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AN ALGORITHM TO COMPUTE GAUSSIAN-TYPE QUADRATURE FORMULAE THAT INTEGRATE POLYNOMIALS AND SOME SPLINE FUNCTIONS EXACTLY

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ABSTRACT. It is well-known that for sufficiently smooth integrands on an interval, numerical integration can be performed stably and efficiently via the classical (polynomial) Gauss quadrature formulae. However, for many other sets of integrands these quadrature formulae do not perform well. A very natural way of avoiding this problem is to include a wide class among arbitrary functions (not necessary polynomials) to be integrated exactly. The spline functions are natural candidates for such problems. In this paper, after studying Gaussian type quadrature formulae which are exact for *spline* functions and which contain boundary terms involving derivatives at both end points, we present a fast algorithm for computing their nodes and weights. It is also shown, taking advantage of the close connection with ordinary Gauss quadrature formula, that the latter are computed, via eigenvalues and eigenvectors of real symmetric tridiagonal matrices. Hence a new class of quadrature formulae can then be computed directly by standard software for ordinary Gauss quadrature formula. Comparative results with classical Gauss quadrature formulae are given to illustrate the numerical performance of the approach.

Key words and phrases: Quadrature formulae, Gaussian quadrature formulae, Splines, Quasi-orthogonal polynomials, Three-term relation, Algorithms.

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1. INTRODUCTION AND MOTIVATIONS

Many commonly quadrature formulae consist of approximating the integrand by a polynomial and then integrating this polynomial exactly. For integrand functions that are better approximated by polynomials, the classical (polynomial) Gaussian quadrature formulae (i. e., those which integrate polynomials of maximum possible degree) have traditionally been more finding a stable, reliable and, more importantly, inexpensive procedure for their construction is still open. Recently, Ma, Rokhlin and Wandzura [19] have presented a numerical algorithm for the construction of Gaussian quadrature formulae for systems of arbitrary functions, but their method, which requires the solution of large nonlinear systems, lacks the rapid convergence and elegance of the polynomial case.

Spline functions are piecewise polynomials which satisfy certain continuity constraints at the joints (called the 'nodes' of the spline). They can be represented in a compact way using the so-called B-splines. An advantage of spline quadrature formulae over classical polynomials methods is that they make it possible to use irregular (adaptively derived) refinement of the mesh, which allows to branch more points in places where the integrand is not smooth and use less points where it is. They often help to find a fast and accurate evaluation, differentiation or integration, see [6] and [28].

This paper studies quadrature formulae which have a *maximum degree of exactness* (abbr. MDE) and that integrate many monomials and some spline functions exactly, it presents a rapid algorithm for their computations. This approach does not seem to have been used *numerically* in the literature. An exception is [24], where the simplest cases of ordinary splines of degree 1 with arbitrary nodes and degree 2 for the case of equidistant nodes are considered. For the general case, surprisingly no efficient algorithm is known to compute such quadrature formulae.

Let n and r be positive integers, $\mathbf{M} = (m_1, ..., m_r)$ a vector of integers with $1 \leq m_i \leq n$, $i = 1, ..., r$, and $\mathbf{\Delta} = (\zeta_0, ..., \zeta_{r+1})$ a sequence of nodes with

$$-1 = \zeta_0 < \zeta_1 < ... < \zeta_r < \zeta_{r+1} = 1.$$

Let $\mathcal{S}(\mathcal{P}_{n-1}; \mathbf{\Delta}; \mathbf{M})$ be the linear space of spline functions of degree $n-1$ $(n \geq 1)$ with nodes $\zeta_1, ..., \zeta_r$ having multiplicities $m_1, ..., m_r$ respectively. That is, every $S \in \mathcal{S}(\mathcal{P}_{n-1}; \mathbf{\Delta}; \mathbf{M})$ is a polynomial with real coefficients and of degree $n-1$ in each of intervals

$$(-\infty, \zeta_0), [\zeta_0, \zeta_1), ..., [\zeta_{r+1}, \infty),$$

and S belongs to $\mathcal{C}^{n-m_i-1}$ in a neighborhood of ζ_i but not in $\mathcal{C}^{n-m_i}$. It follows that any $S \in \mathcal{S}(\mathcal{P}_{n-1}; \mathbf{\Delta}; \mathbf{M})$ has a representation of the form

$$S(t) = \sum_{i=0}^{n-1} a_i t^i + \sum_{i=1}^{r} \sum_{j=1}^{m_i} b_{ij} (t - \zeta_i)_+^{n-j}$$

where all the coefficients a_i and b_{ij} are real numbers and $(t - \zeta_i)_+^{n-j}$ is the truncated power defined as $(t - \zeta_i)^{n-j}$ if $t \geq \zeta_i$, and zero otherwise.

Let $n, k, r, s, m_1, m_2, ..., m_r$ be given such that $n \geq 1$ and

$$2s := n + \sum_{i=1}^{r} m_i, \tag{1.1}$$

it is well known that, for fixed $\zeta_1, ..., \zeta_r$, the linear space $\mathcal{S}(\mathcal{P}_{n-1}; \mathbf{\Delta}; \mathbf{M})$ is of dimension $2s$ (see, for example, [16]).

In this paper, we propose to compute the nodes $x_{i,s} = x_{i,s}(d\sigma;\mathbf{\Delta};\mathbf{M})$, the weights $\lambda_{i,s} = \lambda_{i,s}(d\sigma;\mathbf{\Delta};\mathbf{M})$ and $\omega_{j,s} = \omega_{j,s}(d\sigma;\mathbf{\Delta};\mathbf{M})$ (provided they exist) of the general quadrature formula

$$\begin{aligned} \int_{-1}^{1} f(t)\,d\sigma &= Q_{k,s}(f;d\sigma;\mathbf{\Delta};\mathbf{M}) + R_{k,s}(f;d\sigma;\mathbf{\Delta};\mathbf{M}) \\ &= \sum_{i=1}^{s} \lambda_{i,s} f(x_{i,s}) + \sum_{j=1}^{k} \omega_{j,s} C_j(f) + R_{k,s}(f;d\sigma;\mathbf{\Delta};\mathbf{M}), \end{aligned} \tag{1.2}$$

which integrates each element of the spline space $\mathcal{S}(\mathcal{P}_{n+k-1};\mathbf{\Delta};\mathbf{M})$ exactly, that is,

$$\int_{-1}^{1} f(t)\,d\sigma = \sum_{i=1}^{s} \lambda_{i,s} f(x_{i,s}) + \sum_{j=1}^{k} \omega_{j,s} C_j(f), \qquad \forall f \in \mathcal{S}(\mathcal{P}_{n+k-1};\mathbf{\Delta};\mathbf{M}).$$

Here and subsequently, C_l, $l = 1,...,k$, are given linear functionals of the form

$$C_l(f) = \sum_{m=0}^{q_l-1} a_{lm} f^{(m)}(-1) + \sum_{p=0}^{q_l'-1} b_{lp} f^{(p)}(1), \qquad l = 1,...,k. \tag{1.3}$$

In analogy with some well-known polynomial case [11], we call these functionals *generalized Gauss-Lobatto-Birkhoff* quadrature formulae for spline functions. Indeed, since $2s = n + \sum_{i=1}^{r} m_i$, the MDE$(Q_{k,s})$ is equal to the number of "free" parameters appearing on $Q_{k,s}$, and that in the limit case $r = 0$ (i. e. ordinary polynomial case) choosing appropriately the functional C_j we obtain the well-known Gauss quadrature formula, as well as the quadrature formulae usually associated with the names of Radau, Lobatto and Birkhoff. For the general *polynomial* case, details of existence, uniqueness, and a more general discussion, including theoretical results, applications and numerical approximations of (1.2), may be found in [9], and in fact that work is the source of the present work.

The main contribution of this paper is a procedure for computing specifically the nodes $x_{i,s}$ and the weights $\lambda_{i,s}$ and $\omega_{j,s}$ of (1.2) The great advantage of the new algorithm lies in the fact that it offers an efficient way of reducing the computation of (1.2) to the well-studied problem of computing ordinary Gaussian quadrature formulae from recurrence coefficients, and can therefore be brought into the realm of stable modern methods of constructing orthogonal polynomials; see Gautschi [11]. In fact, we shall show, under additional assumptions on the functionals C_j which guarantee existence of $Q_{k,s}$, that all *nodes* and *weights* of (1.2) can again be computed as eigenvalues and first component of the orthonormalized eigenvectors of specific real symmetric tridiagonal matrices. Also, our approach is conceptually simple and more amenable than the method given in [19] and leads to considerable savings in computational time, since our construction makes no appeal to solving large nonlinear systems. Finally, comparisons of the new quadrature formulae with the ordinary Gauss quadrature formulae indicate that the former are much better than the later.

The remainder of this paper is organized as follows. In the next section, we state and prove, under certain assumptions about boundary conditions, existence and uniqueness of (1.2). We also develop some of their properties. The main results given in section 3 show how such quadrature formulae can be constructed, and in section 4, examples of such quadrature formulae are given. section 5 presents results of numerical experiments involving polynomial $\mathcal{C}^0-$spline Gauss quadrature formulae. Finally, in section 6 we summarize this research and discuss possible future work.

Throughout this paper, we take the integration interval $[-1,1]$ as a matter of convenience and we assume that $\mathbf{\Delta}$, $\mathbf{M}$ and the measure σ are fixed. Therefore, for the sake of simplicity, we shall freely omit $\mathbf{\Delta}$, $\mathbf{M}$ and $d\sigma$ from our notation.

2. EXISTENCE OF THE QUADRATURES AND SOME OF THEIR PROPERTIES

In this section, with additional assumptions on the functionals C_l, we show that there exists one and only one quadrature formula of type (1.2). This result was originally proved by Micchelli and Pinkus [21, Theorem 3.1] in their development of the general theory of the moment theory; however the proof given there is nonconstructive and uses this theory as the primary tool. In fact, we show that the existence and uniqueness of (1.2) is an immediate consequence of a recent result of Bojanov, Grozev and Zhensykbaev [4]. We also show that (1.2) possesses most of the desirable properties of the classical polynomial Gaussian quadrature formulae, for example the free nodes are all located in the support of the measure, the interior nodes have the interlacing property and importantly the weights $\lambda_{i,s}$ are all positive. These, as is known, are important properties that numerical quadrature formulae required to have. We close this section with a result concerning the remainder of the quadrature formula (1.2).

We first introduce some basis notations that will be used in the subsequent sections. Let the function f in (1.2) be differentiable on $[-1,1]$ as many times as needed. For given linear functionals

$$C_l(f) = \sum_{m=0}^{q_l-1} a_{lm} f^{(m)}(-1) + \sum_{p=0}^{q_l'-1} b_{lp} f^{(p)}(1), \qquad l = 1, ..., k,$$

we shall denote by $\mathcal{T}_k(\mathcal{P}_{N+k-1};\mathbf{\Delta};\mathbf{M})$ the subspace defined by

$$\mathcal{T}_k(P_{N+k-1};\mathbf{\Delta};\mathbf{M}) = \left\{S \in \mathcal{S}(\mathcal{P}_{N+k-1};\mathbf{\Delta};\mathbf{M}),\ C_l(S) = 0,\ l = 1, ..., k\right\}.$$

We next state some hypotheses which we shall require to hold for the approximation subspace $\mathcal{T}_k(P_{N+k-1};\mathbf{\Delta};\mathbf{M})$. We assume that (see [21])

(2.1) $$\mathcal{T}_k(\mathcal{P}_{N+k-1};\mathbf{\Delta};\mathbf{M}) \text{ is a weak Chebyshev system of dimension } N + \sum_{i=1}^{r} m_i.$$

Let $\{u_j\}_{j=1}^{N+\sum_{l=1}^r m_l}$ be a basis for $\mathcal{T}_k(\mathcal{P}_{N+k-1};\mathbf{\Delta};\mathbf{M})$. We also assume that for every integer N the set of the linear functionals $\{C_l\}_{l=1}^{k}$ is independent over $\mathcal{T}_k(\mathcal{P}_{N+k-1};\mathbf{\Delta};\mathbf{M})$, that is

(2.2) $$\text{rank } \|C_i(u_j)\|_{i=1,\, j=1}^{k,\, N+\sum_{l=1}^r m_l} = k,$$

Recall that a linear subspace $\mathcal{U}_{m-1}$ of $C[-1,1]$ of dimension m spanned by the functions $u_1, ..., u_m$ is called a *Chebyshev system* on $[-1,1]$ if

$$\begin{vmatrix} u_1(t_1) & ... & u_1(t_m) \\ \vdots & & \vdots \\ u_m(t_1) & ... & u_m(t_m) \end{vmatrix} > 0$$

for all $-1 \le t_1 < ... < t_m \le 1$.

The set $\mathcal{U}_{m-1}$ is called a *weak Chebyshev system* on $[-1,1]$ if for all points $-1 \le t_1 < ... < t_m \le 1$,

$$\begin{vmatrix} u_1(t_1) & ... & u_1(t_m) \\ \vdots & & \vdots \\ u_m(t_1) & ... & u_m(t_m) \end{vmatrix} \ge 0.$$

We follow Micchelli and Pinkus [21] by considering the convexity cone generated by $\mathcal{U}_{m-1}$; that is, $f \in \mathcal{K}(\mathcal{U}_{m-1})$ if f is a function defined on $[-1,1]$ which satisfies the inequality

$$\begin{vmatrix} u_1(t_1) & \dots & u_1(t_{m+1}) \\ \vdots & \vdots & \vdots \\ u_m(t_1) & \dots & u_m(t_{m+1}) \\ f(t_1) & \dots & f(t_{m+1}) \end{vmatrix} \geq 0,$$

for all $-1 < t_1 < \dots < t_{m+1} < 1$.

Let us suppose $N + \sum_{i=1}^{r} m_i = 2s$. As in [4, Theorem 3], we shall next assume that, there exist points $-1 < x_1^* < \dots < x_s^* < 1$ such that the system of equations,

$$\begin{aligned} u(x_i^*) &= 0, \qquad i = 1, \dots, s, \\ u'(x_i^*) &= 0 \qquad i = 1, \dots, s, \end{aligned} \tag{2.3}$$

has only the trivial solution in $\mathcal{T}_k(\mathcal{P}_{N+k-1}; \mathbf{\Delta}; \mathbf{M})$. We also make the following restrictions on the cone $\mathcal{K}(\mathcal{T}_k(\mathcal{P}_{N+k-1}; \mathbf{\Delta}; \mathbf{M}))$: for each points $-1 < t_1 < \dots < t_s < 1$ the set

$$\{(f(t_1), f'(t_1), \dots, f(t_s), f'(t_s)), \quad f \in \mathcal{K}(\mathcal{T}_k(\mathcal{P}_{N+k-1}; \mathbf{\Delta}; \mathbf{M}))\} \tag{2.4}$$

contains a basis for R^{2s}.

These standing hypotheses, which are used in [4] and [21], will not be mentioned explicitly in the results of this paper.

In order to illustrate assumptions (2.1) and (2.2), we list below a few known functionals which satisfy these hypotheses. We refer the interested reader to [20] and [21, p. 216] where these functionals have been studied in detail.

Example 2.1. *Functionals with anti-symmetric boundary conditions*

$$C_i(f) = f^{(i)}(-1) - f^{(i)}(1), \quad i = 0, 1, \dots, k-1. \tag{2.5}$$

Example 2.2. *Functionals with Birkhoff boundary conditions*

$$\begin{aligned} C_\mu(f) &= f^{(i_\mu)}(-1), \quad \mu = 1, \dots, p, \\ C_\mu(f) &= f^{(j_\mu)}(1), \qquad \mu = p+1, \dots, k, \end{aligned} \tag{2.6}$$

where $0 \leq i_1 \leq \dots \leq i_p \leq N+k-1$, $0 \leq j_1 \leq \dots \leq j_p \leq N+k-1$, *and* $M'_{\nu-1} + 2s \geq \nu$, $\nu = 2s+1, \dots, N+k$, *where* M'_ν *counts the number of integers in* $\{i_1, \dots, i_p, j_1, \dots, j_q\}$ *less than or equal to* ν. *Another important class of functionals, which satisfy certain determinantal conditions (see* [21, p. 216, Example 3.5]*), is the set of functionals with separate boundary conditions*

$$\begin{aligned} C_l(f) &= \textstyle\sum_{j=0}^{q_l-1} a_{lj} f^{(j)}(-1), \quad l = 1, \dots, p, \\ C_l(f) &= \textstyle\sum_{j=0}^{q'_l-1} b_{lj} f^{(j)}(1), \qquad l = p+1, \dots, k. \end{aligned} \tag{2.7}$$

Also, for the general case (1.3) examples are given in [20] *and* [21, p. 214].

We shall now discuss, under assumptions (2.1), (2.2), (2.3) and (2.4) the existence and uniqueness results of (1.2). We show that the latter follow from a recent result of [4, Theorem 3], which ensures that there exist unique nodes $\{x_{i,s}\}_{i=1}^s$ and unique coefficients $\{\lambda_{i,s}\}_{i=1}^s$ such that

$$\int_{-1}^{1} f(t)\, d\sigma = \sum_{i=1}^{s} \lambda_{i,s} f(x_{i,s}), \; \forall f \in \mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M}), \tag{2.8}$$

with $x_{i,s} \in (-1,1)$ and $\lambda_{i,s} > 0$, $i = 1, \dots, s$. This simplifies the proof considerably since $n + \sum_{i=1}^{r} m_i = 2s$ and then [4, Theorem 3] representation theorem is applicable.

We first recall a lemma which will be useful in the proof of the existence of (1.2). This result is given in [4, Lemma 3].

Lemma 2.1. *If $x_{i,s}, i = 1, ..., s$ are the nodes of (2.8), then the interpolation problem,*

$$\begin{array}{rcll} u(x_{i,s}) & = & 0, & i = 1, ..., s, \\ u'(x_{i,s}) & = & 0, & i = 1, ..., s, \end{array} \tag{2.9}$$

has only the trivial solution in $\mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$.

Remark 2.1. It is an elementary fact that the linear system (2.9) is equivalent to the homogeneous set of equations in $\mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$

$$\begin{array}{rcll} u(x_{i,s}) & = & 0, & i = 1, ..., s, \\ u'(x_{i,s}) & = & 0, & i = 1, ..., s, \\ C_l(u) & = & 0, & l = 1, ..., k. \end{array} \tag{2.10}$$

Then (2.9) is equivalent to saying that the linear system (2.10) has only the zero solution in $\mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$.

Theorem 2.2. *Given a nonnegative measure $d\sigma$ and nonnegative integers $n, k, r, s, m_1, m_2, ..., m_r$ with $n \geq 1$ such that $n + \sum_{i=1}^{r} m_i = 2s$, then there exists a unique quadrature formula of type (1.2), which exactly integrates all spline functions of $\mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$. The nodes $x_{1,s}, \ldots, x_{s,s}$ are all located in the open interval $(-1, 1)$, and their weights $\lambda_{1,s}, \ldots, \lambda_{s,s}$ are all positive.*

Proof. I**)** *Uniqueness result.* In order to prove uniqueness we suppose that there is another quadrature formula of the form

$$\int_{-1}^{1} f(t)\, d\sigma = \sum_{i=1}^{s} \bar{\lambda}_{i,s} f(\bar{x}_{i,s}) + \sum_{j=1}^{k} \bar{\omega}_{j,s} C_j(f) + \bar{R}_{k,s}(f),$$

having the desired property, and which also is exact for all splines from $\mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$. On account of the uniqueness of the quadrature formula (2.8), it suffices to prove the equality of quadrature weights $\omega_{j,s} = \bar{\omega}_{j,s},\ j = 1, ..., k$. This fact follows fairly easily from the rank property (2.2).

II) Existence result. We require of a good quadrature formula that its nodes be in the support of the measure. We show that, for (1.2), this holds true. To this end, let $x_{i,s}$, $i = 1, \ldots, s$, be the nodes of the quadrature formula (2.8). It follows from our previous remark that for all real data $\{y_i^j\}_{i=1,j=0}^{s,1}$ and $\{c_l\}_{l=1}^{k}$, there exists only one spline $S \in \mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ such that

$$\begin{array}{rcll} S(x_{i,s}) & = & y_i^0, & i = 1, ..., s, \\ S'(x_{i,s}) & = & y_i^1, & i = 1, ..., s, \\ C_l(S) & = & c_l, & l = 1, ..., k. \end{array} \tag{2.11}$$

Let $I_{n+k-1}(f)$ be the interpolation operator of $\mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ based on the data

$$\{f(x_{i,s}),\ f'(x_{i,s}),\ \ i = 1, \ldots, s;\ C_j(f),\ \ j = 1, \ldots, k\}.$$

Then, it is well-known that

$$I_{n+k-1}(f)(t) = \sum_{j=0}^{1} \sum_{i=1}^{s} f^{(j)}(x_{i,s}) h_{i,s}^{(j)}(t) + \sum_{j=1}^{k} C_j(f) l_{j,s}(t),$$

where $h_{i,s}^{(j)}$ and $l_{j,s}$ are the so-called fundamental spline functions. Since $I_{n+k-1}(f) = f$ for all $f \in \mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$, an integration of $I_{n+k-1}(f)$ leads to the following quadrature formula,

$$\int_{-1}^{1} f(t)\, d\sigma = \sum_{j=0}^{1} \sum_{i=1}^{s} \lambda_{i,s}^{(j)} f^{(j)}(x_{i,s}) + \sum_{j=1}^{k} \omega_{j,s} C_j(f) + R_{k,s}(f), \tag{2.12}$$

where

$$\lambda_{i,s}^{(j)} = \int_{-1}^{1} h_{i,s}^{(j)}(t)\, d\sigma, \text{ and } \omega_{j,s} = \int_{-1}^{1} l_{j,s}(t)\, d\sigma.$$

Note that $h_{i,s}^{'} \in \mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$, for all $i = 1, ..., s$, and vanishes at the nodes of (2.8), then applying (2.8) to $h_{i,s}^{'}$, we obtain $\lambda_{i,s}^{(1)} = 0$, $i = 1, ..., s$. This establishes existence. ∎

The proof shows in particular that the interior nodes of (1.2) are those of the quadrature formula (2.8).

Our next theorem shows, as it is known for the classical polynomial case, that the interior nodes of (1.2) have the interlacing property. To simplify the discussion, we first state and prove a theorem which works for the case where the functionnals $C_j(.)$, $j = 1, ..., k$ are given such that $\mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ is a Chebyshev system on $[-1, 1]$, and then we mention how we can apply this result to give a simple proof in the case when $\mathcal{I}_k$ $(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ is a weak Chebyshev system.

Theorem 2.3. *Let $n, s, k, r, m_1, m_2, ..., m_r$ be given as in Theorem 2.2. Assume that $\mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ is a Chebyshev system on $[-1, 1]$, and*

$$\int_{-1}^{1} f(t)\, d\sigma = \sum_{i=1}^{s} \lambda_{i,s} f(x_{i,s}) + \sum_{j=1}^{k} \omega_{j,s} C_j(f) + R_{k,s}(f),$$

$$\int_{-1}^{1} f(t)\, d\sigma = \sum_{i=1}^{s+1} \lambda_{i,s+1} f(x_{i,s+1}) + \sum_{j=1}^{k} \omega_{j,s+1} C_j(f) + R_{k,s+1}(f),$$

with $R_{k,s}(\mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})) = 0$, $R_{k,s+1}(\mathcal{S}(\mathcal{P}_{n+k+1}; \mathbf{\Delta}; \mathbf{M})) = 0$, $-1 < x_{1,s} < \ldots < x_{s,s} < 1$ and $-1 < x_{1,s+1} < \ldots < x_{s+1,s+1} < 1$. Then the following interlacing property holds:

$$-1 < x_{1,s+1} < x_{1,s} < x_{2,s+1} < x_{2,s} < \ldots < x_{s,s} < x_{s+1,s+1} < 1.$$

Proof. Arguing by contradiction, we suppose there exists μ with

$$x_{\mu,s} \notin [x_{\mu,s+1}, x_{\mu+1,s+1}].$$

Define the spline function $S_\mu \in \mathcal{S}(P_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ by the interpolation conditions

$$\begin{array}{rcl} S_\mu(x_{i,s}) & = & 0, \quad i = 1, \ldots, s, \\ C_j(S_\mu) & = & 0, \quad j = 1, \ldots, k, \\ S_\mu^{'}(x_{j,s+1}) & = & 0, \quad j = 1, ..., s+1,\ j \neq \mu \text{ and } j \neq \mu + 1, \\ S_\mu(x_{\mu,s+1}) & = & 1. \end{array}$$

The existence of S_μ follows from the fact that $\mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ is a Chebyshev system of dimension $2s$. Since S_μ has the maximal number of zeros allowed in $[-1, 1]$, we have necessarily $S_\mu(x_{\mu+1,s+1}) > 0$. Then,

$$\int_{-1}^{1} S_\mu(t)\, d\sigma = 0$$

by the first quadrature formula, and

$$\int_{-1}^{1} S_\mu(t)\, d\sigma = \lambda_{\mu,s+1} S_\mu(x_{\mu,s+1}) + \lambda_{\mu+1,s+1} S_\mu(x_{\mu+1,s+1}) > 0$$

by the second. This contradiction proves the interlacing property. ∎

Remark 2.2. In the case when $\mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ is a weak Chebyshev system. It is not difficult to see that this result remains valid, the basic idea of the proof is to "smooth", by convolution with the Gaussian kernel see, e. g., [21, pp. 221], the weak Chebyshev system $\mathcal{T}_k(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ into a Chebyshev system and then apply the previous result for Chebyshev systems. The very technical proof of this fact is simple and will be omitted.

We conclude this section with a remark about the remainder of the quadrature formula (1.2). We recall first of all that the monosplines and the polynomial quadrature formulae are closely connected, as was shown by Schoenberg [28]. In fact, it is well known that the error, when applying ordinary quadrature formula to differentiable functions, is given by an integral formula whose kernel is a monospline.

This property has been extended to the case of quadrature formulae for splines; see the paper by Micchelli and Pinkus [21, Section 4]. We also mention that the latter has been examined in the case of a wide class of quadrature formulae including quadratures of Hermitian type, see [3, Theorem 6.4]. For more details in the polynomial case, we refer to [15].

3. THE CHARACTERIZATION THEOREM

Our aim in this section is to show that the nodes, which are used by (1.2) in each subinterval $\left(\zeta_{i-1,}\zeta_i\right)$, $i = 1, ..., r+1$, are zeros of a certain quasi-orthogonal polynomial. Then, we establish that the latter can be represented as characteristic polynomial of a symmetric tridiagonal matrix. As we will see such a representation is particular useful, since it offers an efficient way of reducing the computation of (1.2) to the well-studied problem of computing a polynomial Gaussian quadrature formula from recurrence coefficients, and can then be computed directly by standard software for polynomial Gaussian quadrature formulae.

In order to formulate the problem precisely, we first give some formal definitions as well as some known results. Let $d\sigma$ be a nonnegative measure with support in the interval $[-1, 1]$, and let $\{\pi_l(.) = \pi_l(.; d\sigma)\}_{l=0,1,2,...}$ be the unique sequence of (monic) orthogonal polynomials with respect to $d\sigma$,

$$\begin{array}{lcl} \pi_l(t) & = & t^l + \text{lower-degree terms}, \qquad l = 0, 1, 2, ..., \\ \int_{-1}^{1} \pi_i(t)\pi_j(t) d\sigma & = & 0, \text{ if } i \neq j. \end{array}$$

It is well known that every sequence of monic orthogonal polynomials satisfies a three-term recurrence relation

$$\begin{array}{lcl} \pi_{l+1}(t) & = & (t - \alpha_l)\pi_l(t) - \beta_l \pi_{l-1}(t), \qquad l = 0, 1, 2, ... \\ \pi_{-1}(t) & = & 0, \ \pi_0(t) = 1, \end{array} \tag{3.1}$$

with coefficients $\alpha_l = \alpha_l(d\sigma)$, $\beta_l = \beta_l(d\sigma) > 0$, that are uniquely determined by the measure $d\sigma$ and by convention $\beta_0 = \beta_0(d\sigma) = \int_{-1}^{1} d\sigma$. These coefficients define a symmetric tridiagonal matrix, $J_n = J_n(d\sigma)$, known as Jacobi matrix, with α_l, $l = 0, 1, ..., n-1$, on the main diagonal and $\sqrt{\beta_l}$, $l = 1, 2, ..., n-1$, on the side diagonals. Golub and Welsch [12] have shown that for all $n \geq 1$ the nodes of the polynomial Gaussian quadrature formula, which has MDE $= 2n-1$, are the eigenvalues, and the weights are proportional to the squares of the first components of J_n. In fact, we can express π_n as the characteristic polynomial of the n-th order Jacobi matrix J_n, $\pi_n(t) = \det(tI_n - J_n)$, where we have denoted I_n the n-th order identity matrix.

This fact is an immediate consequence of the three-term recurrence relation (3.1) conveniently rewritten in matrix notation. For numerical implementation of this method, several algorithms have been proposed, the reader is referred to the definitive one which is given in the recent software package of Gautschi [11]. A thorough survey of the history, theoretical properties and important extensions of these results may be found in Gautschi [10]. See also the paper of Watkins [29].

We shall say that a polynomial $q_{n,r} \in \mathcal{P}_n$ generates a $(2n-r-1, n, d\sigma)$ positive polynomial quadrature formula (that is a quadrature formula which has n-nodes $t_{1,n} < \ldots < t_{n,n}$, positive weights and MDE $= 2n-r-1)$ if all nodes $t_{1,n}, \ldots, t_{n,n}$ are zeros of $q_{n,r}$ and are all located in the open interval $(-1,1)$. Since MDE is $2n-r-1$, it is easy to see that the underlying polynomial $q_{n,r}$ must be orthogonal to $\mathcal{P}_{n-r-1}$ with respect to the measure $d\sigma$. Hence, apart from a multiplicative constant, $q_{n,r}$ must be of the form

$$q_{n,r} = \pi_n + \rho_1 \pi_{n-1} + \ldots + \rho_r \pi_{n-r}, \tag{3.2}$$

where $\rho_1, \ldots, \rho_r$ are real constants. Such a polynomial is called a quasi-orthogonal polynomial of degree n and order r. The quasi-orthogonal polynomials and the positive quadrature formulae have been studied by many authors. For the historical development and a number of practical applications and numerical approximations we refer to Askey [1], Guessab and Rahman [14] and Ezzirani and Guessab [9]. A complete characterization has been done by Peherstorfer [25, 26, 27] and has produced some very interesting theoretical and computational results. For an earlier paper on the subject, see Micchelli and Rivlin [22]. Recently, Xu [31] and Ezzirani and Guessab [9] showed that a large class of quasi-orthogonal polynomials can be expressed as characteristic polynomials of a symmetric tridiagonal matrix.

Now, we restrict our discussion by recalling how quasi-orthogonal polynomials of degree n and order $r = 1, 2, 3$ or 4 can be expressed as characteristic polynomials of symmetric tridiagonal matrices. The following Theorem will be useful in section 4.

Theorem 3.1. *Let $q_{n,4}$ be a quasi-orthogonal polynomial of the form*

$$q_{n,4} = \pi_n + \rho_1 \pi_{n-1} + \ldots + \rho_4 \pi_{n-4}.$$

Then $q_{n,4}$ has a symmetric tridiagonal matrix representation of the form

$$q_{n,4}(t) = \det(tI_n - J_n^*) \tag{3.3}$$

with

$$J_n^* = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & \\ & \sqrt{\beta_2} & & \ddots & & \\ & & \ddots & \ddots & \sqrt{\beta_{n-2}-b_2} & \\ & & & \sqrt{\beta_{n-2}-b_2} & \alpha_{n-2}-a_2 & \sqrt{\beta_{n-1}-b_1} \\ 0 & & & & \sqrt{\beta_{n-1}-b_1} & \alpha_{n-1}-a_1 \end{bmatrix}$$

if and only if

$$\begin{array}{rcl} \rho_4 & < & \beta_{n-3}\beta_{n-2}, \\ b_1 & \leq & \beta_{n-1}, \end{array} \tag{3.4}$$

and

$$\begin{aligned} a_1 &= \rho_1 - a_2, \\ a_2 &= \frac{\rho_3 - b_2(\alpha_{n-3} - \alpha_{n-1}) - b_2\rho_1}{\beta_{n-2} - b_2}, \\ b_1 &= \rho_2 - b_2 - a_2(a_1 - \alpha_{n-1} + \alpha_{n-2}), \\ b_2 &= \frac{\rho_4}{\beta_{n-3}}. \end{aligned}$$

For the details of computations that lead to the results above, see Ezzirani and Guessab [9, Theorem 3.3]. The main results of Theorem 3.1 were given for the particular case $r \leq 4$. Of course, it could have been carried over to $r \geq 5$, but the problem is mainly computational. A more general form of this result has already appeared in [31].

We now consider a polynomial quadrature formula of the form

$$G_{n,r}(f) = \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n}),\ \lambda_{i,n} \in \mathbb{R},\ x_{i,n} \in (-1,1) \tag{3.5}$$

with MDE$(Q_{n,r}) = 2n - r - 1$. Using the symmetric tridiagonal matrix representation of quasi-orthogonal polynomials, we recall the following characterization of positive quadrature formulae, which is due to Xu [31, Theorem 4.1].

Theorem 3.2. *Let $q_{n,r}$ generate a $(2n-r-1, n, d\sigma)$ quadrature formula $G_{n,r}$ of the form (3.5). Then $G_{n,r}$ is a positive quadrature formula if and only if $q_{n,r}$ is a quasi-orthogonal polynomial of degree n and order r that has a symmetric tridiagonal matrix representation.*

To focus our discussion, we concentrate here only on the problem of determining the nodes and weights of (1.2), which exactly integrates each element of the spline space $\mathcal{S}(\mathcal{P}_{n+k-1}; \mathbf{\Delta}; \mathbf{M})$ and uses separate boundary conditions of the form

$$\begin{aligned} C_l(f) &= \textstyle\sum_{j=0}^{q_l-1} a_{lj} f^{(j)}(-1), \quad l = 1, ..., p, \\ C_l(f) &= \textstyle\sum_{j=0}^{q'_l-1} b_{lj} f^{(j)}(1), \quad\ \ l = p+1, ..., k. \end{aligned}$$

This restriction is only done for convenience of presentation. It will be apparent how the statement of theorems must be modified to encompass the more general case.

In order to present some local properties of the quadrature formulae (1.2), we have to introduce some more convenient notations. Let

$$q = \max_{1 \leq l \leq p} q_l \text{ and } q' = \max_{p+1 \leq l \leq q} q'_l.$$

We denote the nodes of (1.2) which are located in $(\zeta_{i-1,}\zeta_i)$ by $x_{1,i,s}, ..., x_{s_i,i,s}$ arranged in increasing order. For the rest of this paper, it is convenient to consider the measures

$$d\sigma_i = k_i d\sigma, \quad \text{on } (-1,1), \qquad i = 1, ..., r+1,$$

and

$$d\overset{\wedge}{\sigma}_i = \hat{k}_i d\sigma, \quad \text{on } (-1,1), \qquad i = 1, ..., r+1,$$

with

$$k_i(t) = \begin{cases} (\zeta_1 - t)_+ & \text{if} \quad i = 1, \\ (t - \zeta_{i-1})_+ (\zeta_i - t)_+ & \text{if} \quad i = 2, ..., r, \\ (t - \zeta_r)_+ & \text{if} \quad i = r+1, \end{cases}$$

and

$$\hat{k}_i(t) = \begin{cases} (1+t)_+^q (\zeta_1 - t)_+ & \text{if} \quad i = 1, \\ (t - \zeta_{i-1})_+ (\zeta_i - t)_+ & \text{if} \quad i = 2, ..., r, \\ (t - \zeta_r)_+ (1-t)_+^{q'} & \text{if} \quad i = r+1. \end{cases}$$

We will also need the following quadrature formulae:

$$\int_{\zeta_{i-1}}^{\zeta_i} f(t)d\sigma_i = G_{i,s_i}(f) = Q_{k,s}(k_i f),\ i = 1, ..., r+1, \tag{3.6}$$

and

$$\int_{\zeta_{i-1}}^{\zeta_i} f(t)d\overset{\wedge}{\sigma}_i = \hat{G}_{i,s_i}(f) = Q_{k,s}(\hat{k}_i f),\ i = 1, ..., r+1. \tag{3.7}$$

It is an elementary fact that $\mathrm{MDE}(G_{i,s_i}) = n + k - u_i - 1$, with

$$u_i = \begin{cases} 1, & \text{if} \quad i = 1, \\ 2, & \text{if} \quad i = 2, ..., r, \\ 1 & \text{if} \quad i = r+1, \end{cases} \tag{3.8}$$

and it is easily seen that, if we define v_i by

$$v_i = \begin{cases} q+1, & \text{if} \quad i = 1, \\ 2, & \text{if} \quad i = 2, ..., r, \\ q'+1 & \text{if} \quad i = r+1, \end{cases} \tag{3.9}$$

then, for $j = 0, ..., n + k - v_i - 1$, we have

$$\begin{aligned} \hat{G}_{i,s_i}((t-\zeta_{i-1})^j) &= Q_{k,s}(\hat{k}_i(t)(t-\zeta_{i-1})_+^j), \\ &= \textstyle\int_{-1}^{1}(t-\zeta_{i-1})_+^j \hat{k}_i(t)d\sigma, \\ &= \textstyle\int_{\zeta_{i-1}}^{\zeta_i}(t-\zeta_{i-1})^j d\overset{\wedge}{\sigma}_i. \end{aligned}$$

Where we have used the fact that $Q_{k,s}$ integrates exactly each element of the spline space $\mathcal{S}(\mathcal{P}_{n+k-1}; \boldsymbol{\Delta}; \mathbf{M})$. Consequently, for $i = 1, ..., r+1$, $\hat{G}_{i,s_i}$ is a $(2s_i - 1 - r_i, s_i, d\overset{\wedge}{\sigma}_i)$ positive quadrature formula, with

$$r_i = 2s_i - n - k + v_i. \tag{3.10}$$

Hence, for all $i = 1, ..., r+1$, the s_i nodes in (1.2), which are located in $\left(\zeta_{i-1,}\zeta_i\right)$, are those of the positive quadrature formula (3.7). Note that the latter are also the "interior" nodes of (3.6). Then, by theorem 3.2, they are the zeros of a quasi orthogonal polynomial of degree s_i and order r_i, where r_i is given by (3.10). Again by Theorem 3.2, this polynomial must have a symmetric tridiagonal matrix representation. Note that, if one can compute the quadrature formula (3.6) (or (3.7)), one is able to determine the quadrature formula (1.2). Thus, the remainder of this section is devoted to the details of computing the quasi-orthogonal polynomial whose zeros are the quadrature nodes.

Let d_i, $i = 1, ..., r+1$, and e_i, $i = 1, ..., r+1$, be the integers given by

(3.11)

$$d_i = \begin{cases} q-p, & \text{if} \quad i = 1, \\ 0, & \text{if} \quad i = 2, ..., r, \\ q'-k+p & \text{if} \quad i = r+1, \end{cases} \qquad \text{and} \qquad e_i = \begin{cases} p, & \text{if} \quad i = 1, \\ 0, & \text{if} \quad i = 2, ..., r, \\ k-p & \text{if} \quad i = r+1. \end{cases}$$

We shall denote respectively for $i = 1$, $i = 2, ..., r$ and $i = r+1$ by E_{i,d_i-1} the spaces defined by

$$E_{i,d_i-1} = \begin{cases} \{P \in \mathcal{P}_{s_i+q-1}, C_j(P) = P(x_{l,i,s}) = 0,\ j = 1, ..., p,\ l = 1, ..., s_i\}, \\ \{P \in \mathcal{P}_{s_i-1}, P(x_{l,i,s}) = 0,\ l = 1, ..., s_i\}, \\ \{P \in \mathcal{P}_{s_i+q'-1}, C_j(P) = P(x_{l,i,s}) = 0,\ j = p+1, ..., k,\ l = 1, ..., s_i\}. \end{cases}$$

It can be easily seen that E_{i,d_i-1} is a space of dimension d_i. Thus, for all $i = 1, ..., r+1$, there exists a set of polynomials $\{\Psi_{i,0}, ..., \Psi_{i,d_i-1}\} \subset E_{i,d_i-1}$ such that

$$\Psi_{i,j}(t) \;=\; t^{s_i+e_i+j} + R_{i,j}(t), \qquad j = 0, ..., d_i - 1, \tag{3.12}$$

with $R_{i,j}$ belonging to $\mathcal{P}_{s_i+e_i-1}$ and

$$E_{i,d_i-1} = \text{span}\left\{\Psi_{i,0}, ..., \Psi_{i,d_i-1}\right\}. \tag{3.13}$$

Bearing in mind the definitions of G_{i,s_i} and $\hat{G}_{i,s_i}$, we are now ready to formulate our main theorem of this section which plays a crucial role in the subsequent development.

Theorem 3.3. *Suppose* u_i, r_i, d_i, e_i, $i = 1, ..., r+1$, *and the set of functions*

$$\{\Psi_{i,0}, ..., \Psi_{i,d_i-1}\}$$

are given respectively as in (3.8), (3.10), (3.11) and (3.12). Suppose further that for all $i = 1, ..., r+1$, $x_{1,i,s}, ..., x_{s_i,i,s}$ *are* s_i *points on the interval* $\left(\zeta_{i-1}, \zeta_i\right)$ *, such that* $x_{j,i,s} \neq x_{j',i,s}$ *for all* $j \neq j'$*. Then the* s_i *nodes* $x_{1,i,s}, ..., x_{s_i,i,s}$ *are the interior nodes belonging to* $\left(\zeta_{i-1}, \zeta_i\right)$ *of the quadrature formula (3.6) if and only if , that for all* $i = 1, ..., r+1$, *they are zeros of a quasi-orthogonal polynomial* q_{i,s_i,r_i} *of degree* s_i *and order* r_i *with respect to* $d\overset{\wedge}{\sigma}_i = \hat{k}_i d\sigma$ *and, for all* $\Psi_{i,j} \in E_{i,d_i-1} \cap \mathcal{P}_{n+k-1-u_i}$,

$$\int_{\zeta_{i-1}}^{\zeta_i} \Psi_{i,j}(t) d\sigma_i \;=\; 0, \qquad j = 0, ..., d_i - 1. \tag{3.14}$$

Proof. Necessity. Assume that for all $i = 1, ..., r+1$, the nodes $x_{j,i,s}$, $j = 1, ..., s_i$; are those of the quadrature formula (3.6). We define

$$q_{i,s_i,r_i}(t) = \prod_{j=1}^{s_i}(t - x_{j,i,s}).$$

Now let p be an arbitrary polynomial of degree $\leq s_i - r_i - 1$; then the polynomial

$$f(t) = q_{i,s_i,r_i}(t)p(t)$$

is a polynomial of degree $\leq 2s_i - r_i - 1$ such that $\hat{G}_{i,s_i}(f) = 0$. Thus, since $\hat{G}_{i,s_i}$ is a $(2s_i - 1 - r_i, s_i, d\overset{\wedge}{\sigma}_i)$ quadrature formula, we have for all $p \in \mathcal{P}_{s_i-r_i-1}$

$$\begin{aligned} \int_{\zeta_{i-1}}^{\zeta_i} q_{i,s_i,r_i}(t)p(t)\, d\overset{\wedge}{\sigma}_i \;&=\; \hat{G}_{i,s_i}(q_{i,s_i,r_i}(t)p(t)), \\ &=\; 0. \end{aligned}$$

This means that q_{i,s_i,r_i} is orthogonal to all polynomials of $\mathcal{P}_{s_i-r_i-1}$ with respect to $d\overset{\wedge}{\sigma}_i$. Therefore, q_{i,s_i,r_i} is a quasi-orthogonal polynomial of degree s_i and order r_i with respect to $d\overset{\wedge}{\sigma}_i$.

For the second result, note that, since for all $i = 1, ..., r+1$, and $j = 0, ..., d_i - 1$, such that the polynomials $\Psi_{i,j} \in E_{i,d_i-1} \cap \mathcal{P}_{n+k-1-u_i}$, we have $G_{i,s_i}(\Psi_{i,j}) = 0$, where G_{i,s_i} is the quadrature formula defined in (3.6), then the exactness of G_{i,s_i} on $\mathcal{P}_{n+k-1-u_i}$ gives

$$\int_{\zeta_{i-1}}^{\zeta_i} \Psi_{i,j}(t) d\sigma_i \;=\; G_{i,s_i}(\Psi_{i,j}) = 0.$$

Thus, the necessity of the condition is proved.

Sufficiency. Assume that (3.14) holds and , for all $i = 1, ..., r+1$, there exist r_i real constants, $\rho_{1,i}, \ldots, \rho_{r_i,i}$, such that the polynomial

$$q_{i,s_i,r_i} = \overset{\wedge}{\pi}_{s_i,i} + \rho_{1,i}\overset{\wedge}{\pi}_{s_i-1,i} + \ldots + \rho_{r_i,i}\overset{\wedge}{\pi}_{s_i-r_i,i}$$

has n distinct zeros $x_{1,i,s},...,x_{s_i,i,s}$ on $\left(\zeta_{i-1},\zeta_i\right)$. Here, and subsequently,

$$\left\{\overset{\wedge}{\pi}_{m,i}(.)=\overset{\wedge}{\pi}_{m,i}(.;d\overset{\wedge}{\sigma}_i)\right\}_{m=0,1,2,...}$$

is the sequence of (monic) orthogonal polynomials with respect to $d\overset{\wedge}{\sigma}_i$ on $\left(\zeta_{i-1},\zeta_i\right)$.

For a given integer $i=1,...,r+1$ and a function f on $\left(\zeta_{i-1},\zeta_i\right)$, we denote by $\mathcal{I}_{s_i+e_i-1,i}(f;.)$, $\{h_{j,i,s},\ l_{j,i,s}\}$ respectively the (s_i+e_i-1)-th degree interpolating polynomial and the set of nodal basis functions with respect to the data

$$\{f(x_{j,i,s}),\ \ j=1,\ldots,s_i;\ C_j(f),\ \ j=1,\ldots,e_i\}\,.$$

Then, we have

$$\mathcal{I}_{s_i+e_i-1,i}(f;t)=\sum_{j=1}^{s_i}f(x_{_{j,i,s}})h_{_{j,i,s}}(t)+\sum_{j=1}^{e_i}C_j(f)l_{_{j,i,s}}(t),\ \text{on}\ \left(\zeta_{i-1},\zeta_i\right).$$

Since $\mathcal{I}_{s_i+e_i-1,i}(f;t)=f(t)$ for all $f\in\mathcal{P}_{s_i+e_i-1}$, we get the following quadrature formula

$$\int_{\zeta_{i-1}}^{\zeta_i}f(t)\,d\sigma_i=\sum_{j=1}^{s_i}\lambda_{_{j,i,s}}f(x_{_{j,i,s}})+\sum_{j=1}^{e_i}\omega_{_{j,i,s}}C_j(f)+R_{i,s,k}(f), \tag{3.15}$$

with $R_{i,s,k}(\mathcal{P}_{s_i+e_i-1})=0$, and where

$$\lambda_{_{j,i,s}}=\int_{\zeta_{i-1}}^{\zeta_i}h_{_{j,i,s}}(t)\,d\sigma_i,\ \omega_{_{j,i,s}}=\int_{\zeta_{i-1}}^{\zeta_i}l_{_{j,i,s}}(t)\,d\sigma_i.$$

Now, let P be an arbitrary polynomial of $\mathcal{P}_{n+k-u_i-1}$. We define the polynomials

$$\Phi_{i,j}(t)=t^{j-s_i-e_i-d_i}q_{i,s_i,r_i}(t)h_i(t),\ j=s_i+e_i+d_i,...$$

where

$$h_i(t)=\begin{cases}(1+t)^q & \text{if}\quad i=1,\\ 0 & \text{if}\quad i=2,...,r,\\ (1-t)^{q'} & \text{if}\quad i=r+1.\end{cases}$$

Le m_i^* be the integer defined by

$$m_i^*=\sup\left\{m:\Phi_{i,m}\in\mathcal{P}_{n+k-u_i-1}\right\}.$$

It can be easily proved that the collection of the polynomials

$$\left\{t^i,\ i=0,...,s_i+e_i-1;\ \Psi_{i,j},\ j=0,...,d_{i-1},\Phi_{i,j},\ j=s_i+e_i+d_i,...,m_i^*\right\},$$

form a basis for $\mathcal{P}_{n+k-u_i-1}$, here $\Psi_{i,j}$, $j=0,...,d_i-1$, are the polynomials defined in (3.12). Then P can be represented uniquely in the form

$$P(t)=\sum_{j=0}^{d_i-1}a_{i,j}\Psi_{i,j}(t)+\sum_{j=s_i+e_i+d_i}^{m_i^*}a_{i,j}\Phi_{i,j}(t)+R_i(t), \tag{3.16}$$

where R_i is a polynomial of $\mathcal{P}_{s_i+e_i-1}$. From this, it follows

$$\begin{aligned}\int_{\zeta_{i-1}}^{\zeta_i}P(t)\,d\sigma_i &= \sum_{j=0}^{d_i-1}a_{i,j}\int_{\zeta_{i-1}}^{\zeta_i}\Psi_{i,j}(t)\,d\sigma_i\\ &\quad+\sum_{j=s_i+e_i+d_i}^{m_i^*}a_{i,j}\int_{\zeta_{i-1}}^{\zeta_i}\Phi_{i,j}(t)\,d\sigma_i+\int_{\zeta_{i-1}}^{\zeta_i}R_i(t)\,d\sigma_i.\end{aligned} \tag{3.17}$$

Since, for all $j=s_i+e_i+d_i,...,m_i^*$,

$$\int_{\zeta_{i-1}}^{\zeta_i}\Phi_{i,j}(t)\,d\sigma_i=G_{i,s_i}(\Phi_{i,j})=0,$$

therefore, from (3.14) and (3.17) we have

$$\int_{\zeta_{i-1}}^{\zeta_i} P(t)\, d\sigma_i = \int_{\zeta_{i-1}}^{\zeta_i} R_i(t)\, d\sigma_i.$$

Consequently, from (3.15)

$$\int_{\zeta_{i-1}}^{\zeta_i} P(t)\, d\sigma_i = \sum_{j=1}^{s_i} \lambda_{_{j,i,s}} R_i(x_{_{j,i,s}}) + \sum_{j=1}^{e_i} \omega_{_{j,i,s}} C_j(R_i).$$

From (3.16), however we have for all $i = 1, ..., n$ and $j = 1, ..., k$

$$P(x_{_{j,i,s}}) = R_i(x_{_{j,i,s}}),\ C_j(P) = C_j(R_i);$$

hence we have the following quadrature formula

$$\int_{\zeta_{i-1}}^{\zeta_i} P(t)\, d\sigma_i = \sum_{j=1}^{s_i} \lambda_{_{j,i,s}} P(x_{_{j,i,s}}) + \sum_{j=1}^{e_i} \omega_{_{j,i,s}} C_j(P) + R_{i,s,k}(P), \tag{3.18}$$

with $R_{i,s,k}(\mathcal{P}_{n+k-u_i-1}) = 0$, which shows that (3.18) is a quadrature formula of the form (3.6). This completes the proof of Theorem 3.3. ∎

Remark 3.1. Theorem 3.3 characterizes all quadrature formulae (3.6). Moreover, it assures that the nodes used by (1.2), which are located in each interval $\left(\zeta_{i-1}, \zeta_i\right)$, $i = 1, ..., r+1$, are zeros of a quasi-orthogonal polynomial of the form

$$q_{i,s_i,r_i} = \overset{\wedge}{\pi}_{s_i,i} + \rho_{1,i}\overset{\wedge}{\pi}_{s_i-1,i} + \ldots + \rho_{r_i,i}\overset{\wedge}{\pi}_{s_i-r_i,i}.$$

This polynomial satisfies the orthogonality relations (3.14) (via the formulae (3.14); recall that the polynomials $\Psi_{i,j}$ are of the form $\Psi_{i,j} = q_{i,s_i,r_i} Q_{j,i}$ with $Q_{j,i}$ belonging to $\mathcal{P}_{e_i+d_i-1}$). To make the procedure described in Theorem 3.3 computationally feasible, we must compute the coefficients $\rho_{1,i}, \ldots, \rho_{r,i}$. These coefficients are solutions of a linear system of r_i equations and r_i unknowns. This result can be obtained by choosing appropriate splines. To clarify this fact, we will discuss this more fully in section 4, in the case of spline Gauss quadrature formulae. In addition, a detailed analysis of the location and nature of the nodes relative to each subinterval will also appear in the next section. As we have previously pointed out, the underlying polynomial q_{i,s_i,r_i} must have a symmetric tridiagonal matrix representation. Hence, we can use the existing routine [11] for determining the weights and nodes in (1.2). These observations play a central role in the construction of (1.2).

We close this section with a result concerning the distribution of the nodes of (1.2). In the case of separate boundary conditions (2.7), the following result, which has been proved in [20] (see also[21, pp. 221]), gives some additional information about the distribution of the node of (1.2) in $(-1, 1)$. As usual with splines, here and in the sequel, we use the extended nodes sequences corresponding to $\mathbf{\Delta}$, M, and $x_{i,s}, i = 1, ..., s$, which are defined by

$$x^*_{2i-1,s} = x^*_{2i,s} = x_{i,s}, \qquad i = 1, ..., s,$$

$$\zeta^*_{\sum_{j=1}^{i-1} m_j + 1} = ... = \zeta^*_{\sum_{j=1}^{i} m_j} = \zeta_i, \qquad i = 1, ..., r.$$

Theorem 3.4. *The nodes $x_{i,s}, i = 1, ..., s$ of the quadrature formula (1.2) satisfy the interlacing condition*

$$x^*_{i-p,s} < \zeta^*_i < x^*_{i+n-p,s}, \qquad i = 1, ..., \sum_{j=1}^{r} m_j.$$

4. COMPUTATION OF SPLINE GAUSS QUADRATURES

In this section, we turn our attention to the numerical problem of computing a class of quadrature formulae of type (1.2). We will illustrate, by means of a simple example, how our characterization of the "interior" nodes of (1.2) can be used to evaluate the free nodes as the eigenvalues of symmetric tridiagonal matrices. We also establish that the weights are proportional to the squares of the first components of the orthonormal eigenvectors.

To focus our discussion, we concentrate here only on the problem of determining the nodes and weights of (1.2), which do not use boundary conditions and which is exact on $\mathcal{S}(\mathcal{P}_{2n-1};\mathbf{\Delta};\mathbf{M})$. Hence, we seek a quadrature formula of the form,

$$\int_{-1}^{1} f(t)\,d\sigma = Q_{2n-1}(f) + R(f),$$

where,

$$Q_{2n-1}(f) = \sum_{j=1}^{s} w_{j,n} f(x_{j,s}) \tag{4.1}$$

such that,

$$\int_{-1}^{1} f(t)\,d\sigma = Q_{2n-1}(f),$$

for each element of the spline space $\mathcal{S}(\mathcal{P}_{2n-1};\mathbf{\Delta};\mathbf{M})$, with $\mathbf{M} = (m_1, ..., m_r)$ is a vector of integers of the form

$$m_i = 2n-1, \qquad i = 1, ..., r.$$

Note, in this case, that from equation (1.1) we have

$$2s = 2n + r(2n-1), \tag{4.2}$$

therefore the number of the nodes $\zeta_i, i = 1, ..., r,$ must be even. We also observe that every element of $\mathcal{S}(\mathcal{P}_{2n-1};\mathbf{\Delta};\mathbf{M})$ is continuous on $[-1,1]$. Thus, the important point to note here is the fact that (4.1) is necessary based on *continuous* piecewise polynomials interpolation.

From the fundamental theorem of determinants for polynomial splines [17], one can infer that the assumptions (2.1), (2.2), (2.3) and (2.4) are trivially fulfilled for each n. Consequently, Theorem 2.2 guarantees the existence of a unique quadrature formula of the form (4.1), with respect to $d\sigma$ on the interval $[-1,1]$. In order to show how (4.1) can be obtained numerically, we adopt the following notations. For all $i = 1, ..., r+1$, let $\left\{\overset{\wedge}{\pi}_{l,i}(.) = \overset{\wedge}{\pi}_{l,i}(.; d\overset{\wedge}{\sigma}_i)\right\}_{l=0,1,2,...}$ be the unique sequence of (monic) orthogonal polynomials with respect to $d\overset{\wedge}{\sigma}_i = \hat{k}_i d\sigma$, where

$$\hat{k}_i(t) = \begin{cases} (\zeta_1 - t)_+ & \text{if } \; i = 1, \\ (t-\zeta_{i-1})_+(\zeta_i - t)_+ & \text{if } \; i = 2, ..., r, \\ (t-\zeta_r)_+ & \text{if } \; i = r+1, \end{cases}$$

and let

$$\begin{aligned} \overset{\wedge}{\alpha}_{l,i} &= \overset{\wedge}{\alpha}_l(\overset{\wedge}{\sigma}_i), \qquad & 0 \le l \le s_i - 1, \\ \overset{\wedge}{\beta}_{l,i} &= \overset{\wedge}{\beta}_l(\overset{\wedge}{\sigma}_i), \qquad & 1 \le l \le s_i - 1, \end{aligned}$$

where s_i denotes the number of the nodes in (4.1) which are located in $(\zeta_{i-1,}\zeta_i)$.

Note that, in this particular case, we have (using the notations introduced in the previous section) $k = q = q' = 0$, and

$$u_i = v_i = \begin{cases} 1, & \text{if} \quad i = 1, \\ 2, & \text{if} \quad i = 2, ..., r, \\ 1 & \text{if} \quad i = r + 1. \end{cases} \tag{4.3}$$

The basic difficulty that we face here is how to assess the exact value of s_i. But before considering the computational details of determining this value, let us point out the following obvious lower bounds

$$\begin{cases} s_i \geq n, & \text{if} \quad i = 1, \\ s_i \geq n - 1, & \text{if} \quad i = 2, ..., r, \\ s_i \geq n & \text{if} \quad i = r + 1. \end{cases} \tag{4.4}$$

These bounds follow easily from the fact that $Q_{2n-1}(\hat{k}_i f)$ is a $(2s_i - 1 - r_i, s_i, d\overset{\wedge}{\sigma}_i)$ quadrature formula and that $2s_i - 1 - r_i = 2n - v_i - 1$.

Now upon applying theorem 3.4 to (4.1) we obtain that the nodes $x_{j,s}, j = 1, ..., s$ of the quadrature formula (4.1) satisfy the interlacing condition

$$x^*_{i,s} < \zeta^*_i < x^*_{i+2n,s}, \qquad i = 1, ..., r(2n-1). \tag{4.5}$$

We are now ready to establish that

$$s_i = \begin{cases} n, & \text{if} \quad i \text{ is odd }, \\ n - 1, & \text{if} \quad i \text{ is even.} \end{cases}$$

First we prove that $s_1 = n$. Since $s_1 \geq n$ and by (4.5) for $i = 1$, $\zeta^*_1 = \zeta_1 < x^*_{2n+1,s} = x_{n+1,s}$, we conclude that the first interval $[\zeta_0, \zeta_1]$ contains exactly n nodes. Therefore $s_1 = n$. If we apply now (4.5) for $i = 2n, ..., 4n - 3$, then we obtain $\zeta_1 < x_{n+1,s} < ... < x_{2n-1,s} < \zeta_2$ which, combined with $\zeta_2 = \zeta^*_{2n} < x^*_{4n,s} = x_{2n,s}$ obtained from (4.5) for $i = 2n$, implies $s_2 = n - 1$. By an analogous argument and by induction the exact number of the nodes in each interval is similarly proven.

On the basis of this result, we conclude that

$$r_i = \begin{cases} 1 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i = 2, ..., r \text{ and } i \text{ is even} \\ 2 & \text{if} \quad i = 2, ..., r \text{ and } i \text{ is odd} \\ 1 & \text{if} \quad i = r + 1. \end{cases} \tag{4.6}$$

It is convenient at this stage to rewrite the quadrature formula (4.1) as

$$Q_{2n-1}(f) = \sum_{i=1}^{r} \sum_{j=1}^{s_i} w_{j,i,s} f(x_{j,i,s}),$$

here $x_{j,i,s}, j = 1, ..., s_i$, denote the nodes in (4.1) which are located in (ζ_{i-1}, ζ_i). According the Theorem 3.3, we also know that, for all $i = 1, ..., r$, the nodes $x_{j,i,s}, j = 1, ..., s_i$, are the zeros of a certain quasi-orthogonal polynomial q_{i,s_i,r_i} of degree s_i and order r_i. Therefore q_{i,s_i,r_i} takes also the form

$$q_{i,s_i,r_i} = \overset{\wedge}{\pi}_{s_i,i} + \sum_{j=1}^{r_i} \hat{a}_{j,i} \overset{\wedge}{\pi}_{s_i-j,i}.$$

We next show how the quadrature formula (4.1) can be constructed by means suitable modifications of classical Gauss quadrature formulae with modified weight functions. For this, we consider separately three cases.

4.1. ***Case I:*** $i = 2, ..., r$ **and** i **is even.** This is the simplest case, since, using the fact that $s_i = n-1$ and $r_i = 0$, we have, therefore, the quasi-orthogonal polynomial q_{i,s_i,r_i} is the classical orthogonal polynomial $\overset{\wedge}{\pi}_{n-1,i}$. This means that the nodes in (4.1), which are located in the interval $\left(\zeta_{i-1,}\zeta_i\right)$, are precisely those of the classical polynomial Gauss quadrature formula

$$\int_{\zeta_{i-1}}^{\zeta_i} f(t)(t-\zeta_{i-1})(\zeta_i - t)d\sigma = \sum_{j=1}^{n-1} w_{j,i,s}^G f(x_{j,i,s}) + R_n^G(f), \qquad R_n^G(\mathcal{P}_{2n-3}) = 0, \tag{4.7}$$

and the weights $w_{j,i,s}$ corresponding to $x_{j,i,s}$ in (4.1) are simply obtained from (4.7) in term of the Christoffel numbers $w_{j,i,s}^G$ by

$$w_{j,i,s} = \frac{w_{j,i,s}^G}{(x_{j,i,s} - \zeta_{i-1})(\zeta_i - x_{j,i,s})}, \qquad j = 1, ..., n-1.$$

4.2. ***Case II:*** $i = 1$ **or** $i = r+1$**.** We are now in the case where q_{i,s_i,r_i} is a quasi-orthogonal polynomial of degree $s_i = n$ and order $r_i = 1$. Therefore, we have

$$q_{i,s_i,r_i} = q_{i,n,1} = \overset{\wedge}{\pi}_{n,i} + \hat{a}_{n,i}\overset{\wedge}{\pi}_{n-1,i}. \tag{4.8}$$

To compute $\hat{a}_{n,i}$, for $i = 1$ and $i = r+1$, we consider respectively two splines S_1 and S_{r+1} defined by

$$S_1(t) = \begin{cases} q_{1,n,1}(t), & \text{if} \quad -1 \le t \le \zeta_1 \\ \frac{q_{1,n,1}(\zeta_1)}{(\zeta_2 - \zeta_1) q_{2,s_2,r_2}(\zeta_1)}(\zeta_2 - t) q_{2,s_2,r_2}(t), & \text{if} \quad \zeta_1 \le t \le \zeta_2, \\ 0, & \text{if} \quad \zeta_2 \le t \le 1, \end{cases}$$

and

$$S_{r+1}(t) = \begin{cases} 0, & \text{if} \quad -1 \le t \le \zeta_{r-1} \\ \frac{q_{r+1,n,1}(\zeta_r)}{(\zeta_r - \zeta_{r-1}) q_{r,s_r,r_r}(\zeta_r)}(t - \zeta_{r-1}) q_{r,s_r,r_r}(t), & \text{if} \quad \zeta_{r-1} \le t \le \zeta_r, \\ q_{r+1,n,1}(t), & \text{if} \quad \zeta_r \le t \le 1. \end{cases}$$

Evidently such functions belong to $\mathcal{S}(\mathcal{P}_{2n-1}; \boldsymbol{\Delta}; \mathbf{M})$ and they satisfy $Q_{2n-1}(S_1) = Q_{2n-1}(S_{r+1}) = 0$. Consequently, if we apply (4.1) to S_1 and S_{r+1} then we obtain

$$\int_{-1}^{1} S_1(t)\, d\sigma = \int_{-1}^{1} S_{r+1}(t)\, d\sigma = 0.$$

Hence, for example for the case $i = 1$, the coefficient $\hat{a}_{n,1}$ defined in (4.8) must be a solution of

$$\int_{-1}^{\zeta_1} q_{1,n,1}(t) d\sigma = -\frac{q_{1,s_1,1}(\zeta_1)}{(\zeta_2 - \zeta_1) q_{2,s_2,r_2}(\zeta_1)} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - t) q_{2,s_2,r_2}(t) d\sigma, \tag{4.9}$$

then by an elementary calculation one can show that

$$\hat{a}_{n,1} = -\frac{\int_{-1}^{\zeta_1} \overset{\wedge}{\pi}_{n,1}(t) d\sigma + \frac{\overset{\wedge}{\pi}_{n,1}(\zeta_1)}{(\zeta_2 - \zeta_1) q_{2,s_2,r_2}(\zeta_1)} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - t) q_{2,s_2,r_2}(t) d\sigma}{\int_{-1}^{\zeta_1} \overset{\wedge}{\pi}_{n-1,1}(t) d\sigma + \frac{\overset{\wedge}{\pi}_{n-1,1}(\zeta_1)}{(\zeta_2 - \zeta_1) q_{2,s_2,r_2}(\zeta_1)} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - t) q_{2,s_2,r_2}(t) d\sigma}. \tag{4.10}$$

Recall from Case I that, we know that $q_{2,s_2,r_2} \equiv \overset{\wedge}{\pi}_{n-1,2}$ then the parameter $\hat{a}_{n,1}$ can easily be calculated by an explicit formula or at least numerically. We shall indicate how such formula may be obtained in the case of Jacobi weights.

Now by Theorem 3.1, the polynomial admits the following matrix representation

$$q_{1,n,1}(t) = \det(tI_n - J_n(\overset{\wedge}{\sigma}_1)),$$

where

$$
(4.11) \qquad J_n(\overset{\wedge}{\sigma}_1) = \begin{bmatrix} \overset{\wedge}{\alpha}_{0,1} & \sqrt{\overset{\wedge}{\beta}_{1,1}} & & & 0 \\ \sqrt{\overset{\wedge}{\beta}_{1,1}} & \overset{\wedge}{\alpha}_{1,1} & \ddots & & \\ & \ddots & \ddots & \sqrt{\overset{\wedge}{\beta}_{n-2,1}} & \\ & & \sqrt{\overset{\wedge}{\beta}_{n-2,1}} & \overset{\wedge}{\alpha}_{n-2,1} & \sqrt{\overset{\wedge}{\beta}_{n-1,1}} \\ 0 & & & \sqrt{\overset{\wedge}{\beta}_{n-1,1}} & \overset{\wedge}{\alpha}_{n-1,1} - \hat{a}_{n,1} \end{bmatrix}
$$

Thus, the n nodes of the quadrature formula Q_{2n-1}, which are located in the first interval $(-1,\zeta_1)$, are the eigenvalues of $J_n(\overset{\wedge}{\sigma}_1)$.

We now consider the following quadrature formula based on the zeros of $q_{1,n,1}$ and of the form

$$
(4.12) \qquad \int_{-1}^{\zeta_1} (\zeta_1 - t) f(t) d\sigma = \sum_{j=1}^{n} (\zeta_1 - x_{j,1,s}) w_{j,1,s} f(x_{j,1,s}).
$$

Since (4.12) is positive and exactly integrates every polynomial of degree $2n-2$, then as a direct consequence of [30, Theorem 6.1], the weights $(\zeta_1 - x_{j,1,s})w_{j,1,s}$ are given by

$$
(4.13) \qquad (\zeta_1 - x_{j,1,s}) w_{j,1,s} = \frac{1}{K_n(x_{j,1,s}, x_{j,1,s})} \text{ for all } j = 1, ..., n,
$$

where

$$
K_n(x,y) = \sum_{k=0}^{n-2} \tilde{\pi}_{k,1}(x)\tilde{\pi}_{k,1}(y) + \tilde{\pi}_{n-1,1}(x)\tilde{\pi}_{n-1,1}(y)
$$

with $\left\{ \tilde{\pi}_{k,1}(.) = \tilde{\pi}_{k,1}(.;\overset{\wedge}{\sigma}_1) \right\}_{k=0,1,2,...}$ being the set of *orthonormal* polynomials with respect to $d\overset{\wedge}{\sigma}_1$.

We now suppose that the eigenvectors of $J_n(\overset{\wedge}{\sigma}_1)$ are calculated such that

$$
J_n(\overset{\wedge}{\sigma}_1) V_{j,1} = x_{j,1,s} V_{j,1}, \qquad j = 1, ..., n,
$$

with $V_{j,1}^T V_{j,1} = 1$ and $V_{j,1}^T = (v_{1,j,1}, ..., v_{n,j,1})$. Then, as in the ordinary Gauss quadrature formula, it follows from (4.13) that the coefficients $w_{j,1,s}$ are expressible in terms of the first components $v_{1,j,1}$ of $V_{j,1}$ by

$$
w_{j,1,s} = \frac{v_{1,j,1}^2}{(\zeta_1 - x_{j,1,s})} \beta_{0,1}(d\overset{\wedge}{\sigma}_1), \qquad j = 1, ..., n.
$$

It is clear that a virtually identical argument works in the case $i = r+1$ and gives the following result. Let $\hat{a}_{n,r+1}$and $J_n(\overset{\wedge}{\sigma}_{r+1})$ be defined respectively by

$$
(4.14) \qquad \hat{a}_{n,r+1} = - \frac{\int_{\zeta_r}^{1} \overset{\wedge}{\pi}_{n,r+1}(t) d\sigma + \frac{\overset{\wedge}{\pi}_{n,r+1}(\zeta_r)}{(\zeta_r - \zeta_{r-1}) q_{r,s_r,r_r}(\zeta_r)} \int_{\zeta_{r-1}}^{\zeta_r} (t - \zeta_{r-1}) q_{r,s_r,r_r}(t) d\sigma}{\int_{\zeta_r}^{1} \overset{\wedge}{\pi}_{n-1,r+1}(t) d\sigma + \frac{\overset{\wedge}{\pi}_{n-1,r+1}(\zeta_r)}{(\zeta_r - \zeta_{r-1}) q_{r,s_r,r_r}(\zeta_r)} \int_{\zeta_{r-1}}^{\zeta_r} (t - \zeta_{r-1}) q_{r,s_r,r_r}(t) d\sigma},
$$

and

$$(4.15)\qquad J_n(\overset{\wedge}{\sigma}_{r+1}) = \begin{bmatrix} \overset{\wedge}{\alpha}_{0,r+1} & \sqrt{\overset{\wedge}{\beta}_{1,r+1}} & & & 0 \\ \sqrt{\overset{\wedge}{\beta}_{1,r+1}} & \overset{\wedge}{\alpha}_{1,r+1} & \ddots & & \\ & \ddots & \ddots & \sqrt{\overset{\wedge}{\beta}_{n-2,r+1}} & \\ & & \sqrt{\overset{\wedge}{\beta}_{n-2,r+1}} & \overset{\wedge}{\alpha}_{n-2,r+1} & \sqrt{\overset{\wedge}{\beta}_{n-1,r+1}} \\ 0 & & & \sqrt{\overset{\wedge}{\beta}_{n-1,r+1}} & \overset{\wedge}{\alpha}{}^{*}_{n-1,r+1} \end{bmatrix},$$

with $\overset{\wedge}{\alpha}{}^{*}_{n-1,r+1} = \overset{\wedge}{\alpha}_{n-1,r+1} - \hat{a}_{n,r+1}$. Then, the n nodes $x_{j,r+1,s}$, $j = 1, ..., n$, which are located in the last interval $(\zeta_r, 1)$, are the n distinct real eigenvalues of $J_n(\overset{\wedge}{\sigma}_{r+1})$ and the respective weights in (4.1) are given by

$$w_{j,r+1,s} = \frac{v_{1,j,r+1}^2}{(x_{j,r+1,s} - \zeta_r)}\beta_{0,r+1}(d\overset{\wedge}{\sigma}_{r+1}), \qquad j = 1, ..., n,$$

with $v_{1,j,r+1}$ being the first components of the normalized eigenvectors of $J_n(\overset{\wedge}{\sigma}_{r+1})$ corresponding to the eigenvalues $x_{j,r+1,s}$.

4.3. ***Case III:*** $i = 3, ..., r-1$ **and** i **is odd.** Here q_{i,s_i,r_i} is a quasi-orthogonal polynomial of degree $s_i = n$ and order 2. Thus q_{i,n,r_i} has the form

$$(4.16)\qquad q_{i,s_i,r_i} = q_{i,n,2} = \overset{\wedge}{\pi}_{n,i} + \hat{a}_{n,i}\overset{\wedge}{\pi}_{n-1,i} + \hat{b}_{n,i}\overset{\wedge}{\pi}_{n-2,i}.$$

As in Case II, to compute $\hat{a}_{n,i}$ and $\hat{b}_{n,i}$, we consider respectively two splines $S_{1,i}$ and $S_{2,i}$ defined by

$$S_{1,i}(t) = \begin{cases} (t-\zeta_{i-2})q_{i-1,s_{i-1},r_{i-1}}(t), & \text{if } \zeta_{i-2} \le t \le \zeta_{i-1} \\ \frac{(\zeta_{i-1}-\zeta_{i-2})q_{i-1,s_{i-1},r_{i-1}}(\zeta_{i-1})}{(\zeta_i-\zeta_{i-1})q_{i,s_i,r_i}(\zeta_{i-1})}(\zeta_i - t)q_{i,s_i,r_i}(t), & \text{if } \zeta_{i-1} \le t \le \zeta_i, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$S_{2,i}(t) = \begin{cases} (t-\zeta_{i-1})q_{i,s_i,r_i}(t), & \text{if } \zeta_{i-1} \le t \le \zeta_i \\ \frac{(\zeta_i-\zeta_{i-1})q_{i,s_i,r_i}(\zeta_i)}{(\zeta_{i+1}-\zeta_i)q_{i+1,s_{i+1},r_{r+1}}(\zeta_i)}(\zeta_{i+1} - t)q_{i+1,s_{i+1},r_{r+1}}(t), & \text{if } \zeta_i \le t \le \zeta_{i+1}, \\ 0 & \text{otherwise}. \end{cases}$$

Recall that since $i-1$ and $i+1$ are even then from the case I, we have $q_{i-1,s_{i-1},r_{i-1}} = \overset{\wedge}{\pi}_{n-1,i-1}$ and $q_{i+1,s_{i+1},r_{i+1}} = \overset{\wedge}{\pi}_{n-1,i+1}$. It is staightforward to cheek that $S_{1,i}$ and $S_{2,i}$ belong to $\mathcal{S}(\mathcal{P}_{2n-1}; \mathbf{\Delta}; \mathbf{M})$ and that $Q_{2n-1}(S_{1,i}) = Q_{2n-1}(S_{2,i}) = 0$. Putting $S_{1,i}$ respectively $S_{2,i}$ in (4.1) we get

$$\int_{-1}^{1} S_{1,i}(t)\,d\sigma = \int_{-1}^{1} S_{2,i}(t)\,d\sigma = 0.$$

Then, with the help of a simple computation, we can show that these equations give rise to a 2×2 linear system with unknowns $\hat{a}_{n,i}$, $\hat{b}_{n,i}$ of the form

$$(4.17)\qquad \begin{cases} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2. \end{cases}$$

Once we have determined $\hat{a}_{n,i}$ and $\hat{b}_{n,i}$ by solving (4.17), we can rewrite, using Theorem 3.1, the polynomial q_{i,s_i,r_i} in matrix notation

$$q_{i,s_i,r_i}(t) = \det(tI_n - J_n(\overset{\wedge}{\sigma}_i))$$

where

$$J_n(\overset{\wedge}{\sigma}_i) = \begin{bmatrix} \overset{\wedge}{\alpha}_{0,i} & \sqrt{\overset{\wedge}{\beta}_{1,i}} & & & 0 \\ \sqrt{\overset{\wedge}{\beta}_{1,i}} & \overset{\wedge}{\alpha}_{1,i} & \ddots & & \\ & \ddots & \ddots & \sqrt{\overset{\wedge}{\beta}_{n-2,i}} & \\ & & \sqrt{\overset{\wedge}{\beta}_{n-2,i}} & \overset{\wedge}{\alpha}_{n-2,i} & \sqrt{\overset{\wedge}{\beta}^{*}_{n-1,i}} \\ 0 & & & \sqrt{\overset{\wedge}{\beta}^{*}_{n-1,i}} & \overset{\wedge}{\alpha}^{*}_{n-1,i} \end{bmatrix}, \tag{4.18}$$

where $\overset{\wedge}{\alpha}^{*}_{n-1,i} = \overset{\wedge}{\alpha}_{n-1,i} - \hat{a}_{n,i}$, and $\overset{\wedge}{\beta}^{*}_{n-1,i} = \overset{\wedge}{\beta}_{n-1,i} - \hat{b}_{n,i}$. Thus, the n nodes of the quadrature formula Q_{2n-1}, which are located in $\left(\zeta_i, \zeta_{i+1}\right)$, are the eigenvalues of $J_n(\overset{\wedge}{\sigma}_i)$.

We now consider the following quadrature formula based on the zeros of q_{i,s_i,r_i} and of the form

$$\int_{\zeta_i,}^{\zeta_{i+1}} (t-\zeta_i)(\zeta_{i+1}-t)f(t)d\sigma = \sum_{j=1}^{n} (x_{j,i,s}-\zeta_i)(\zeta_{i+1}-x_{j,i,s})w_{j,i,s}f(x_{j,i,s}). \tag{4.19}$$

Since (4.19) exactly integrates every polynomial of degree $2n-3$, then as a direct consequence of [30, Theorem 6.1], the weights $(x_{j,i,s}-\zeta_i)(\zeta_{i+1}-x_{j,i,s})w_{j,i,s}$ are given by

$$(x_{j,i,s}-\zeta_i)(\zeta_{i+1}-x_{j,i,s})w_{j,i,s} = \frac{A_{n,i}}{K_{n,i}(x_{j,i,s}, x_{j,i,s})} \text{ for all } j = 1, ..., n, \tag{4.20}$$

where

$$K_{n,i}(x,y) = A_{n,i}\sum_{k=0}^{n-2} \widetilde{\pi}_{k,i}(x)\widetilde{\pi}_{k,i}(y) + \widetilde{\pi}_{n-1,i}(x)\widetilde{\pi}_{n-1,i}(y)$$

with $A_{n,i} = 1 - \dfrac{\hat{b}_{n,i}}{\overset{\wedge}{\beta}_{n-1,i}}$ and $\left\{\widetilde{\pi}_{k,i}(.) = \widetilde{\pi}_{k,i}(.;\overset{\wedge}{\sigma}_i)\right\}_{k=0,1,2,...}$ being the set of *orthonormal* polynomials with respect to $d\overset{\wedge}{\sigma}_i$.

We now suppose that the eigenvectors of $J_n(\overset{\wedge}{\sigma}_i)$ are calculated such that

$$J_n(\overset{\wedge}{\sigma}_i)V_{j,i} = x_{j,i,s}V_{j,i}, \qquad j = 1, ..., n,$$

with $V_{j,i}^T V_{j,i} = 1$ and $V_{j,i}^T = (v_{1,j,i}, ..., v_{n,j,i})$. Then, as in the ordinary Gauss quadrature formula, it follows from (4.20) that $w_{j,i,s}$ are expressible in terms of the first components $v_{1,j,i}$ of $V_{j,i}$ by

$$w_{j,i,s} = \frac{v_{1,j,i}^2}{(x_{j,i,s}-\zeta_i)(\zeta_{i+1}-x_{j,i,s})}\beta_{0,i}(d\overset{\wedge}{\sigma}_i), \qquad j = 1, ..., n.$$

We summarize this construction process in the following theorem.

Theorem 4.1. *Given* $\mathbf{\Delta} = (\zeta_0, ..., \zeta_{r+1})$ *a sequence of nodes with*

$$-1 = \zeta_0 < \zeta_1 < ... < \zeta_r < \zeta_{r+1} = 1,$$

and $\mathbf{M} = (m_1, ..., m_r)$ *a vector of integers of the form*

$$m_i = 2n - 1, \qquad i = 1, ..., r.$$

Define

$$s_i = \begin{cases} n, & \text{if } \ i \text{ is odd }, \\ n-1, & \text{if } \ i \text{ is even.} \end{cases}$$

Let i *be an odd integer such that* $1 \leq i \leq r+1$ *and let* $J_n(\overset{\wedge}{\sigma}_1)$, $J_n(\overset{\wedge}{\sigma}_i)$ *and* $J_n(\overset{\wedge}{\sigma}_{r+1})$ *be the matrices defined, respectively, as in (4.11), (4.15) and (4.18). Then, there exists a unique quadrature formula of the form,*

$$Q_{2n-1}(f) = \sum_{i=1}^{r+1} \sum_{j=1}^{s_i} w_{j,i,s} f(x_{j,i,s}), \tag{4.21}$$

which exactly integrates all spline functions of $\mathcal{S}(\mathcal{P}_{2n-1}; \mathbf{\Delta}; \mathbf{M})$*. With, if* i *is even and* $2 \leq i \leq r$*, the nodes* $x_{j,i,s}$, $j = 1, ..., n-1$, *in (4.21) are those of the classical polynomial Gauss quadrature formula* $(2n-3, n-1, (t-\zeta_{i-1})(\zeta_i - t)d\sigma)$ *and the weights* $w_{j,i,s}$ *corresponding to* $x_{j,i,s}$ *are given in term of the Christoffel numbers* $w^G_{j,i,s}$ *of* $(2n-3, n-1, (t-\zeta_{i-1})(\zeta_i - t)d\sigma)$ *by*

$$w_{j,i,s} = \frac{w^G_{j,i,s}}{(x_{j,i,s} - \zeta_i)(\zeta_{i+1} - x_{j,i,s})}, \qquad j = 1, ..., n-1.$$

If i *is odd and* $1 \leq i \leq r+1$*, the nodes* $x_{j,i,s}$, $j = 1, ..., n$, *in (4.21) are the eigenvalues of* $J_n(\overset{\wedge}{\sigma}_i)$ *and the respective weights are given by*

$$w_{j,i,s} = \frac{v^2_{1,j,i}}{\hat{k}_i(x_{j,i,s})} \beta_{0,i}(d\overset{\wedge}{\sigma}_i), \qquad j = 1, ..., n.$$

with $v_{1,j,i}$ *being the first components of the normalized eigenvectors of* $J_n(\overset{\wedge}{\sigma}_i)$ *corresponding to the eigenvalues* $x_{j,i,s}$.

There are several important corollaries of Theorem 4.1 that are of interest in applications, the most important one relates to the *Legendre weight function* $w(t) = 1$ on $[-1, 1]$; the corresponding quadrature rule will be referred to as the spline *Gauss-Legendre* quadrature formula (SGL). In this particular case, we can prove results that contain more information about the coefficients of the quasi-orthogonal polynomials (4.8) and (4.16). In fact, the latter can be determined explicitly and obtained from the known relations of the Jacobi orthogonal polynomials and the following formula [2, Formula 4, p. 263],

$$\int_{-1}^{1} w_{\rho,\beta}(t)\, p_n^{(\alpha,\beta)}(t)dt = \frac{2^{\beta+\rho+1}\Gamma(\rho+1)\Gamma(\beta+n+1)\Gamma(\alpha-\rho+n)}{n!\Gamma(\alpha-\rho)\Gamma(\beta+\rho+n+2)}, \ \rho < \alpha,$$

where $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ $(\alpha, \beta > -1)$, and $p_n^{(\alpha,\beta)}$ being the Jacobi polynomial on $[-1, 1]$ with parameters α, β and Γ the Gamma function. We leave the details to the reader.

Theorem 4.2. *Let* $\mathbf{\Delta}$, $\mathbf{M}$,*and* $s_i, i = 1, ..., r+1$, *be defined as in Theorem 4.1. Define*

$$h_i = \zeta_i - \xi_{i-1}, \qquad i = 1, ..., r+1.$$

Let i be an odd integer such that $1 \leq i \leq r+1$, and let $J_n(\overset{\wedge}{\sigma}_i)$ be the (n)-th order tridiagonal matrix defined by $\alpha_{k,i}$ on the main diagonal and $\sqrt{\beta_{k,i}}$ on the side diagonals, where

$$\begin{aligned} \alpha_{k,i} &= 0, & k=0,...,n-2, \\ \beta_{k,i} &= \frac{k(k+4)}{(2k+3)(2k+5)}, & k=1,...,n-2, \end{aligned}$$

and

$$\alpha_{n-1,i} = -a_{n,i},\ \beta_{n-1,i} = \frac{(n-1)(n+3)}{(2n+1)(2n+3)} - b_{n,i},$$

with

$$\begin{aligned} a_{n,1} &= -\frac{n+1}{2n+1}\frac{nh_1+(n+1)h_2}{(n+1)h_1+nh_2}, \qquad b_{n,1}=0, \\ a_{n,i} &= \frac{n(n+2)}{(n+1)(2n+1)}\frac{c_1b_3-b_1c_3}{b_2c_3-c_2b_3}, \quad b_{n,i} = \frac{(n+2)(n-1)}{(2n-1)(2n+1)}, \quad i=1,...,r, \\ a_{n,r+1} &= \frac{n+1}{2n+1}\frac{nh_{r+1}+(n+1)h_r}{(n+1)h_{r+1}+nh_r}, \quad b_{n,r+1}=0, \end{aligned}$$

and

$$c_1 = \frac{h_{2i}}{n} + \frac{h_{2i+1}}{n+2},\ c_2 = -\frac{(h_{2i}+h_{2i+1})}{n+1},\ c_3 = \frac{(n-1)h_{2i}+(n+1)h_{2i+1}}{n(n+1)},$$

$$b_1 = \frac{h_{2i+2}}{n} + \frac{h_{2i+1}}{n+2},\quad b_2 = \frac{(h_{2i+2}+h_{2i+1})}{n+1},\quad b_3 = \frac{(n-1)h_{2i+2}+(n+1)h_{2i+1}}{n(n-1)}.$$

Then, there exists a unique quadrature formula of the form,

$$Q_{2n-1}(f) = \sum_{i=1}^{r+1}\sum_{j=1}^{s_i} w_{j,i,s} f(x_{j,i,s}), \tag{4.22}$$

which exactly integrates all spline functions of $\mathcal{S}(\mathcal{P}_{2n-1};\mathbf{\Delta};\mathbf{M})$. With, if i is even and $2 \leq i \leq r$, the nodes $x_{j,i,s}$, $j=1,...,n-1$, are those of the classical polynomial Gauss quadrature formula $(2n-3, n-1, (t-\zeta_{i-1})(\zeta_i - t)dt)$ and the weights $w_{j,i,s}$ corresponding to $x_{j,i,s}$ are given in term of the Christoffel numbers $w^G_{j,i,s}$ of $(2n-3, n-1, (t-\zeta_{i-1})(\zeta_i - t)dt)$by

$$w_{j,i,s} = \frac{w^G_{j,i,s}}{(x_{j,i,s}-\zeta_i)(\zeta_{i+1}-x_{j,i,s})}, \qquad j=1,...,n-1.$$

If i is odd and $1 \leq i \leq r+1$, the nodes located in $(\zeta_{i-1},\zeta_i)$are of the form

$$x_{j,i,s} = \frac{(h_i x^*_{i,j,s} + \zeta_{i-1} + \xi_i)}{2},\ j=1,...,n$$

with $x^*_{j,i,s}$, $j=1,...,s_i$ are the eigenvalues of $J(\overset{\wedge}{\sigma}_i)$ and the respective weights are given by

$$w_{j,i,s} = \frac{4v^2_{1,j,i}}{3(1-x^{*2}_{j,i,s})}, \qquad j=1,...,n.$$

with $v_{1,j,i}$ being the first components of the normalized eigenvectors of $J_n(\overset{\wedge}{\sigma}_i)$ corresponding to the eigenvalues $x_{j,i,s}$.

The construction of quadrature formulae of type (4.22), with Jacobi weight functions, presents no extra difficulties; a detailed discussion on such quadrature formulae can be found in [8]. The latter can be done in precisely the same way that these results were established for the Gauss-Legendre case. We therefore omit this development here. For reasons of clarity we have considered only on the problem of determining the nodes and weights of (1.2), which do not use boundary. We mention that the concepts are generalized to include Gauss-Radau and Gauss-Lobatto quadrature type formulae for splines. The construction becomes somewhat more complicated in the presence of boundaries. Space limitation prevents us from presenting this generalization here.

5. COMPARATIVE NUMERICAL RESULTS

As expressed by Christoffel (cf. [10, p. 86]), the use of preassigned nodes in quadrature formulae, chosen judiciously at locations where the integrand function is predominant, should be advantageous. In order to demonstrate that the use of preassigned nodes can indeed be helpful, we compare the spline quadrature formula (4.22) developed in Theorem 4.2 and the polynomial Gauss-Legendre quadrature formula, that uses the same number of evaluations of integrand. For illustration, we experiment with numerical examples which involve three different kinds of integration problems. They differ mostly in the specific properties of the integrand functions. The first one is a parametrized family of functions which have a peak at a point in $[-1, 1]$ with a severity that is controlled by two parameters. In the second example, in another context, we will choose a whole family of integrands where the oscillations increase. We conclude with one final numerical example in which the integrand has a logarithmic endpoint singularity. The three examples are as follows:

$$\begin{aligned} I_1(u,v) &= \int_{-1}^{1} \tfrac{10^{-u}}{(x-v)^2+10^{-2u}} dx, \\ &= \arctan(10^u(1-v)) - \arctan(10^u(-1-v)), \end{aligned}$$

$$I_2(m) = \int_0^{2\pi} x\cos(50x)\sin(mx)dx = \frac{2m\pi}{2500-m^2}, \ (m \neq 50),$$

$$I_3 = \int_0^1 \frac{\log x}{(1+x)^2} dx = -\log 2.$$

All the computations described in this paper were carried out on a personal IBM computer in a double precision.

Example 5.1. *In the first example, we take the following test function* $f_{u,v}(x) = \frac{10^{-u}}{(x-v)^2+10^{-2u}}$. *The parameter* v *is equal to the location of the peak, while the parameter* u *determines the height* 10^u *of the peak at* $x = v$. *One would expect that the difficulty of the integrand* f *depends heavily on the location of the peak, i. e. on the selection of the parameter* v, *and on the height of the peak, i. e. on the size of the parameter* u. *It is natural, then, to employ our quadrature formula SGL given in (4.22), choosing the preassigned nodes of splines judiciously at the location where the integrand function is predominant. This suggests to choose the nodes* ζ_i *in the neighborhood of the peak* v. *The integral of* $f_{u,v}$ *was approximated numerically by using the new quadrature formulae SGL*$_{140}$ *and the ordinary Gauss quadrature formula having the same number of the nodes* (140). *The results of SGL in this case have been obtained by using* $r = 6$ *and* $\zeta_i, i = 1, ..., 6$, *chosen in a neighborhood of* 0.

In Table 5.1, the results of SGL are shown for $u = 1.5, 2, 2.5$ *and* $v = 0, 0.5, 0.9$. *Also shown in the last two columns are the respective integration errors. As we can see from Table 5.1, for this family of functions, the new quadrature formulae compare favorably with the ordinary Gaussian quadrature formulae and in all cases give better results by several orders of magnitude. It is worth noting that the integration error of the ordinary quadrature formula for* $f_{u,v}$ *depends heavily on the location of the peak. This weakness is accentuated when the peak is moderately high. We also have observed that the numerical results of GL, as the peak becomes relatively important, converge rather slowly. It is interesting to note the catastrophic loss in accuracy produced by GL in this case, particularly when the peak* v *is very close to* 0. *The latter*

u	*v*	GL_{140}	SGL_{140}
	0	*8.7(-4)*	*1.5(-11)*
1.5	*0.5*	*1.7(-3)*	*2.7(-9)*
	0.9	*6.1(-9)*	*4.8(-15)*
	0	*3.5(-1)*	*2.6(-8)*
2	*0.5*	*1.6(-1)*	*1.2(-6)*
	0.9	*9.4(-3)*	*4.1(-9)*
	0	*1.8*	*1.3(-5)*
2.5	*0.5*	*2.2*	*1.5(-5)*
	0.9	*6.6(-1)*	*1.2(-6)*

Table 5.1: Numerical results of $I_1(u, v)$ and comparison with Gauss quadrature. (Numbers in parentheses denote decimal exponents.)

failure is easily explained: it is due to the closing of the nodes of GL towards the boundary. This fact is also confirmed by the "good" results of GL in the "corner peak" case. This emphasizes that when a polynomial Gauss quadrature formula is employed for integrand with a moderate peak, care must taken. In contrast, SGL does not suffer from any numerical instability and gives best accuracy.

Example 5.2. *Here $f(x) = x\cos(50x)\sin(mx)$ is highly oscillatory. In Table 5.2 we compare the performance of our quadrature formula SGL with Gauss quadrature formula. The results for SGL have been obtained by using $r = 10$, with $\zeta_i = -1 + 2i/11$, $i = 1, ..., 10$. We tabulate the results for $m = 20, 30, 40$; $N = 66, 110, 165, 220, 275, 330$. It is interesting to note the poor quality and the unreliability of GL as the oscillations become important. However, for small n, the GL seems to be competitive, as is seen in Table 5.2. The convergence of GL for large is very slow, and the comparison with SGL is striking. We also have observed that the numerical results of SGL is extremely sensitive with respect to the correct choice of the node ζ_i . As expected, the best results of SGL were achieved by choosing the nodes ζ_i close to the zeros of f.*

Example 5.3. *The integrand here is a function having a logarithmic endpoint singularity of the type $f(x) = \frac{\log x}{(1+x)^2}$. Our approach to deal with the singularity and retain the high accuracy of our quadrature formula SGL is to impose a relatively fine mesh subdivision in a neighborhood of 0. In Table 5.3, we compare the results of SGL with those obtained, at the same number of the nodes, by the GL quadrature formula. We did our computations for $N = 50, 100, 150$ and $r = 4$. A reasonable choice of $\zeta_i, i = 1, ..., 4$ would be $\zeta_1 = 0.00005$, $\zeta_2 = 0.0005$, $\zeta_3 = 0.005$ and $\zeta_4 = 0.05$, since this choice makes a great possible use of nodes closet to 0. It can be seen that the new quadrature formula produces results which are in several orders of magnitude larger that those furnished by the ordinary Gauss quadrature formula.*

6. CONCLUDING REMARKS

In this paper, we have shown how to modify the Jacobi matrix to obtain an efficient algorithm to compute a new class of Gaussian type quadrature formulae for splines. An important practical aspect of these quadratures is, as in polynomial case, that the latter are computed via eigenvalues and eigenvectors of real symmetric tridiagonal matrices. Therefore this algorithm is simple from an implementation standpoint.

The results of this paper can be improved upon and extended in several directions:

m	*N*	*err. GL*	*err. SGL*
20	66	9.0(-1)	2.3(-1)
	110	3.2(-1)	1.9(-1)
	165	5.9(-2)	1.4(-4)
	220	5.9(-2)	1.2(-9)
	275	5.9(-2)	1.4(-14)
	330	5.9(-2)	3.4(-17)
30	66	5.7(-1)	9.4(-1)
	110	1.8(-1)	1.6(-2)
	165	1.1(-1)	1.1(-3)
	220	1.1(-1)	5.1(-8)
	275	1.1(-1)	1.2(-13)
	330	1.1(-1)	3.9(-15)
40	66	6.5(-1)	9.9(-1)
	110	9.0(-1)	2.1(-2)
	165	2.7(-1)	7.4(-3)
	220	2.7(-1)	1.5(-6)
	275	2.7(-1)	1.3(-11)
	330	2.7(-1)	7.7(-15)

Table 5.2: Numerical results of $I_2(m)$ and comparison with Gauss quadrature.

N	*err. GL*	*err. SGL*
50	2.4(-4)	3.8(-6)
100	6.2(-5)	2.7(-9)
150	2.7(-5)	2.6(-10)

Table 5.3: Numerical results of I_3 and comparison with Gauss quadrature.

(1) The characterizations presented in this paper have also been applied to the computation of a new family of quadrature formulae (1.2) that use end conditions common in applications. We remark that the difficulty in extending the method used in this paper to the general boundary conditions lies primarily in determination of the exact number of points required by the quadrature formula between two nodes of splines. This difficulty can be, in general, overcome on some boundary conditions by using theorem 3.4. We refer to some recent results which were found together with Ezzirani, see the thesis Ezzirani (1997) for more information in this subject. We also have done construction (not published) for $\mathcal{C}^d-$spline Gaussian quadrature formulae with $d = 1$. It is tempting to go further than this, for example to consider the case $d \geq 2$, but we believe that the calculation of such quadrature formulae is really a difficult task.

(2) Many integrands of practical interest are characterized by small regions in which they have complex and varying peaks surrounded by regions where they are relatively smooth. Efficient quadrature formulae for such integrands required an adaptive mesh refinement. In our approach, we have some freedom in choosing the mesh which defines the spline spaces. The main advantage of using the class of new quadrature formulae lies in its great flexibility which offers the user various selection of the latter. As we note with the examples presented in section 5, the results are extremely sensitive with respect to such choice and good results can often be obtained by carefully adjusting the mesh. Hence, the construction of adaptive nonuniform

meshes is a crucial part of these quadrature formulae, we did not use this approach here. Thus the next logical step in this work is to develop an algorithm for refinement strategies that is necessary to perform efficiently such quadrature formulae. For a numerical point of view, this technique is much more efficient than the use of uniform meshes when the integrand is changing much more rapidly. Such methods have been examined by many researchers in the case of polynomial quadrature formulae, see, for example, [18] for more information in this subject.

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