AN ALTERNATIVE PROOF OF MONOTONICITY FOR THE EXTENDED MEAN VALUES

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ABSTRACT. An alternative proof of monotonicity for the extended mean values is given.

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1. Introduction

The generalized logarithmic mean $L_r(x, y)$ of two positive numbers $x, y$ is introduced in [1], [7], [8] for $x = y$ by $L_r(x, y) = x$ and for $x \neq y$ by

$$L_r(x, y) = \left( \frac{y^{r+1} - x^{r+1}}{(r+1)(y - x)} \right)^{1/r}, \quad r \neq -1, 0; \quad (1.1)$$

$$L_{-1}(x, y) = \frac{y - x}{\ln y - \ln x} = L(x, y); \quad (1.2)$$

$$L_0(x, y) = \frac{1}{e} \left( \frac{y^y}{x^x} \right)^{1/(y - x)} = I(x, y). \quad (1.3)$$

$L(x, y)$ and $I(x, y)$ are respectively called the logarithmic mean and exponential mean of two positive numbers $x, y$. When $x \neq y$, $L_r(x, y)$ is a strictly increasing function of $r$. In particular,

$$\lim_{r \to -\infty} L_r(x, y) = \min\{x, y\}, \quad \lim_{r \to +\infty} L_r(x, y) = \max\{x, y\}. \quad (1.4)$$

For $x \neq y$, the following well known inequality holds clearly:

$$G(x, y) < L(x, y) < I(x, y) < A(x, y), \quad (1.5)$$

where $A(x, y)$ and $G(x, y)$ are the arithmetic and geometric means of two positive numbers $x, y$, respectively.

Stolarsky defined in [7] the extended mean values $E(r, s; x, y)$ by

$$E(r, s; x, y) = \left( \frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \right)^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0; \quad (1.6)$$

$$E(r, 0; x, y) = \left( \frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x} \right)^{1/r}, \quad r(x-y) \neq 0;$$

$$E(r, r; x, y) = \frac{1}{e^{1/r}} \left( \frac{x^x}{y^y} \right)^{1/(x^r-y^r)}, \quad r(x-y) \neq 0;$$

$$E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y;$$

$$E(r, s; x, x) = x, \quad x = y$$

and proved that it is continuous on the domain $\{(r, s; x, y) : r, s \in \mathbb{R}, x, y > 0\}$.

Leach and Sholander showed in [2], [3] that $E(r, s; x, y)$ is increasing with both $r$ and $s$, and with both $x$ and $y$. The monotonicities of $E$ has also been researched in [5], [6] using different ideas and simpler methods. See also [4], an expository paper. Clearly, $L_r(x, y)$ is special case of $E(r, s; x, y)$ since $E(0, r; x, y) = L_r(x, y)$.

The aim of this note is to give an alternative proof of monotonicity for the extended mean values $E(r, s; x, y)$.

Theorem. $E(r, s; x, y)$ is strictly increasing with both $r$ and $s$, and with both $x$ and $y$.

2. An alternative proof of monotonicity of $E(r, s; x, y)$

First, we prove that $E(r, s; x, y)$ is strictly increasing with both $r$ and $s$. We define the function $\psi$ by

$$\psi(r, s; x, y) = \left( s - r \right)^2 \frac{\partial E(r, s; x, y)}{E(r, s; x, y)} \frac{\partial E(r, s; x, y)}{\partial s} \quad (2.1)$$

$$= - \ln \left( \frac{r(y^r - x^r)}{s(y^r - x^r)} + (s - r) \left( \frac{y^s \ln y - x^s \ln x}{y^s - x^s} - \frac{1}{s} \right) \right)$$
for $rs(r - s)(x - y) \neq 0$, and

$$\psi(r, 0; x, y) = \lim_{s \to 0} \psi(r, s; x, y) = \frac{L(x^r, y^r)}{G(x^r, y^r)},$$

$$\psi(r, r; x, y) = \lim_{s \to r} \psi(r, s; x, y) = 0.$$  \hspace{1cm} (2.2)

Then, by direct calculation, we have

$$\frac{\partial \psi(r, s; x, y)}{\partial s} = (s - r) \left( \frac{1}{s^2} - \frac{x^s y^s (\ln y - \ln x)^2}{(y^s - x^s)^2} \right)$$

$$= (s - r) \left( y^s - x^s \right)^2 - x^s y^s (\ln y^s - \ln x^s)^2$$

$$= \frac{(s - r)(\ln y^s - \ln x^s)^2}{s^2(y^s - x^s)^2} \left( \frac{(y^s - x^s)^2}{(\ln y^s - \ln x^s)^2} - x^s y^s \right)$$

$$= \frac{(s - r)[L(x^s, y^s)]^2 - [G(x^s, y^s)]^2}{s^2[L(x^s, y^s)]^2},$$  \hspace{1cm} (2.3)

for $rs(r - s)(x - y) \neq 0$.

By the well-known fact that $L(x^s, y^s) > G(x^s, y^s)$ for $s(x - y) \neq 0$, it is easy to see that $\psi(r, s; x, y)$ takes its unique minimum $\psi(r, r; x, y) = 0$ at $s = r$. This implies $\psi(r, s; x, y) > 0$ and $\frac{\partial E(r, s; x, y)}{\partial s} > 0$ for $rs(r - s)(x - y) \neq 0$. Thus, $E(r, s; x, y)$ is strictly increasing with respect to $s$.

The same monotonicity can be applied to the variable $r$ since the property of symmetry $E(r, s; x, y) = E(s, r; x, y)$.

Next, we prove that $E(r, s; x, y)$ is strictly increasing with both $x$ and $y$. Evaluating the partial derivative of $E(r, s; x, y)$ with respect to $x$ yields for $rs(r - s)(x - y) \neq 0$

$$
\frac{1}{E(r, s; x, y)} \frac{\partial E(r, s; x, y)}{\partial x} = \frac{1}{s - r} \left( \frac{sx^{s-1}}{x^s - y^s} - \frac{rx^{r-1}}{x^r - y^r} \right).
$$  \hspace{1cm} (2.4)

Differentiation yields

$$
\left( \frac{tx^{t-1}}{x^t - y^t} \right)'_t = \frac{x^{t-1}[(x^t - y^t) - y^t(\ln x^t - \ln y^t)]}{(x^t - y^t)^2}
$$

$$= \frac{x^{t-1}(\ln x^t - \ln y^t)}{(x^t - y^t)^2} \left( \frac{x^t - y^t}{\ln x^t - \ln y^t} - y^t \right)
$$

$$= \frac{x^{t-1}L(x^t, y^t)}{L(x^t, y^t)} \frac{y^t}{x^t - y^t} > 0$$  \hspace{1cm} (2.5)

for $t(x - y) \neq 0$, which implies

$$\frac{\partial E(r, s; x, y)}{\partial x} > 0$$  \hspace{1cm} (2.6)

for $rs(r - s)(x - y) \neq 0$. Thus, $E(r, s; x, y)$ is strictly increasing with respect to $x$.

The same monotonicity can be applied to the variable $y$ since the property of symmetry $E(r, s; x, y) = E(r, s; y, x)$. The proof is complete.

REFERENCES


