The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 1, Issue 2, Article 11, pp. 1-4, 2004

AN ALTERNATIVE PROOF OF MONOTONICITY FOR THE EXTENDED MEAN VALUES

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Received 23 June, 2004; accepted 23 November, 2004; published 15 December, 2004.

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ABSTRACT. An alternative proof of monotonicity for the extended mean values is given.

Key words and phrases: Monotonicity, Extended mean values.

2000 *Mathematics Subject Classification.* Primary 26A48.

ISSN (electronic): 1449-5910

The authors were supported in part by SF of Henan Innovation Talents at Universities, China.

1. INTRODUCTION

The generalized logarithmic mean $L_r(x,y)$ of two positive numbers x,y is introduced in [1], [7], [8] for $x=y$ by $L_r(x,y)=x$ and for $x\neq y$ by

$$L_r(x,y)=\left(\frac{y^{r+1}-x^{r+1}}{(r+1)(y-x)}\right)^{1/r},\quad r\neq -1,0; \tag{1.1}$$

$$L_{-1}(x,y)=\frac{y-x}{\ln y-\ln x}=L(x,y); \tag{1.2}$$

$$L_0(x,y)=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}=I(x,y). \tag{1.3}$$

$L(x,y)$ and $I(x,y)$ are respectively called the logarithmic mean and exponential mean of two positive numbers x,y. When $x\neq y$, $L_r(x,y)$ is a strictly increasing function of r. In particular,

$$\lim_{r\to-\infty}L_r(x,y)=\min\{x,y\},\qquad \lim_{r\to+\infty}L_r(x,y)=\max\{x,y\}. \tag{1.4}$$

For $x\neq y$, the following well known inequality holds clearly:

$$G(x,y)<L(x,y)<I(x,y)<A(x,y), \tag{1.5}$$

where $A(x,y)$ and $G(x,y)$ are the arithmetic and geometric means of two positive numbers x,y, respectively.

Stolarsky defined in [7] the extended mean values $E(r,s;x,y)$ by

$$\begin{aligned}
E(r,s;x,y)&=\left(\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, && rs(r-s)(x-y)\neq 0;\\
E(r,0;x,y)&=\left(\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, && r(x-y)\neq 0;\\
E(r,r;x,y)&=\frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, && r(x-y)\neq 0;\\
E(0,0;x,y)&=\sqrt{xy}, && x\neq y;\\
E(r,s;x,x)&=x, && x=y
\end{aligned} \tag{1.6}$$

and proved that it is continuous on the domain $\{(r,s;x,y):r,s\in\mathbb{R},x,y>0\}$.

Leach and Sholander showed in [2], [3] that $E(r,s;x,y)$ is increasing with both r and s, and with both x and y. The monotonicities of E has also been researched in [5], [6] using different ideas and simpler methods. See also [4], an expository paper. Clearly, $L_r(x,y)$ is special case of $E(r,s;x,y)$ since $E(0,r;x,y)=L_r(x,y)$.

The aim of this note is to give an alternative proof of monotonicity for the extended mean values $E(r,s;x,y)$.

Theorem. *$E(r,s;x,y)$ is strictly increasing with both r and s, and with both x and y.*

2. AN ALTERNATIVE PROOF OF MONOTONICITY OF $E(r,s;x,y)$

First, we prove that $E(r,s;x,y)$ is strictly increasing with both r and s. We define the function ψ by

$$\begin{aligned}
\psi(r,s;x,y)&=\frac{(s-r)^2}{E(r,s;x,y)}\frac{\partial E(r,s;x,y)}{\partial s}\\
&=-\ln\frac{r(y^s-x^s)}{s(y^r-x^r)}+(s-r)\left(\frac{y^s\ln y-x^s\ln x}{y^s-x^s}-\frac{1}{s}\right)
\end{aligned} \tag{2.1}$$

for $rs(r-s)(x-y) \neq 0$, and

$$\psi(r,0;x,y) = \lim_{s\to 0}\psi(r,s;x,y) = \frac{L(x^r,y^r)}{G(x^r,y^r)}, \quad \psi(r,r;x,y) = \lim_{s\to r}\psi(r,s;x,y) = 0. \tag{2.2}$$

Then, by direct calculation, we have

$$\begin{aligned} \frac{\partial\psi(r,s;x,y)}{\partial s} &= (s-r)\left(\frac{1}{s^2} - \frac{x^s y^s(\ln y - \ln x)^2}{(y^s-x^s)^2}\right) \\ &= (s-r)\frac{(y^s-x^s)^2 - x^s y^s(\ln y^s - \ln x^s)^2}{s^2(y^s-x^s)^2} \\ &= \frac{(s-r)(\ln y^s - \ln x^s)^2}{s^2(y^s-x^s)^2}\left(\frac{(y^s-x^s)^2}{(\ln y^s - \ln x^s)^2} - x^s y^s\right) \\ &= (s-r)\frac{[L(x^s,y^s)]^2 - [G(x^s,y^s)]^2}{s^2[L(x^s,y^s)]^2} \end{aligned} \tag{2.3}$$

for $rs(r-s)(x-y) \neq 0$.

By the well-known fact that $L(x^s,y^s) > G(x^s,y^s)$ for $s(x-y)\neq 0$, it is easy to see that $\psi(r,s;x,y)$ takes its unique minimum $\psi(r,r;x,y)=0$ at $s=r$. This implies $\psi(r,s;x,y)>0$ and $\frac{\partial E(r,s;x,y)}{\partial s} > 0$ for $rs(r-s)(x-y)\neq 0$. Thus, $E(r,s;x,y)$ is strictly increasing with respect to s.

The same monotonicity can be applied to the variable r since the property of symmetry $E(r,s;x,y) = E(s,r;x,y)$.

Next, we prove that $E(r,s;x,y)$ is strictly increasing with both x and y. Evaluating the partial derivative of $E(r,s;x,y)$ with respect to x yields for $rs(r-s)(x-y)\neq 0$

$$\frac{1}{E(r,s;x,y)}\frac{\partial E(r,s;x,y)}{\partial x} = \frac{1}{s-r}\left(\frac{sx^{s-1}}{x^s-y^s} - \frac{rx^{r-1}}{x^r-y^r}\right). \tag{2.4}$$

Differentiation yields

$$\begin{aligned} \left(\frac{tx^{t-1}}{x^t-y^t}\right)'_t &= \frac{x^{t-1}[(x^t-y^t) - y^t(\ln x^t - \ln y^t)]}{(x^t-y^t)^2} \\ &= \frac{x^{t-1}(\ln x^t - \ln y^t)}{(x^t-y^t)^2}\left(\frac{x^t-y^t}{\ln x^t - \ln y^t} - y^t\right) \\ &= \frac{x^{t-1}}{L(x^t,y^t)}\frac{L(x^t,y^t)-y^t}{x^t-y^t} > 0 \end{aligned} \tag{2.5}$$

for $t(x-y)\neq 0$, which implies

$$\frac{\partial E(r,s;x,y)}{\partial x} > 0 \tag{2.6}$$

for $rs(r-s)(x-y)\neq 0$. Thus, $E(r,s;x,y)$ is strictly increasing with respect to x.

The same monotonicity can be applied to the variable y since the property of symmetry $E(r,s;x,y) = E(r,s;y,x)$. The proof is complete.

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