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ERROR ESTIMATES FOR APPROXIMATIONS OF THE FOURIER TRANSFORM OF FUNCTIONS IN L_p SPACES

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ABSTRACT. New inequalities concerning the Fourier transform are given. Estimates of the difference between two Fourier transforms and even bounds to the associated numerical quadrature formulae are provided as well.

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1. INTRODUCTION

The Fourier transform $\mathcal{F}(g)(x)$ of Lebegue integrable mapping $g:[a,b]\to\mathbb{R}$ is defined by

$$\mathcal{F}(g)(x)=\int_a^b g(t)\,e^{-2\pi ixt}dt. \tag{1.1}$$

The inverse Fourier transform $\mathcal{F}^{-1}(g)(x)$ is defined by

$$\mathcal{F}^{-1}(g)(x)=\int_a^b g(t)\,e^{2\pi ixt}dt.$$

In the recent paper [3] the following theorem has been proved:

Theorem 1.1. *Let $g:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$, then for all $x\neq 0$ we have the inequality*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|\leq\frac{3}{4}(b-a)\bigvee_a^b(g). \tag{1.2}$$

Here $\bigvee_a^b(g)$ is total variation of g on $[a,b]$, $E(z,w)$ is exponential mean of z and w

$$E(z,w)=\begin{cases}\frac{e^z-e^w}{z-w}, & \textit{if } z\neq w,\\ e^w, & \textit{if } z=w.\end{cases}\qquad z,w\in\mathbb{C} \tag{1.3}$$

Also, the next theorem has been established in [2]:

Theorem 1.2. *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. Then for all $x\neq 0$ we have the inequality*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|$$
$$\leq\begin{cases}\frac{1}{3}(b-a)^2\|g'\|_\infty, & \textit{if } g'\in L_\infty[a,b],\\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|g'\|_p, & \textit{if } g'\in L_p[a,b],\\ (b-a)\|g'\|_1 & \textit{if } g'\in L_1[a,b],\end{cases} \tag{1.4}$$

where $\mathcal{F}(g)$ and $E(z,w)$ are given by (1.1) and (1.3).

In this paper we give other error estimates to the same approximations of Fourier transform (Section 2). Some new inequalities concerning the estimate of difference between two Fourier transforms (Section 3) as well as the associated numerical quadrature rules (Section 4) are given.

2. ANOTHER ERROR ESTIMATES OF INEQUALITIES (1.2) AND (1.4)

Theorem 2.1. *Let $g:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$, then for all $x\neq 0$ we have the inequality .*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|\leq\frac{1}{\pi|x|}\bigvee_a^b(g), \tag{2.1}$$

where $\bigvee_a^b(g)$ is total variation of g on $[a,b]$ and $\mathcal{F}(g)$, $E(z,w)$ are given by (1.1) and (1.3).

Proof. Montgomery identity states (see [4]):

$$g(t)=\frac{1}{b-a}\int_a^b g(s)\,ds+\int_a^b P(t,s)\,dg(s)\,, \tag{2.2}$$

where $P(t,s)$ is the Peano kernel, defined by

$$P(t,s)=\begin{cases}\frac{s-a}{b-a}, & a\le s\le t,\\ \frac{s-b}{b-a} & t<s\le b.\end{cases}$$

Using this identity we have

$$\begin{aligned}\mathcal{F}(g)(x)&=\int_a^b g(t)\,e^{-2\pi ixt}dt\\ &=\frac{1}{b-a}\int_a^b\left[\int_a^b g(s)\,ds+\int_a^t (s-a)\,dg(s)+\int_t^b (s-b)\,dg(s)\right]e^{-2\pi ixt}dt.\end{aligned}$$

By an interchange of the order of integration we get

$$\begin{aligned}\int_a^b\left(\int_a^b g(s)\,ds\right)e^{-2\pi ixt}dt&=\int_a^b\left(\int_a^b e^{-2\pi ixt}dt\right)g(s)\,ds\\ &=\int_a^b\left(\frac{e^{-2\pi ixb}-e^{-2\pi ixa}}{-2\pi ix}\right)g(s)\,ds\\ &=E(-2\pi ixa,-2\pi ixb)\,(b-a)\int_a^b g(s)\,ds,\end{aligned}$$

$$\begin{aligned}\int_a^b\left(\int_a^t (s-a)\,dg(s)\right)e^{-2\pi ixt}dt&=\int_a^b\left(\int_s^b e^{-2\pi ixt}dt\right)(s-a)\,dg(s)\\ &=\int_a^b\left(\frac{e^{-2\pi ixb}-e^{-2\pi ixs}}{-2\pi ix}\right)(s-a)\,dg(s)\,,\end{aligned}$$

$$\begin{aligned}\int_a^b\left(\int_t^b (s-b)\,dg(s)\right)e^{-2\pi ixt}dt&=\int_a^b\left(\int_a^s e^{-2\pi ixt}dt\right)(s-b)\,dg(s)\\ &=\int_a^b\left(\frac{e^{-2\pi ixs}-e^{-2\pi ixa}}{-2\pi ix}\right)(s-b)\,dg(s)\,.\end{aligned}$$

So we have

$$\begin{aligned}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds&=\int_a^b\frac{e^{-2\pi ixs}}{2\pi ix}dg(s)\\ +\left[\int_a^b\frac{e^{-2\pi ixb}}{-2\pi ix}\left(\frac{s-a}{b-a}\right)dg(s)+\int_a^b\left(\frac{e^{-2\pi ixa}}{-2\pi ix}\right)\left(\frac{b-s}{b-a}\right)dg(s)\right]\end{aligned} \tag{2.3}$$

and the proof follows since

$$\begin{aligned}
&\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \\
&= \left|\int_a^b \left[\frac{e^{-2\pi ixs}}{2\pi ix} + \left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix} + \left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix}\right] dg(s)\right| \\
&\le \sup_{s\in[a,b]}\left|\frac{e^{-2\pi ixs}}{2\pi ix} + \left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix} + \left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix}\right| \bigvee_a^b(g) \\
&\le \frac{1}{2\pi|x|}\left|1 + \frac{s-a}{b-a} + \frac{b-s}{b-a}\right| \bigvee_a^b(g) = \frac{2}{2\pi|x|}\bigvee_a^b(g).
\end{aligned}$$

∎

Remark 2.1. Whenever it holds that $|x| > \frac{4}{3\pi(b-a)}$ for $x \in [a,b]$, the Theorem 2.1 is an improvement of the Theorem 1.1.

Next theorem provides a similar result for L_p spaces.

Theorem 2.2. *Assume (p,q) is a pair of conjugate exponents. Let $g : [a,b] \to \mathbb{R}$ be absolutely continuous such that $g' \in L_p[a,b]$. Then for $1 < p \le \infty$, and for all $x \ne 0$ we have the inequality*

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \le \frac{(b-a)^{\frac{1}{q}}}{\pi|x|}\|g'\|_p, \tag{2.4}$$

while for $p = 1$ we have

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \le \frac{1}{\pi|x|}\|g'\|_1. \tag{2.5}$$

Proof. Using formula (2.3) with $dg(s) = g'(s)\,ds$ we have

$$\begin{aligned}
&\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds = \int_a^b \frac{e^{-2\pi ixs}}{2\pi ix} g'(s)\,ds \\
&+ \left[\int_a^b \frac{e^{-2\pi ixb}}{-2\pi ix}\left(\frac{s-a}{b-a}\right) g'(s)\,ds + \int_a^b \left(\frac{e^{-2\pi ixa}}{-2\pi ix}\right)\left(\frac{b-s}{b-a}\right) g'(s)\,ds\right].
\end{aligned}$$

For $1 < p \le \infty$, by applying Hölder inequality we obtain

$$\begin{aligned}
&\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \\
&= \left|\int_a^b \left[\frac{e^{-2\pi ixs}}{2\pi ix} + \left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix} + \left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix}\right] g'(s)\,ds\right| \\
&\le \left\|\frac{e^{-2\pi ixs}}{2\pi ix} + \left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix} + \left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix}\right\|_q \|g'\|_p \\
&\le \left\|\frac{1}{2\pi ix}\left(1 + \frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_q \|g'\|_p \le \left\|\frac{1}{\pi ix}\right\|_q \|g'\|_p \\
&= \frac{(b-a)^{\frac{1}{q}}}{\pi|x|}\|g'\|_p.
\end{aligned}$$

Similarly for $p=1$ we have

$$\begin{aligned} & \left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right| \\ = & \left|\int_a^b\left[\frac{e^{-2\pi ixs}}{2\pi ix}+\left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix}+\left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix}\right]g'(s)\,ds\right| \\ \leq & \sup_{s\in[a,b]}\left|\frac{e^{-2\pi ixs}}{2\pi ix}+\left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix}+\left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix}\right|\|g'\|_1 \\ \leq & \frac{1}{2\pi|x|}\left|1+\frac{s-a}{b-a}+\frac{b-s}{b-a}\right|\|g'\|_1=\frac{1}{\pi|x|}\|g'\|_1, \end{aligned}$$

and the proof is completed. ∎

Remark 2.2. Whenever it holds that $|x|>\frac{1}{\pi(b-a)}\left[\frac{1}{2}(q+1)(q+2)\right]^{\frac{1}{q}}$(if $1<p\leq\infty$) or $|x|>\frac{1}{\pi(b-a)}$ (if $p=1$) for $x\in[a,b]$, the Theorem 2.2 is an improvement of the Theorem 1.2.

Remark 2.3. We have

$$\mathcal{F}(g)(0)=\int_a^b g(t)\,dt$$

and for $x=0$ the left-hand side of the inequalities (2.1), (2.4) and (2.5) reduces to

$$\left|\mathcal{F}(g)(0)-E(0,0)\int_a^b g(s)\,ds\right|=0.$$

Remark 2.4. The inequality (2.5) can also be obtained from (2.1) since for g such that $g'\in L_1[a,b]$ it holds that

$$\bigvee_a^b(g)=\int_a^b|g'(t)|\,dt=\|g'\|_1.$$

3. ESTIMATES OF THE DIFFERENCE BETWEEN TWO FOURIER TRANSFORMS

Let weighted function $w:[a,b]\to\mathbb{R}$ be integrable such that $\int_a^b w(t)\,dt\neq 0$ and $W(x)=\int_a^x w(t)\,dt$, $x\in[a,b]$. Then weighted Montgomery identity states (see [5])

$$f(x)-\frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)\,w(t)\,dt=\int_a^b P_w(x,t)\,f'(t)\,dt \tag{3.1}$$

where $P_w(t,s)$ the weighted Peano kernel, is defined by

$$P_w(x,t)=\begin{cases}\frac{W(t)}{W(b)}, & a\leq s\leq x,\\ \frac{W(t)}{W(b)}-1, & x<s\leq b.\end{cases} \tag{3.2}$$

Lemma 3.1. *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]\cup[c,d]$, $w:[a,b]\to\mathbb{R}$ and $u:[c,d]\to\mathbb{R}$ some weighted functions, such that $\int_a^b w(t)\,dt\neq 0$, $\int_c^d u(t)\,dt\neq 0$ and*

$$W(x)=\begin{cases}0, & t<a,\\ \int_a^x w(t)\,dt, & a\leq t\leq b,\\ \int_a^b w(t)\,dt, & t>b,\end{cases}\qquad U(x)=\begin{cases}0, & t<c,\\ \int_c^x u(t)\,dt & c\leq t\leq d,\\ \int_c^d u(t)\,dt, & t>d,\end{cases}$$

and $[a,b] \cap [c,d] \neq \emptyset$*. Then, for both cases* $[c,d] \subseteq [a,b]$ *and* $[a,b] \cap [c,d] = [c,b]$*, (and also for* $[a,b] \subseteq [c,d]$ *and* $[a,b] \cap [c,d] = [a,d]$*) the next formula is valid*

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt - \frac{1}{\int_c^d u(t)\,dt}\int_c^d u(t) f(t)\,dt = \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) f'(t)\,dt \tag{3.3}$$

where

$$K(t) = P_u(x,t) - P_w(x,t),\ t \in [\min\{a,c\}, \max\{b,d\}]$$

and $P_u(x,t)$*,* $P_w(x,t)$ *are given by*

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \le s \le x, \\ \frac{W(t)}{W(b)} - 1, & x < s \le b, \end{cases} \quad , \quad P_u(x,t) = \begin{cases} \frac{U(t)}{U(b)}, & c \le s \le x, \\ \frac{U(t)}{U(b)} - 1, & x < s \le d. \end{cases}$$

Proof. For $x \in [a,b] \cap [c,d]$, we subtract identities

$$f(x) - \frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t) w(t)\,dt = \int_a^b P_w(x,t) f'(t)\,dt$$

and

$$f(x) - \frac{1}{\int_c^d u(t)\,dt}\int_c^d f(t) u(t)\,dt = \int_c^d P_u(x,t) f'(t)\,dt.$$

Then put

$$K(x,t) = P_u(x,t) - P_w(x,t),\ t \in [\min\{a,c\}, \max\{b,d\}].$$

$K(x,t)$ doesn't depend on x, so we put $K(t)$ instead:

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c], \\ -\frac{W(t)}{W(b)} + \frac{U(t)}{U(d)}, & t \in (c,d), \\ 1 - \frac{W(t)}{W(b)}, & t \in [d,b], \end{cases} \quad \text{if} \quad [c,d] \subseteq [a,b], \tag{3.4}$$

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c], \\ -\frac{W(t)}{W(b)} + \frac{U(t)}{U(d)}, & t \in (c,b), \\ \frac{U(t)}{U(d)} - 1, & t \in [b,d]. \end{cases} \quad \text{if} \quad [a,b] \cap [c,d] = [c,b]. \tag{3.5}$$

∎

Remark 3.1. In the special case for normalized weighted functions. i.e., when it holds that $\int_a^b w(t)\,dt = \int_c^d u(t)\,dt = 1$, this result coincides with the result from [1].

We can suppose that function $g:[a,b]\to\mathbb{R}$ is normalized, i.e. $\int_a^b g(t)\,dt=1$ (since we can always have $\frac{g(t)}{\int_a^b g(t)dt}$ instead of $g(t)$). In this way the results in the next theorem are simpler, but they hold also with $\frac{g(t)}{\int_a^b g(t)dt}$ instead of $g(t)$ and with $\frac{1}{\int_a^b g(t)dt}\mathcal{F}(g)(x)$ instead of $\mathcal{F}(g)(x)$ since

$$\mathcal{F}\left(\frac{g}{\int_a^b g(t)\,dt}\right)(x)=\frac{1}{\int_a^b g(t)\,dt}\mathcal{F}(g)(x).$$

Theorem 3.2. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $g,h:[a,b]\to\mathbb{R}$ be normalized, absolutely continuous function on $[a,b]$ and $G(t)=\int_a^t g(s)\,ds$, $H(t)=\int_a^t h(s)\,ds$, $G,H\in L_p[a,b]$. Then for $1<p\le\infty$, and for all $x\neq 0$ we have the inequality*

$$|\mathcal{F}(g)(x)-\mathcal{F}(h)(x)|\le 2\pi|x|(b-a)^{\frac{1}{q}}\|H(t)-G(t)\|_p, \tag{3.6}$$

while for $p=1$ we have

$$|\mathcal{F}(g)(x)-\mathcal{F}(h)(x)|\le 2\pi|x|\|H(t)-G(t)\|_1. \tag{3.7}$$

Proof. If we apply identity (3.3) with $a=c$, $b=d$, $w(t)=g(t)$, $u(t)=h(t)$ and $f(t)=e^{-2\pi ixt}$, then we obtain

$$\mathcal{F}(g)(x)-\mathcal{F}(h)(x)=(-2\pi ix)\int_a^b K(t)\,e^{-2\pi ixt}dt,$$

where $K(t)=H(t)-G(t)$. By taking the modulus and applying Hölder inequality for $1<p\le\infty$ we have

$$|\mathcal{F}(g)(x)-\mathcal{F}(h)(x)|\le 2\pi|x|\left\|e^{-2\pi ixt}\right\|_q\|K(t)\|_p,$$

and

$$\left\|e^{-2\pi ixt}\right\|_q=\left(\int_a^b\left|e^{-2\pi ixt}\right|^q dt\right)^{\frac{1}{q}}=(b-a)^{\frac{1}{q}},$$

$$\|K(t)\|_p=\left(\int_a^b|H(t)-G(t)|^p\,dt\right)^{\frac{1}{p}}$$

so the inequality (3.6) follows. Similarly, for $p=1$ we have

$$\left\|e^{-2\pi ixt}\right\|_\infty=\sup_{t\in[a,b]}\left\{\left|e^{-2\pi ixt}\right|\right\}=1,$$

and the proof is completed. ∎

Theorem 3.3. *Assume (p,q) is a pair of conjugate exponents. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous function on $[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $1<p\le\infty$, and for $x\neq 0$ we have the inequality*

$$\begin{aligned}&\left|\frac{d-c}{b-a}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt\right|\\ \le\ &(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\|g'\|_p,\end{aligned} \tag{3.8}$$

while for $p=1$ and $x\neq 0$ we have

$$\left|\frac{d-c}{b-a}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt\right|\le 2(d-c)\|g'\|_1, \tag{3.9}$$

where $E(z,w)$ *is given by (1.3).*

Proof. If we apply identity (3.3) with $f(t)=g(t)$, $w(t)=e^{-2\pi ixt}$, $u(t)=\frac{1}{d-c}$ then we have $W(t)=\int_a^t e^{-2\pi ixs}ds=(t-a)E(-2\pi ixa,-2\pi ixt)$,

$$\frac{1}{(b-a)E(-2\pi ixa,-2\pi ixb)}\mathcal{F}(g)(x)-\frac{1}{d-c}\int_c^d g(t)\,dt=\int_a^b K(t)\,g'(t)\,dt,$$

and, since $[c,d]\subseteq[a,b]$ by using (3.4)

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c],\\ -\frac{W(t)}{W(b)}+\frac{t-c}{d-c}, & t\in(c,d),\\ 1-\frac{W(t)}{W(b)}, & t\in[d,b].\end{cases}$$

Thus

$$\frac{d-c}{b-a}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt=\frac{d-c}{b-a}W(b)\int_a^b K(t)\,g'(t)\,dt$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt\right|\le\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q\left\|g'\right\|_p.$$

Now, for $1<p\le\infty$ (for $1\le q<\infty$) we have

$$\begin{aligned}\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q &= \left(\int_a^c\left|\frac{d-c}{b-a}W(t)\right|^q dt\right.\\ &\left.+\int_c^d\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|^q dt+\int_d^b\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|^q dt\right)\end{aligned}$$

and since $|W(t)|=\left|\int_a^t e^{-2\pi ixs}ds\right|\le\int_a^t\left|e^{-2\pi ixs}\right|ds=\int_a^t ds=t-a$ for $t\in[a,b]$, we have

$$\int_a^c\left|\frac{d-c}{b-a}W(t)\right|^q dt\le\int_a^c\left(\frac{d-c}{b-a}(t-a)\right)^q dt=\left(\frac{d-c}{b-a}\right)^q\frac{(c-a)^{q+1}}{(q+1)},$$

$$\begin{aligned}&\int_c^d\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|^q dt\\ \le\ &\int_c^d\left(\left|\frac{d-c}{b-a}W(t)\right|+\left|\frac{t-c}{b-a}W(b)\right|\right)^q dt\le\int_c^d\left(\frac{d-c}{b-a}(t-a)+t-c\right)^q dt\\ =\ &\frac{1}{(b-a)^q}\int_c^d\left((b-a+d-c)\,t-c\,(b-a)-a\,(d-c)\right)^q dt.\end{aligned}$$

If we denote

$$\lambda(t)=(b-a+d-c)\,t-c\,(b-a)-a\,(d-c) \tag{3.10}$$

we have $\lambda(c) = (d-c)(c-a)$ and $\lambda(d) = (d-c)(b+d-2a)$ so

$$\begin{aligned} &\frac{1}{(b-a)^q}\int_c^d ((b-a+d-c)t - c(b-a) - a(d-c))^q\,dt \\ = &\frac{\left(\lambda(d)^{q+1} - \lambda(c)^{q+1}\right)}{(b-a)^q(q+1)(b-a+d-c)} \\ = &\frac{(d-c)^{q+1}\left((b+d-2a)^{q+1} - (c-a)^{q+1}\right)}{(b-a)^q(q+1)(b-a+d-c)} \le \frac{2^q(d-c)^q(b-a)}{(q+1)}. \end{aligned}$$

Also

$$\begin{aligned} &\int_d^b \left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right|^q dt \\ \le &\int_d^b \left(\frac{d-c}{b-a}(b-t)\right)^q dt = \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}. \end{aligned}$$

Thus

$$\begin{aligned} &\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q \\ \le &\left(\left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q(d-c)^q(b-a)}{(q+1)} + \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}} \\ \le &\left(\left(\frac{d-c}{b-a}\right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q(d-c)^q(b-a)}{(q+1)}\right)^{\frac{1}{q}} = (d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}} \end{aligned}$$

and inequality (3.4) is proved. For $p = 1$ we have

$$\begin{aligned} \left\|\frac{d-c}{b-a}W(b)K(t)\right\|_\infty = \max\Bigg\{&\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right|, \\ &\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right|, \sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right|\Bigg\} \end{aligned}$$

and

$$\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right| \le \frac{(d-c)(c-a)}{(b-a)},$$

$$\begin{aligned} &\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right| \le \sup_{t\in[c,d]}\left\{\left|\frac{d-c}{b-a}W(t)\right| + \left|\frac{t-c}{b-a}W(b)\right|\right\} \\ = &\frac{d-c}{b-a}(d-a) + d - c = (d-c)\frac{b+d-2a}{b-a}, \end{aligned}$$

$$\sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right| \le \frac{(d-c)(b-d)}{(b-a)}.$$

Thus

$$\|U(d)K(t)\|_\infty \le \frac{d-c}{b-a}\max\{(c-a),(b+d-2a),(b-d)\} \le 2(d-c)$$

and the proof is completed. ∎

Theorem 3.4. *Assume (p,q) is a pair of conjugate exponents. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous function on $[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $1<p\le\infty$, and for $x\neq 0$ we have the inequality*

$$\left|\frac{d-c}{b-a}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt\right| \le (d-c)\frac{(b-a)^{\frac{1}{q}-1}}{\pi|x|}\|g'\|_p, \tag{3.11}$$

while for $p=1$ and $x\neq 0$ we have

$$\left|\frac{d-c}{b-a}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt\right|\le\frac{d-c}{(b-a)\pi|x|}\|g'\|_1, \tag{3.12}$$

where $E(z,w)$ is given by (1.3).

Proof. If we apply identity (3.3) with $f(t)=g(t)$, $w(t)=e^{-2\pi ixt}$, $u(t)=\frac{1}{d-c}$ again we have $W(t)=\int_a^t e^{-2\pi ixs}ds=(t-a)E(-2\pi ixa,-2\pi ixt)$,

$$\frac{1}{(b-a)E(-2\pi ixa,-2\pi ixb)}\mathcal{F}(g)(x)-\frac{1}{d-c}\int_c^d g(t)\,dt=\int_a^b K(t)g'(t)\,dt,$$

and, since $[c,d]\subseteq[a,b]$ by using (3.4)

$$K(t)=\begin{cases}-\frac{t-a}{b-a}\frac{E(-2\pi ixa,-2\pi ixt)}{E(-2\pi ixa,-2\pi ixb)}, & t\in[a,c],\\[2ex] -\frac{t-a}{b-a}\frac{E(-2\pi ixa,-2\pi ixt)}{E(-2\pi ixa,-2\pi ixb)}+\frac{t-c}{d-c}, & t\in(c,d),\\[2ex] 1-\frac{t-a}{b-a}\frac{E(-2\pi ixa,-2\pi ixt)}{E(-2\pi ixa,-2\pi ixb)}, & t\in[d,b].\end{cases}$$

Thus

$$\begin{aligned}&\frac{d-c}{(b-a)}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt\\ =\ &(d-c)E(-2\pi ixa,-2\pi ixb)\int_a^b K(t)g'(t)\,dt\end{aligned}$$

and by taking the modulus and applying Hölder inequality we obtain

$$\begin{aligned}&\left|\frac{d-c}{b-a}\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,dt\right|\\ \le\ &(d-c)\|E(-2\pi ixa,-2\pi ixb)K(t)\|_q\|g'\|_p.\end{aligned}$$

Now, for $1<p\le\infty$ (for $1\le q<\infty$) we have

$$\begin{aligned}\|E(-2\pi ixa,-2\pi ixb)K(t)\|_q=&\left(\int_a^c\left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)\right|^q dt\right.\\ &+\int_c^d\left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)-\frac{t-c}{d-c}E(-2\pi ixa,-2\pi ixb)\right|^q dt\\ &\left.+\int_d^b\left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)-E(-2\pi ixa,-2\pi ixb)\right|^q dt\right)\end{aligned}$$

and since $|E(-2\pi ixr,-2\pi ixs)| \leq \left|\frac{e^{-2\pi ixr}-e^{-2\pi ixs}}{-2\pi ix(r-s)}\right| \leq \frac{2}{2\pi|x||r-s|}$ we have

$$\begin{aligned}
&\int_a^c \left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)\right|^q dt \leq \int_a^c \left|\frac{1}{(b-a)\pi|x|}\right|^q dt \\
&= \frac{c-a}{((b-a)\pi|x|)^q},
\end{aligned}$$

$$\begin{aligned}
&\int_c^d \left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)-\frac{t-c}{d-c}E(-2\pi ixa,-2\pi ixb)\right|^q dt \\
&= \frac{1}{(2(b-a)\pi|x|)^q}\int_c^d \left|\frac{d-t}{d-c}e^{-2\pi ixa}+\frac{t-c}{d-c}e^{-2\pi ixb}-e^{-2\pi ixt}\right|^q dt \\
&\leq \frac{1}{(2(b-a)\pi|x|)^q}\int_c^d \left|\frac{d-t}{d-c}+\frac{t-c}{d-c}+1\right|^q dt \\
&= \frac{1}{(2(b-a)\pi|x|)^q}\int_c^d |2|^q\,dt = \frac{d-c}{((b-a)\pi|x|)^q},
\end{aligned}$$

$$\begin{aligned}
&\int_d^b \left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)-E(-2\pi ixa,-2\pi ixb)\right|^q dt \\
&= \frac{1}{(2(b-a)\pi|x|)^q}\int_d^b \left|e^{-2\pi ixb}-e^{-2\pi ixt}\right|^q dt \\
&\leq \frac{1}{(2(b-a)\pi|x|)^q}\int_d^b |2|^q\,dt = \frac{b-d}{((b-a)\pi|x|)^q}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|E(-2\pi ixa,-2\pi ixb)K(t)\|_q \leq \left(\frac{c-a+d-c+b-d}{((b-a)\pi|x|)^q}\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{q}-1}}{\pi|x|}
\end{aligned}$$

and inequality (3.4) is proved. For $p=1$ we have

$$\begin{aligned}
&\|E(-2\pi ixa,-2\pi ixb)K(t)\|_\infty = \max\left\{\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)\right|,\right. \\
&\sup_{t\in[c,d]}\left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)-\frac{t-c}{d-c}E(-2\pi ixa,-2\pi ixb)\right|, \\
&\left.\sup_{t\in[d,b]}\left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)-E(-2\pi ixa,-2\pi ixb)\right|\right\}
\end{aligned}$$

and

$$\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}E(-2\pi ixa,-2\pi ixt)\right| \leq \frac{1}{(b-a)\pi|x|},$$

$$\begin{aligned}
&\sup_{t\in[c,d]}\left|\frac{t-a}{b-a}E\left(-2\pi ixa,-2\pi ixt\right)-\frac{t-c}{d-c}E\left(-2\pi ixa,-2\pi ixb\right)\right| \\
=\ &\frac{1}{2\left(b-a\right)\pi\left|x\right|}\sup_{t\in[c,d]}\left|\frac{d-t}{d-c}e^{-2\pi ixa}+\frac{t-c}{d-c}e^{-2\pi ixb}-e^{-2\pi ixt}\right| \\
\leq\ &\frac{1}{2\left(b-a\right)\pi\left|x\right|}\sup_{t\in[c,d]}\left|\frac{d-t}{d-c}+\frac{t-c}{d-c}+1\right|=\frac{1}{\left(b-a\right)\pi\left|x\right|},
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[d,b]}\left|\frac{t-a}{b-a}E\left(-2\pi ixa,-2\pi ixt\right)-E\left(-2\pi ixa,-2\pi ixb\right)\right| \\
=\ &\frac{1}{2\left(b-a\right)\pi\left|x\right|}\sup_{t\in[d,b]}\left|e^{-2\pi ixb}-e^{-2\pi ixt}\right|\leq\frac{1}{\left(b-a\right)\pi\left|x\right|}.
\end{aligned}$$

Thus

$$\left\|E\left(-2\pi ixa,-2\pi ixb\right)K\left(t\right)\right\|_{\infty}\leq\frac{1}{\left(b-a\right)\pi\left|x\right|}$$

and the proof is completed. ∎

Theorem 3.5. *Assume (p,q) is a pair of conjugate exponents. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous function on $[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $1<p\leq\infty$, and for $x\neq0$ we have the inequality*

$$\begin{aligned}
&\left|\frac{d-c}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\int_a^b g\left(t\right)dt-\int_c^d e^{-2\pi ixt}g\left(t\right)dt\right| \\
&\leq\ \left(d-c\right)\left(\frac{\left(2^q+1\right)\left(b-a\right)}{\left(q+1\right)}\right)^{\frac{1}{q}}\left\|g'\right\|_p, \qquad (3.13)
\end{aligned}$$

while for $p=1$ and $x\neq0$ we have

$$\left|\frac{d-c}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\int_a^b g\left(t\right)dt-\int_c^d e^{-2\pi ixt}g\left(t\right)dt\right|\leq2\left(d-c\right)\left\|g'\right\|_1, \qquad (3.14)$$

where $E(z,w)$ is given by (1.3).

Proof. If we apply identity (3.3) with $f(t)=g(t)$, $w(t)=\frac{1}{b-a}$, $u(t)=e^{-2\pi ixt}$ we have $U(t)=\int_c^t e^{-2\pi ixs}ds=(t-c)E(-2\pi ixc,-2\pi ixt)$,

$$\begin{aligned}
&\frac{1}{\left(b-a\right)}\int_a^b g\left(t\right)dt-\frac{1}{\left(d-c\right)E\left(-2\pi ixc,-2\pi ixd\right)}\int_c^d e^{-2\pi ixt}g\left(t\right)dt \\
=\ &\int_a^b K\left(t\right)g'\left(t\right)dt,
\end{aligned}$$

and, since $[c,d]\subseteq[a,b]$ by using (3.4)

$$K\left(t\right)=\begin{cases}-\frac{t-a}{b-a}, & t\in[a,c]\,, \\ \frac{U(t)}{U(d)}-\frac{t-a}{b-a}, & t\in(c,d)\,, \\ \frac{b-t}{b-a}, & t\in[d,b]\,.\end{cases}$$

Thus

$$\begin{aligned} & \frac{d-c}{b-a} E(-2\pi ixc, -2\pi ixd) \int_a^b g(t)\,dt - \int_c^d e^{-2\pi ixt} g(t)\,dt \\ = \;& U(d) \int_a^b K(t)\, g'(t)\,dt \end{aligned}$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left| \frac{d-c}{b-a} E(-2\pi ixc, -2\pi ixd) \int_a^b g(t)\,dt - \int_c^d e^{-2\pi ixt} g(t)\,dt \right| \le \|U(d)\,K(t)\|_q \, \|g'\|_p \,.$$

Now, for $1 < p \le \infty$ (for $1 \le q < \infty$) we have

$$\begin{aligned} \|U(d)\,K(t)\|_q = & \left(\int_a^c \left| \frac{t-a}{b-a} U(d) \right|^q dt \right. \\ & \left. + \int_c^d \left| U(t) - \frac{t-a}{b-a} U(d) \right|^q dt + \int_d^b \left| \frac{b-t}{b-a} U(d) \right|^q dt \right) \end{aligned}$$

and since $|U(t)| = \left| \int_c^t e^{-2\pi ixs} ds \right| \le \int_c^t |e^{-2\pi ixs}|\, ds = \int_c^t ds = t - c$ for $t \in [c,d]$, we have

$$\int_a^c \left| \frac{t-a}{b-a} U(d) \right|^q dt \le \int_a^c \left(\frac{t-a}{b-a} (d-c) \right)^q dt = \left(\frac{d-c}{b-a} \right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\begin{aligned} & \int_c^d \left| U(t) - \frac{t-a}{b-a} U(d) \right|^q dt \le \int_c^d \left(|U(t)| + \left| \frac{t-a}{b-a} U(d) \right| \right)^q dt \\ \le \;& \int_c^d \left(t - c + \frac{d-c}{b-a} (t-a) \right)^q dt \\ \le \;& \frac{1}{(b-a)^q} \int_c^d ((b-a+d-c)\,t - c\,(b-a) - a\,(d-c))^q\, dt \\ = \;& \frac{\left(\lambda(d)^{q+1} - \lambda(c)^{q+1} \right)}{(b-a)^q (q+1)(b-a+d-c)} \\ = \;& \frac{(d-c)^{q+1} \left((b+d-2a)^{q+1} - (c-a)^{q+1} \right)}{(b-a)^q (q+1)(b-a+d-c)} \le \frac{2^q (d-c)^q (b-a)}{(q+1)}, \end{aligned}$$

where $\lambda(t)$ is given by (3.10) and

$$\int_d^b \left| \frac{b-t}{b-a} U(d) \right|^q dt \le \int_d^b \left(\frac{b-t}{b-a} (d-c) \right)^q dt = \left(\frac{d-c}{b-a} \right)^q \frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\begin{aligned} & \|U(d)\,K(t)\|_q \\ \le \;& \left(\left(\frac{d-c}{b-a} \right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} + \left(\frac{d-c}{b-a} \right)^q \frac{(b-d)^{q+1}}{(q+1)} \right)^{\frac{1}{q}} \\ \le \;& \left(\left(\frac{d-c}{b-a} \right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} \right)^{\frac{1}{q}} = (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \end{aligned}$$

and inequality (3.4) is proved. For $p = 1$ we have

$$\begin{aligned} & \|U(d) K(t)\|_\infty \\ = & \max\left\{\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}U(d)\right|, \sup_{t\in[c,d]}\left|U(t)-\frac{t-a}{b-a}U(d)\right|, \sup_{t\in[d,b]}\left|\frac{b-t}{b-a}U(d)\right|\right\}\end{aligned}$$

and

$$\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}U(d)\right| \le \frac{(c-a)(d-c)}{(b-a)},$$

$$\begin{aligned} \sup_{t\in[c,d]}\left|U(t)-\frac{t-a}{b-a}U(d)\right| &= \sup_{t\in[c,d]}\left\{|U(t)|+\left|\frac{t-a}{b-a}U(d)\right|\right\} \\ \le \sup_{t\in[c,d]}\left|d-c+\frac{d-a}{b-a}(d-c)\right| &= (d-c)\frac{b+d-2a}{b-a}, \end{aligned}$$

$$\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}U(d)\right| \le \frac{(b-d)(d-c)}{(b-a)}.$$

Thus

$$\|U(d) K(t)\|_\infty \le \frac{d-c}{b-a}\max\{(c-a),(b+d-2a),(b-d)\} \le 2(d-c)$$

and the proof is completed. ∎

Theorem 3.6. *Assume (p,q) is a pair of conjugate exponents. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous function on $[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $1<p\le\infty$, and for $x\ne 0$ we have the inequality*

$$\left|\frac{d-c}{b-a}E(-2\pi ixc,-2\pi ixd)\int_a^b g(t)\,dt-\int_c^d e^{-2\pi ixt}g(t)\,dt\right| \le \frac{(b-a)^{\frac{1}{q}}}{\pi|x|}\|g'\|_p, \tag{3.15}$$

while for $p=1$ and $x\ne 0$ we have

$$\left|\frac{d-c}{b-a}E(-2\pi ixc,-2\pi ixd)\int_a^b g(t)\,dt-\int_c^d e^{-2\pi ixt}g(t)\,dt\right| \le \frac{1}{\pi|x|}\|g'\|_1, \tag{3.16}$$

where $E(z,w)$ is given by (1.3).

Proof. We apply identity (3.3) again with $f(t)=g(t)$, $w(t)=\frac{1}{b-a}$, $u(t)=e^{-2\pi ixt}$, so have $U(t)=\int_c^t e^{-2\pi ixs}ds=(t-c)E(-2\pi ixc,-2\pi ixt)$,

$$\begin{aligned} & \frac{1}{(b-a)}\int_a^b g(t)\,dt-\frac{1}{(d-c)E(-2\pi ixc,-2\pi ixd)}\int_c^d e^{-2\pi ixt}g(t)\,dt \\ = & \int_a^b K(t)\,g'(t)\,dt, \end{aligned}$$

and, since $[c,d]\subseteq[a,b]$ by using (3.4)

$$K(t)=\begin{cases} -\frac{t-a}{b-a}, & t\in[a,c], \\ \frac{t-c}{d-c}\frac{E(-2\pi ixc,-2\pi ixt)}{E(-2\pi ixc,-2\pi ixd)}-\frac{t-a}{b-a}, & t\in(c,d), \\ \frac{b-t}{b-a}, & t\in[d,b]. \end{cases}$$

Thus

$$\begin{aligned}&\frac{d-c}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\int_a^b g\left(t\right)dt-\int_c^d e^{-2\pi ixt}g\left(t\right)dt\\=\;&\left(d-c\right)E\left(-2\pi ixc,-2\pi ixd\right)\int_a^b K\left(t\right)g'\left(t\right)dt\end{aligned}$$

and by taking the modulus and applying Hölder inequality we obtain

$$\begin{aligned}&\left|\frac{d-c}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\int_a^b g\left(t\right)dt-\int_c^d e^{-2\pi ixt}g\left(t\right)dt\right|\\\leq\;&\left(d-c\right)\left\|E\left(-2\pi ixc,-2\pi ixd\right)K\left(t\right)\right\|_q\left\|g'\right\|_p.\end{aligned}$$

Now, for $1<p\leq\infty$ (for $1\leq q<\infty$) we have

$$\begin{aligned}&\left\|E\left(-2\pi ixc,-2\pi ixd\right)K\left(t\right)\right\|_q=\left(\int_a^c\left|\frac{t-a}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\right|^q dt\right.\\&+\int_c^d\left|\frac{t-c}{d-c}E\left(-2\pi ixc,-2\pi ixt\right)-\frac{t-a}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\right|^q dt\\&\left.+\int_d^b\left|\frac{b-t}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\right|^q dt\right)\end{aligned}$$

and since $\left|E\left(-2\pi ixr,-2\pi ixs\right)\right|\leq\left|\frac{e^{-2\pi ixr}-e^{-2\pi ixs}}{-2\pi ix(r-s)}\right|\leq\frac{2}{2\pi|x||r-s|}$ we have

$$\begin{aligned}&\int_a^c\left|\frac{t-a}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\right|^q dt=\left|E\left(-2\pi ixc,-2\pi ixd\right)\right|^q\int_a^c\left(\frac{t-a}{b-a}\right)^q dt\\=\;&\left|E\left(-2\pi ixc,-2\pi ixd\right)\right|^q\frac{\left(c-a\right)^{q+1}}{\left(q+1\right)\left(b-a\right)^q}\leq\frac{\left(c-a\right)^{q+1}}{q+1}\frac{1}{\left(\pi\left|x\right|\left(d-c\right)\left(b-a\right)\right)^q},\end{aligned}$$

$$\begin{aligned}&\int_c^d\left|\frac{t-c}{d-c}E\left(-2\pi ixc,-2\pi ixt\right)-\frac{t-a}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\right|^q dt\\=\;&\frac{1}{\left(2\left(d-c\right)\pi\left|x\right|\right)^q}\int_c^d\left|\frac{b-t}{b-a}e^{-2\pi ixc}+\frac{t-a}{b-a}e^{-2\pi ixd}-e^{-2\pi ixt}\right|^q dt\\\leq\;&\frac{1}{\left(2\left(d-c\right)\pi\left|x\right|\right)^q}\int_c^d\left|\frac{b-t}{b-a}+\frac{t-a}{b-a}+1\right|^q dt\\=\;&\frac{1}{\left(2\left(d-c\right)\pi\left|x\right|\right)^q}\int_c^d\left|2\right|^q dt=\frac{d-c}{\left(\left(d-c\right)\pi\left|x\right|\right)^q},\end{aligned}$$

$$\begin{aligned}&\int_d^b\left|\frac{b-t}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\right|^q dt=\left|E\left(-2\pi ixc,-2\pi ixd\right)\right|^q\int_d^b\left(\frac{b-t}{b-a}\right)^q dt\\=\;&\left|E\left(-2\pi ixc,-2\pi ixd\right)\right|^q\frac{\left(b-d\right)^{q+1}}{\left(q+1\right)\left(b-a\right)^q}\leq\frac{\left(b-d\right)^{q+1}}{q+1}\frac{1}{\left(\pi\left|x\right|\left(d-c\right)\left(b-a\right)\right)^q}.\end{aligned}$$

Thus

$$\begin{aligned}&\left\|E\left(-2\pi ixc,-2\pi ixd\right)K\left(t\right)\right\|_q\\\leq\;&\left(\frac{\frac{(c-a)^{q+1}}{q+1}+\left(d-c\right)\left(b-a\right)^q+\frac{(b-d)^{q+1}}{q+1}}{\left(\left(d-c\right)\left(b-a\right)\pi\left|x\right|\right)^q}\right)^{\frac{1}{q}}\leq\frac{\left(b-a\right)^{\frac{1}{q}}}{\left(d-c\right)\pi\left|x\right|}\end{aligned}$$

and inequality (3.4) is proved. For $p = 1$ we have

$$\|E(-2\pi ixc, -2\pi ixd) K(t)\|_\infty = \max\left\{ \sup_{t\in[a,c]} \left| \frac{t-a}{b-a} E(-2\pi ixc, -2\pi ixd) \right|, \right.$$

$$\sup_{t\in[c,d]} \left| \frac{t-c}{d-c} E(-2\pi ixc, -2\pi ixt) - \frac{t-a}{b-a} E(-2\pi ixc, -2\pi ixd) \right|,$$

$$\left. \sup_{t\in[d,b]} \left| \frac{b-t}{b-a} E(-2\pi ixc, -2\pi ixd) \right| \right\}$$

and

$$\sup_{t\in[a,c]} \left| \frac{t-a}{b-a} E(-2\pi ixc, -2\pi ixd) \right| \le \frac{c-a}{(d-c)(b-a)\pi|x|},$$

$$\sup_{t\in[c,d]} \left| \frac{t-c}{d-c} E(-2\pi ixc, -2\pi ixt) - \frac{t-a}{b-a} E(-2\pi ixc, -2\pi ixd) \right|$$

$$= \frac{1}{2(d-c)\pi|x|} \sup_{t\in[c,d]} \left| \frac{b-t}{b-a} e^{-2\pi ixc} + \frac{t-a}{b-a} e^{-2\pi ixd} - e^{-2\pi ixt} \right|$$

$$\le \frac{1}{2(d-c)\pi|x|} \sup_{t\in[c,d]} \left| \frac{d-t}{d-c} + \frac{t-c}{d-c} + 1 \right| = \frac{1}{(d-c)\pi|x|},$$

$$\sup_{t\in[d,b]} \left| \frac{b-t}{b-a} E(-2\pi ixc, -2\pi ixd) \right| \le \frac{b-d}{(d-c)(b-a)\pi|x|}.$$

Thus

$$\|E(-2\pi ixc, -2\pi ixd) K(t)\|_\infty \le \frac{\max\{(c-a),(b-a),(b-d)\}}{(d-c)(b-a)\pi|x|}$$

$$= \frac{1}{(d-c)\pi|x|}$$

and the proof is completed. ∎

Corollary 3.7. *Assume* (p, q) *is a pair of conjugate exponents. Let* $g : [a, b] \to \mathbb{R}$ *be absolutely continuous function on* $[a, b]$. *Then for* $1 < p \le \infty$, *and for all* $x \neq 0$ *we have the inequality*

$$\left| E(-2\pi ixa, -2\pi ixb) \int_a^b g(t)\,dt - \int_a^b e^{-2\pi ixt} g(t)\,dt \right| \tag{3.17}$$
$$\le 2(b-a)^{1+\frac{1}{q}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \|g'\|_p,$$

while for $p = 1$ *we have*

$$\left| E(-2\pi ixa, -2\pi ixb) \int_a^b g(t)\,dt - \int_a^b e^{-2\pi ixt} g(t)\,dt \right| \le 2(b-a)\|g'\|_1. \tag{3.18}$$

Proof. By applying the proof of the Theorems 3.3 or 3.5 in the special case when $c = a$ and $d = b$. ∎

Corollary 3.8. *Assume* (p, q) *is a pair of conjugate exponents. Let* $g : [a, b] \to \mathbb{R}$ *be absolutely continuous function on* $[a, b]$. *Then for* $1 < p \le \infty$, *and for all* $x \neq 0$ *we have the inequality*

$$\left| E(-2\pi ixa, -2\pi ixb) \int_a^b g(t)\,dt - \int_a^b e^{-2\pi ixt} g(t)\,dt \right| \le \frac{(b-a)^{\frac{1}{q}}}{\pi|x|} \|g'\|_p, \tag{3.19}$$

while for $p=1$ *we have*

$$\left|E\left(-2\pi ixa,-2\pi ixb\right)\int_a^b g\left(t\right)dt-\int_a^b e^{-2\pi ixt}g\left(t\right)dt\right|\leq\frac{1}{\pi\left|x\right|}\left\|g'\right\|_1. \tag{3.20}$$

Proof. By applying Theorems 3.4 and 3.6 with $c=a$ and $d=b$, the proof follows directly. ∎

4. A NUMERICAL QUADRATURE FORMULA

Let $I_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$, $h_k:=x_{k+1}-x_k,\ k=0,1,\ldots,n-1$ and $\nu\left(h\right):=\max_k\left\{h_k\right\}$. Following [2], define the sum

$$\mathcal{E}\left(g,I_n,x\right)=\sum_{k=0}^{n-1}E\left(-2\pi ixx_k,-2\pi ixx_{k+1}\right)\int_{x_k}^{x_{k+1}}g\left(t\right)dt \tag{4.1}$$

where $x\neq0$.

The following approximation theorem holds.

Theorem 4.1. *Assume* (p,q) *is a pair of conjugate exponents. Let* $g:[a,b]\to\mathbb{R}$ *be absolutely continuous function on* $[a,b]$. *Then we have the quadrature rule*

$$\mathcal{F}\left(g\right)\left(x\right)=\mathcal{E}\left(g,I_n,x\right)+R\left(g,I_n,x\right)$$

where $x\neq0$, $\mathcal{E}\left(g,I_n,x\right)$ *is given by (4.1) and for* $1<p\leq\infty$ *the reminder* $R\left(g,I_n,x\right)$ *satisfies the estimate*

$$\left|R\left(g,I_n,x\right)\right|\leq2\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}h_k^{q+1}\right]^{\frac{1}{q}}\left\|g'\right\|_p, \tag{4.2}$$

while for $p=1$ *we have*

$$\left|R\left(g,I_n,x\right)\right|\leq2\nu\left(h\right)\left\|g'\right\|_1 \tag{4.3}$$

Proof. For $1<p\leq\infty$ by applying the Corollary 3.7 with $a=x_k$, $b=x_{k+1}$ we have

$$\begin{aligned}&\left|E\left(-2\pi ixx_k,-2\pi ixx_{k+1}\right)\int_{x_k}^{x_{k+1}}g\left(t\right)dt-\int_{x_k}^{x_{k+1}}e^{-2\pi ixt}g\left(t\right)dt\right|\\ \leq\ &2\left(x_{k+1}-x_k\right)^{1+\frac{1}{q}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\int_{x_k}^{x_{k+1}}\left|g'\left(t\right)\right|^pdt\right)^{\frac{1}{p}}.\end{aligned}$$

Summing over k from 0 to $n-1$ and using generalized triangle inequality, we obtain

$$\begin{aligned}\left|R\left(g,I_n,x\right)\right|&=\left|\mathcal{F}\left(g\right)\left(x\right)-\mathcal{E}\left(g,I_n,x\right)\right|\\&\leq\sum_{k=0}^{n-1}2\left(h_k\right)^{1+\frac{1}{q}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\int_{x_k}^{x_{k+1}}\left|g'\left(t\right)\right|^pdt\right)^{\frac{1}{p}}.\end{aligned}$$

Using the Hölder discrete inequality, we get

$$\begin{aligned}&2\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\sum_{k=0}^{n-1}\left(h_k\right)^{1+\frac{1}{q}}\left(\int_{x_k}^{x_{k+1}}\left|g'\left(t\right)\right|^pdt\right)^{\frac{1}{p}}\\ \leq\ &2\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}\left(\left(h_k\right)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}\left(\left(\left(\int_{x_k}^{x_{k+1}}\left|g'\left(t\right)\right|^pdt\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}\\ =\ &2\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}h_k^{q+1}\right]^{\frac{1}{q}}\left\|g'\right\|_p\end{aligned}$$

and the inequality (4.2) is proved. For $p=1$ we have

$$\begin{aligned} |R(g,I_n,x)| &\leq \sum_{k=0}^{n-1} 2h_k \left(\int_{x_k}^{x_{k+1}} |g'(t)|\, dt\right) \\ &\leq 2\nu(h) \sum_{k=0}^{n-1} \left(\int_{x_k}^{x_{k+1}} |g'(t)|\, dt\right) = 2\nu(h) \|g'\|_1 \end{aligned}$$

and the proof is completed. ∎

Corollary 4.2. *Suppose that all assumptions of Theorem 4.1 hold. Additionally suppose* [2]

$$\begin{aligned} \mathcal{E}(g,I_n,x) &= \\ &\int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}} g(t)\, dt \\ &\cdot \sum_{k=0}^{n-1} E\left(-2\pi i x\left(a+k\cdot\frac{b-a}{n}\right), -2\pi i x\left(a+(k+1)\cdot\frac{b-a}{n}\right)\right). \end{aligned} \tag{4.4}$$

Then we have the quadrature rule

$$\mathcal{F}(g)(x) = \mathcal{E}(g,I_n,x) + R(g,I_n,x)$$

where $x \neq 0$ and for $1 < p \leq \infty$ the reminder $R(g,I_n,x)$ satisfies the estimate

$$|R(g,I_n,x)| \leq 2\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|g'\|_p, \tag{4.5}$$

while for $p=1$ we have

$$|R(g,I_n,x)| \leq \frac{2(b-a)}{n} \|g'\|_1. \tag{4.6}$$

Proof. If we apply Theorem 4.1 with $x_j = a + j\cdot\frac{b-a}{n}$, $j=0,1,\ldots,n$ (equidistant partition of $[a,b]$) we have (4.4) and $h_k = \frac{b-a}{n}$, $k=0,1,\ldots,n-1$. For $1<p\leq\infty$ we obtain

$$\begin{aligned} |R(g,I_n,x)| &\leq 2\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \|g'\|_p \\ &= 2\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|g'\|_p, \end{aligned}$$

while for $p=1$, $\nu(h) = \frac{b-a}{n}$ and the claim immediately follows. ∎

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