series $\sum a_{n}\lambda_n$ is summable $|A, p_n; \delta|_k$.

Theorem 2.2. Let $k \geq 1$ and $0 \leq \delta < 1/k$. The summability $|A, p_n; \delta|_k$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (2.5)-(2.11) are satisfied.

It may be remarked that, if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 2.1 and Theorem 2.2, then we get Theorem [1.1] and Theorem [1.2] respectively.

3. PROOF OF THEOREM 2.1

Let $(T_n)$ denotes $A$-transform of the series $\sum a_{n}\lambda_n$. Then we have, by (2.3) and (2.4),

$$\Delta T_n = \sum_{v=0}^{n} \hat{a}_{nv} a_{v}\lambda_v.$$ 

Applying Abel’s transformation to this sum, we get that

$$\Delta T_n = \sum_{v=0}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v s_v + \sum_{v=0}^{n-1} \hat{a}_{n,v+1} \Delta \lambda_v s_v + a_{nn} \lambda_n s_n = T_n(1) + T_n(2) + T_n(3), \text{ say.}$$

Since

$$|T_n(1) + T_n(2) + T_n(3)|^k \leq 3^k([T_n(1)]^k + [T_n(2)]^k + [T_n(3)]^k),$$

to complete the proof of Theorem 2.1 it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{P_n}{p_n} \delta^{k+1} |T_n(r)|^k < \infty \text{ for } r = 1, 2, 3.$$
Since \((s_n)\) is bounded, when \(k > 1\), applying Hölder’s inequality with indices \(k\) and \(k'\), where \(\frac{1}{k} + \frac{1}{k'} = 1\), we have that

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |T_n(1)|^k \leq \sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left\{ \sum_{v=0}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v||s_v| \right\}^k
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left\{ \sum_{v=0}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k \right\} \times \left\{ \sum_{v=0}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1}.
\]

Since

\[
\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}
\]

\[
= a_{nv} - a_{n-1,v} - a_{n,v+1} + a_{n-1,v+1}
\]

\[
= a_{nv} - a_{n-1,v},
\]

by using (2.5) and (2.6)

\[
\sum_{v=0}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=0}^{n-1} (a_{n-1,v} - a_{nv}) = 1 - 1 + a_{nn} = a_{nn},
\]

we get

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |T_n(1)|^k = O(1) \sum_{v=0}^{m} |\lambda_v|^k \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} |\Delta_v(\hat{a}_{nv})|
\]

\[
= O(1) \sum_{v=0}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k - 1} |\lambda_v|^k
\]

\[
= O(1) \text{ as } m \to \infty,
\]

by virtue of the hypothesis of Theorem 2.1.

Again using Hölder’s inequality,

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |T_n(2)|^k \leq \sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left\{ \sum_{v=0}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v||s_v| \right\}^k
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left\{ \sum_{v=0}^{n-1} |\Delta_v(\hat{a}_{nv})||\Delta \lambda_v| \right\} \times \left\{ \sum_{v=0}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1}.
\]
Taking account of (2.5) and (2.6) we have, for \(1 \leq v \leq n - 1\),
\[
\hat{a}_{n,v+1} = a_{n,v+1} - a_{n-1,v+1} = \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i}
\]
\[
= 1 - \sum_{i=0}^{v} a_{ni} - 1 + \sum_{i=0}^{v} a_{n-1,i}
\]
\[
= \sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) \leq \sum_{i=0}^{n-1} (a_{n-1,i} - a_{ni}) = 1 - 1 + a_{nn} = a_{nn},
\]
where
\[
\sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) \geq 0.
\]

Thus,
\[
\sum_{n=1}^{m+1} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k+k-1} |T_{n}(2)|^{k} = O(1) \sum_{v=0}^{m} |\Delta \lambda_{v}| \sum_{n=v+1}^{m+1} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k} |\hat{a}_{n,v+1}|
\]
\[
= O(1) \sum_{v=0}^{m} \left( \frac{P_{v}}{p_{v}} \right)^{\delta k} |\Delta \lambda_{v}| = O(1) \quad \text{as} \quad m \to \infty,
\]
by virtue of the hypothesis of Theorem 2.1.

Finally, we have that
\[
\sum_{n=1}^{m} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k+k-1} |T_{n}(3)|^{k} = O(1) \sum_{n=1}^{m} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k-1} |\lambda_{n}|^{k} = O(1) \quad \text{as} \quad m \to \infty,
\]
by virtue of the hypothesis of Theorem 2.1.

Therefore, we get that
\[
\sum_{n=1}^{m} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k+k-1} |T_{n}(r)|^{k} = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3.
\]
This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

Since the behaviour of the Fourier series for a particular value of \(x\), as far as convergence is concerned, depends on the behaviour of the function in the immediate neighbourhood of this point only, Theorem 2.2 is a necessary consequence of Theorem 2.1.
REFERENCES


