The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 1, Issue 2, Article 12, pp. 1-6, 2004

NEW PROOFS OF THE GRÜSS INEQUALITY

A.MC.D. MERCER AND PETER R. MERCER

Received 27 September, 2004; accepted 23 November, 2004; published 27 December, 2004.

DEPT. MATHEMATICS AND STATISTICS, UNIVERSITY OF GUELPH, GUELPH, ONTARIO, CANADA.
amercer@reach.net

DEPT. MATHEMATICS, S.U.N.Y. COLLEGE AT BUFFALO, BUFFALO, NY, U.S.A.
mercerpr@math.buffalostate.edu

ABSTRACT. We present new proofs of the Grüss inequality in its original form and in its linear functional form.

Key words and phrases: Inequality, Grüss, Andersson.

1991 *Mathematics Subject Classification.* 26D15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

We denote by $L^\infty(a,b)$ the commutative Banach algebra of real-valued functions defined and essentially bounded on (a,b). Let L be a positive linear functional on $L^\infty(a,b)$. Unless L is trivial, there is no loss of generality in taking $L(1) = 1$.

It is well known that if $f, g \in L^\infty(a,b)$ with $m \le f \le M$ and $p \le g \le P$ $a.e.$ then

$$|L(fg) - L(f)L(g)| \le \frac{1}{4}(M-m)(P-p). \tag{1.1}$$

In its original form this reads

$$\left| \tfrac{1}{b-a}\int_a^b f(x)g(x)dx - \tfrac{1}{b-a}\int_a^b f(x)dx \tfrac{1}{b-a}\int_a^b g(x)dx \right| \le \frac{1}{4}(M-m)(P-p), \tag{1.2}$$

and as such is known as Grüss' Inequality. Proofs of it, of other particular cases of (1.1), and of (1.1) itself can be found in the literature. (Chapter 10 of [4] serves as a comprehensive reference.)

The purpose of this note is to present some new proofs of these results, which we believe to be of interest. We begin with some preparation.

It is clear that both (1.1) and (1.2) are invariant under affine transformations of f and g (that is $f \to af + b$, $g \to \alpha g + \beta$). In view of this we may suppose, for example, that m (and/or p) is positive, or is zero. Indeed we could even take $m = p = 0$ and $M = P = 1$. Below we make free use of these devices, whilst staying as close to the original right hand sides of (1.1) and (1.2) as is convenient.

We note also that to prove (1.1), for example, is it sufficient to establish it with the absolute value signs on the left removed. For if the inequality is true without them, for f and g, then it is also true for $M + m - f$ and g, whence the result becomes

$$L(fg) - L(f)L(g) \ge -\frac{1}{4}(M-m)(P-p),$$

which gives (1.1).

Finally, there is no loss of generality if we take $(a,b) = (0,1)$ and we shall do this throughout.

2. TWO PROOFS OF (1.2)

Here we give two proofs of (1.2) without the absolute value signs, and we take $M = P = 1$ and $m = p = 0$. Also, for brevity, we write $\int f$ to mean $\int_0^1 f(x)dx$ etc.

First proof of (1.2). Let f^* and g^* be the non-increasing rearrangements of f and g ([3], Sec 10.13).

Then $\int f = \int f^*$ and $\int g = \int g^*$ but $\int fg \le \int f^* g^*$ so that

$$\int fg - \int f \int g \le \int f^* g^* - \int f^* \int g^* = \int f^*(g^* - \int g^*)$$

Note that the rearranged functions are each non-increasing from 1 to 0 in $(0,1)$.

We suppose that the change of sign of $g^* - \int g^*$ takes place at $x = \alpha$, and then define F via

$$F(x) = \begin{cases} 1 & \text{if } 0 \le x \le \alpha \\ 0 & \text{if } \alpha < x \le 1. \end{cases}$$

Let the change of sign of $F - \int F$ take place at $x = \beta$, say, and define G via

$$G(x) = \begin{cases} 1 & \text{if } 0 \le x \le \beta \\ 0 & \text{if } \beta < x \le 1. \end{cases}$$

Then we have

$$\int f^*(g^* - \int g^*) \le \int F(g^* - \int g^*) = \int g^*(F - \int F) \le \int G(F - \int F).$$

Now supposing for the sake of definiteness that $\alpha \le \beta$, this last expression is equal to

$$\alpha - \alpha\beta.$$

But

$$\alpha - \alpha\beta \le \alpha - \alpha^2 \le \frac{1}{4},$$

which completes a proof of (1.2). □

Second proof of (1.2). Denote by $K \subset L^\infty(0,1)$ the closed convex set of functions which are bounded almost everywhere by 0 and 1. For $f, g \in K$ let

$$\Psi(f,g) = \int fg - \int f \int g.$$

Suppose now that g is fixed. Then Ψ is a continuous linear functional on K. It will take its maximum value at an extreme point of K – that is, when f is a characteristic function taking only values 0 and 1 – say $\phi(g)$. Therefore

$$\Psi(f,g) \le \int \phi(g)g - \int \phi(g) \int g.$$

Next consider

$$\Psi(\phi(g), w) = \int \phi(g)w - \int \phi(g) \int w \qquad (w \in K).$$

In a similar way, this is maximized by

$$\int \phi(g)\theta(g) - \int \phi(g) \int \theta(g)$$

for some characteristic function $\theta(g)$.

We point out that just as ϕ depends on g so does θ depend on $\phi(g)$ and hence also on g. But whatever g may be, the upper bound of $\Psi(f,g)$ has the form

$$\int \phi\theta - \int \phi \int \theta$$

for some characteristic functions ϕ and θ satisfying, say,

$$\phi(x) = 1 \text{ on } E_1 \text{ and } = 0 \text{ elsewhere in } (0,1)$$

and

$$\theta(x) = 1 \text{ on } E_2 \text{ and } = 0 \text{ elsewhere in } (0,1).$$

Hence

$$\Psi(f,g) \le \int \phi\theta - \int \phi \int \theta = m(E_1 \cap E_2) - mE_1 mE_2.$$

We may suppose that $mE_1 \leq mE_2$, and then we get

$$m(E_1 \cap E_2) - mE_1 mE_2 \leq mE_1 - (mE_1)^2 \leq \frac{1}{4},$$

which completes another proof of (1.2). □

3. TWO PROOFS OF (1.1)

For these two proofs we take the bounds of f and g to be M, m and P, p respectively, as there is no advantage in not doing so. We shall prove the versions with the absolute value signs. First we need a little more preparation. For brevity again, we write F for $L(f)$ and G for $L(g)$. There should be no confusion with the F and G used before.

Consider the following quadratic in λ :

$$L((g-\lambda)^2) \text{ where } p \leq \lambda \leq P.$$

This quadratic takes its minimum value at G, so applying the Schwarz inequality for positive linear functionals and remembering that $L(1) = 1$ we get

$$L(|g-G|)^2 \leq L((g-G)^2) \leq L((g-\frac{P+p}{2})^2) \leq \frac{1}{4}(P-p)^2, \tag{3.1}$$

so that

$$L(|g-G|) \leq \frac{1}{2}(P-p). \tag{3.2}$$

First proof of (1.1). Since $L(g-G) = 0$ we have

$$\begin{aligned} |L(fg) - L(f)L(g)| &= |L(f(g-G))| = \frac{1}{2}\left|L((2f-M-m)(g-G))\right| \\ &\leq \frac{1}{2}L((|f-M| + |f-m|)\,|g-G|) \leq (M-m)L(|g-G|). \end{aligned}$$

Interchanging the roles of f and g in the first and last of these we can combine the two results to get a 'pre-Grüss inequality'

$$|L(fg) - L(f)L(g)| \leq \frac{1}{2}Min\left[(M-m)L(|g-G|)\,,\,(P-p)L(|f-F|\right], \tag{3.3}$$

which is a refinement of (1.1).

Now applying (3.2) to the former of these last two expressions we get

$$|L(fg) - L(f)L(g)| \leq \frac{1}{2}(M-m)\frac{1}{2}(P-p),$$

which concludes a proof of (1.1). □

Second proof of (1.1). This proof follows quickly from (3.1). We have

$$|L(fg) - L(f)L(g)| = |L((f-F)(g-G))|\,,$$

and so

$$|L(fg) - L(f)L(g)|^2 \leq \left|L((f-F)^2)\right| \left|L((g-G)^2)\right|,$$

by Schwarz's Inequality. By (3.1), this gives

$$|L(fg) - L(f)L(g)|^2 \leq \frac{(M-m)^2}{4}\frac{(P-p)^2}{4},$$

and taking square roots, we have concluded another proof of (1.1). □

4. Final Remarks

Remark 4.1. Inequality (3.3) is obtained in [2] by a different method. There, the sharpness of the constant $1/2$ is also demonstrated.

Remark 4.2. If in (1.1) we take f and g to be continuous on $[a, b]$, then in view of the form of positive linear functionals defined on $C[a, b]$, it can be written as

$$\left| \tfrac{1}{b-a} \int_a^b f(x)g(x)dw - \tfrac{1}{b-a} \int_a^b f(x)dw \tfrac{1}{b-a} \int_a^b g(x)dw \right| \leq \frac{1}{4}(M-m)(P-p),$$

where w is non-decreasing in $[a, b]$. Now taking $a = 0$, $b = n$ and $w(x) \equiv [|x|]$ (the integer part of x) and writing a_k for $f(k)$ and b_k for $g(k)$, we get the discrete form of Grüss' Inequality:

$$\left| \frac{1}{n} \sum_{k=1}^{n} a_k b_k - \frac{1}{n} \sum_{k=1}^{n} a_k \frac{1}{n} \sum_{k=1}^{n} b_k \right| \leq \frac{1}{4}(M-m)(P-p),$$

in which $m \leq a_k \leq M$, and $p \leq b_k \leq P$ for all k.

Remark 4.3. Referring to the second proof of (1.2) above we remark that the same method can be extended to the case of three or more functions. For example (leaving the details to the reader),

$$\begin{aligned} \int fgh - \int f \int g \int h &\leq m(E_1 \cap E_2 \cap E_3) - mE_1 mE_2 mE_3 \\ &\leq mE_1 - (mE_1)^3 \leq \frac{2}{3\sqrt{3}}. \end{aligned}$$

The absolute value signs can now be added to the left hand side for the same reason as before. It should be noted, however, that the analogue can only be arrived at with $m = p = 0$ because the affine property is absent in the case of more than two functions.

Remark 4.4. Chebyshev's Inequality ([4]) reads: If f and g are both increasing or both decreasing then

$$\int_a^b fg \geq \tfrac{1}{b-a} \int_a^b f \int_a^b g.$$

This can be established using the same idea as in the second proof of (1.2) above. Indeed, looking instead for a minimum for Ψ, we get (with the obvious changed roles for ϕ and θ)

$$\Psi(f, g) \geq \int \phi\theta - \int \phi \int \theta = m(E_1 \cap E_2) - mE_1 mE_2,$$

and since ϕ and θ are both increasing or both decreasing, this is positive. (Each of E_1, E_2 is an interval and one of them is a subset of the other.)

In the same way, and in view of Remark 4.2 above, we can obtain an inequality due to Andersson ([1]): If f_j are convex and increasing on $[0, 1]$ with $f_j(0) = 0$, then

$$\int_0^1 f_1 \cdots f_n \geq \tfrac{2^n}{n+1} \int_0^1 f_1 \cdots \int_0^1 f_n.$$

Here the extreme points are functions of the form αx $(\alpha \geq 0)$, and $\int_0^1 f_1 \cdots f_n - \frac{2^n}{n+1} \int_0^1 f_1 \cdots \int_0^1 f_n$ is zero when each f_j is an extreme point.

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