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MULTIVALUED EQUILIBRIUM PROBLEMS WITH TRIFUNCTION

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ABSTRACT. In this paper, we use the auxiliary principle technique to suggest some new classes of iterative algorithms for solving multivalued equilibrium problems with trifunction. The convergence of the proposed methods either requires partially relaxed strongly monotonicity or pseudomonotonicity. As special cases, we obtain a number of known and new results for solving various classes of equilibrium and variational inequality problems. Since multivalued equilibrium problems with trifunction include equilibrium, variational inequality and complementarity problems as specials cases, our results continue to hold for these problems.

Key words and phrases: Equilibrium problems, Variational inequalities, Auxiliary principle, Iterative methods, Convergence.

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1. INTRODUCTION

Equilibrium problems theory is an interesting and fascinating branch of applicable mathematics with a wide range of applications in industry, physical, regional, social, pure and applied sciences. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields, see [1], [3], [4], [14], [16]-[19]. Equilibrium problems have been generalized and extended in different directions using the novel and innovative techniques. Inspired and motivated by the recent research going on in this area, we introduce and consider a class of equilibrium problems, which is called multivalued general equilibrium with trifunction. It is known [14], [16]-[19] that multivalued equilibrium problems include general equilibrium , variational inequality and complementarity problems as special cases. There are several numerical methods including projection methods, Wiener-Hopf equations, descent and decomposition for solving variational inequalities, see [3]-[13], [15], [18], [20], [21]. On the other hand, there are no such methods for solving equilibrium problems, since it is not possible to find the projection. To overcome these drawbacks, one usually uses the auxiliary principle technique to suggest some iterative methods for solving equilibrium problems. The auxiliary principle technique is mainly due to Lions and Stampacchia [7]. Glowinski, Lions and Tremolieres [5] used this approach to study the existence of a solution of the mixed variational inequalities. In recent years, Noor[14]-[17] has used this technique to study some predictor-corrector methods for various classes of equilibrium and variational inequality problems. In this paper, we again use the auxiliary principle technique to suggest a class of three-step predictor-corrector iterative methods for multivalued equilibrium problems with trifunction. In particular, we show that one can obtain various forward-backward splitting, modified projection, and other methods as special cases from these methods. We also prove that the convergence of the suggested methods requires only the partially relaxed strongly monotonicity. Using the auxiliary principle technique, we also suggest and analyze an inertial proximal method for solving multivalued equilibrium problems. We show that the convergence of the inertial proximal method converges for pseudomonotone functions, which is a weaker condition than monotonicity. It is worth mentioning that inertial proximal method include the classical proximal method as a special case. Consequently, our results represent an improvement and refinement of the previously known results. Our results can be considered as an important and significant extension of the previously known results for solving general equilibrium, variational inequality and complementarity problems.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$, respectively. Let $C(H)$ be the family of all non-empty compact subsets of H. Let $T : H \longrightarrow C(H)$ be a multivalued operator and $g : H \longrightarrow H$ be a single-valued operator. Let K be a nonempty, closed and convex set in H.

For a given single-valued trifunction $F(.,.,.) : H \times H \times H \longrightarrow C(H),$ we consider the problem of finding $u \in H, g(u) \in K, \nu \in T(u),$ such that

$$F(u,\nu,g(v)) \geq 0, \quad \forall g(v) \in K, \tag{2.1}$$

which is called the *multivalued general equilibrium problem with trifunction.* It can be shown that a wide class of problems arising in various branches of pure and applied sciences can be studied in the general framework of multivalued equilibrium problems. For $g = I,$ the identity operator, we obtain the multivalued equilibrium problems considered and studied by Noor and Oettli [19] and Noor [17] using quite different techniques.

If $T : H \longrightarrow H$ is a single-valued operator, then problem 2.1 is equivalent to finding $u \in H$ such that

$$F(u, Tu, g(v)) \geq 0, \quad \forall g(v) \in K, \tag{2.2}$$

which is called the general equilibrium problem with trifunction. If $g = I$, where I is the identity operator, problem 2.2 was introduced and studied by Noor [17].

If $F(u, \nu, g(v)) \equiv F(\nu, g(v))$, then problem 2.1 is equivalent to finding $u \in H : g(u) \in K, \quad \nu \in T(u)$ such that

$$F(\nu, g(v)) \geq 0, \quad \forall g(v) \in K, \tag{2.3}$$

which is known as the multivalued general equilibrium problem, introduced and studied by Noor [16]. If T is a single-valued operator and $g = I$, the identity operator, we obtain the original equilibrium problems considered and studied by Blum and Oettli [1] and Noor and Oettli [19] in 1994.

If $F(u, \nu, g(v)) = \langle \nu, g(v) - g(u) \rangle$, then problem 2.1 is equivalent to finding $u \in H, \nu \in T(u), g(u) \in K$ such that

$$\langle \nu, g(v) - g(u) \rangle \quad \geq \quad 0, \qquad \forall g(v) \in K. \tag{2.4}$$

The inequality of type 2.4 is called the *multivalued variational inequality*. It is known that a wide class of multivalued odd order and nonsymmetric free, obstacle, moving, equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued variational inequalities 2.4, see, for example, Noor [10].

We note that, if $T : H \longrightarrow H$ is a single-valued operator, then problem 2.4 is equivalent to finding $u \in H, \quad g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \quad \geq \quad 0, \quad \forall g(v) \in K, \tag{2.5}$$

which is known as the general variational inequality, introduced and studied by Noor [9] in 1988. Problem 2.5 is a quite general and unified one. It has been shown that a class of quasi-variational inequalities, odd-order and nonsymmetric free, moving, unilateral, obstacle and non-convex programming problems can be studied by the general variational inequality approach, see [10]-[13],[15].

We remark that, if $g \equiv I$, the identity operator, then problem 2.4 is equivalent to finding $u \in K, \nu \in T(u)$ such that

$$\langle \nu, v - u \rangle \quad \geq \quad 0, \qquad \forall v \in K, \tag{2.6}$$

which are called the generalized variational inequalities introduced and studied by Fang and Peterson [2]. For the applications, numerical methods and formulations, see [2], [10], [12] and the references therein.

If $K^* = \{u \in H : \langle u, v \rangle \geq 0, \forall v \in K\}$ is a polar cone of a convex cone K in H, then problem 2.4 is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad \nu \in T(u) \subseteq K^*, \quad \text{and} \quad \langle \nu, g(u) \rangle = 0, \tag{2.7}$$

which is known as the multivalued complementarity problem. We note that if $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem 2.7 is called the multivalued quasi(implicit) complementarity problem.

It is clear that problems 2.2-2.7 are special cases of the multivalued variational inequality 2.1. In brief, for a suitable and appropriate choice of the operators $F(.,.)$, T, g, and the space H, one can obtain a wide class of equilibrium, variational inequalities and complementarity problems. This clearly shows that problem 2.1 is quite general and unifying one. Furthermore, problem 2.1 has many important applications in various branches of pure and applied sciences, see [1]-[21].

We also need the following well known results and concepts.

Lemma 2.1. $\forall u, v \in H$, *we have*

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2 \tag{2.8}$$

Definition 2.1. $\forall u_1, u_2, z \in H, w_1 \in T(u_1), w_2 \in T(u_2)$, the trifunction $F(.,.,.) : H \times H \times H \longrightarrow C(H)$ and the operator T is said to be:

(i) *partially relaxed strongly jointly g-monotone,* iff, there exists a constant $\alpha > 0$, such that

$$F(u_1, w_1, g(u_2)) + F(u_2, w_2, g(z)) \le \alpha\|g(z) - g(u_1)\|^2$$

(ii) *jointly g-monotone,* iff,

$$F(u_1, w_1, g(u_2)) + F(u_2, w_2, g(u_1)) \le 0.$$

(iii) *jointly g-pseudomonotone,* iff,

$$F(u_1, w_1, g(u_2)) \ge 0, \quad \text{implies} \quad F(u_2, w_2, g(u_1)) \le 0.$$

Definition 2.2. $\forall u_1, u_2 \in H, w_1 \in T(u_1), w \in T(u_2)$, the multivalued operator $T : H \longrightarrow C(H)$ is said to be *M-Lipschitz continuous,* iff, there exists a constant $\delta > 0$, such that

$$M(T(u_1), T(u_2)) \le \delta||u_1 - u_2||,$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$.

We remark that, if $z = u_1$, then partially relaxed strongly g-monotonicity is exactly g-monotonicity of $F(.,.,.)$. For $g \equiv I$, the indentity operator, Definition 2.1 reduces to the definition of partially relaxed strongly monotonicity, monotonicity and pseudomonotonicity of the trifunction $F(.,.,.)$.

3. MAIN RESULTS

In this section, we suggest and analyze a class of iterative methods for solving the problem 2.1 by using the auxiliary principle technique.

For a given $u \in H : g(u) \in K, \nu \in T(u)$, consider the problem of finding a solution $w \in H$, $g(w) \in K$, satisfying the auxiliary equilibrium problem

$$\rho F(u, \nu, g(v)) + \langle g(w) - g(u), g(v) - g(w)\rangle \quad \ge \quad 0, \quad \forall g(v) \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that, if $w = u$, then clearly w is a solution of the multivalued equilibrium problem 2.1. This observation enables us to suggest the following predictor-corrector method for solving the multivalued equilibrium problem 2.1.

Algorithm 1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho F(w_n, \eta_n, g(v)) + \langle g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle \ge \quad 0, \quad \forall g(v) \in K \tag{3.2}$$

$$\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \le M(T(w_{n+1}), T(w_n)) \tag{3.3}$$

$$\beta F(y_n, \xi_n, g(v)) + \langle g(w_n) - g(y_n), g(v) - g(w_n)\rangle \quad \ge 0, \quad \forall g(v) \in K \tag{3.4}$$

$$\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \le M(T(y_{n+1}), T(y_n)) \tag{3.5}$$

and

$$\mu F(u_n, \nu_n, g(v)) + \langle g(y_n) - g(u_n), g(v) - g(y_n)\rangle \quad \geq 0, \quad \forall g(v) \in K. \tag{3.6}$$

$$\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots \tag{3.7}$$

where $\rho > 0$, $\mu > 0$ and $\beta > 0$ are constants.

Note that, if $g \equiv I$, the identity operator, then Algorithm 1 reduces to the following predictor-corrector method for solving the multivalued equilibrium problem.

Algorithm 2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned}
&\rho F(w_n, \eta_n, v) + \langle u_{n+1} - w_n, v - u_{n+1}\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(w_{n+1}), T(w_n)) \\
&\beta F(y_n, \xi_n, v) + \langle w_n - y_n, v - w_n\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leq M(T(y_{n+1}), T(y_n)) \\
&\mu F(u_n, \nu_n, v) + \langle y_n - u_n, v - y_n\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2 \ldots
\end{aligned}$$

If $F(u, \nu, g(v)) = \langle \nu, g(v) - g(u)\rangle$, then Algorithm 1 reduces to the following algorithm for solving multivalued variational inequalities 2.3.

Algorithm 3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
&\langle \rho\eta_n + u_{n+1} - w_n, v - u_{n+1}\rangle \quad \geq \quad 0, \quad \forall v \in K, \\
&\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(w_{n+1}), T(w_n)) \\
&\langle \beta\xi_n + w_n - y_n, v - w_n\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leq M(T(y_{n+1}), T(y_n)) \\
&\langle \mu\nu_n + y_n - u_n, v - y_n\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2 \ldots
\end{aligned}$$

which can be written as

Algorithm 4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
g(u_{n+1}) &= P_K[g(w_n) - \rho\eta_n] \\
\eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)) \\
g(w_n) &= P_K[g(y_n) - \beta\xi] \\
\xi_n \in T(y_n) &: \|\xi_{n+1} - \xi_n\| \leq M(T(y_{n+1}), T(y_n)) \\
g(y_n) &= P_K[g(u_n) - \mu\nu] \\
\nu_n \in T(u_n) &: \|\nu_{n+1} - \nu_n\| \leq M(T(u_{n+1}), T(u_n)),
\end{aligned}$$

where P_K is the projection of H onto the closed convex set K. Algorithm 4 is known as the predictor-corrector method for solving the multivalued variational inequalities 2.4, see [12], [13].

If T is a single-valued operator, then Algorithms 3 and 4 reduce to:

Algorithm 5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned}
\langle \rho T(w_n)) + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle &\geq 0, \quad \forall g(v) \in K \\
\langle \beta T(y_n) + g(w_n) - g(y_n), g(v) - g(w_n)\rangle &\geq 0, \quad \forall g(v) \in K \\
\langle \mu T(u_n) + g(y_n) - g(u_n), g(v) - g(y_n)\rangle &\geq 0, \quad \forall g(v) \in K,
\end{aligned}$$

which is called the predictor-corrector method for solving general variational inequalities 2.5, see Noor [11]-[13].

We remark that Algorithm 5 can be written in the following equivalent form as

Algorithm 6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned}
g(y_n) &= P_K[g(u_n) - \mu T u_n] \\
g(w_n) &= P_K[g(y_n) - \beta T(y_n)] \\
g(u_{n+1}) &= P_K[g(w_n) - \rho T(w_n)], \quad n = 0, 1, 2 \ldots
\end{aligned}$$

which can be written in the following form, if g is invertible,

$$g(u_{n+1}) = P_K[I - \rho T g^{-1}]P_K[I - \beta T g^{-1}]P_K[I - \mu T g^{-1}]g(u_n), \quad n = 0, 1, 2 \ldots$$

Algorithm 6 is known as three-step forward-backward splitting algorithms. Algorithm 6 is similar to the so-called θ-scheme of Glowinski and Le Tallec [6], which they suggested by using the Lagrangian multiplier method. It has been shown in [6] that three-step schemes are numerically efficient and are reasonably easy to use for computations as compared with one-step and two-step iterative methods for solving nonlinear problems arising in elasticity and mechanics. The convergence analysis of Algorithm 6 has been considered by Noor [11]-[13].

We now rewrite Algorithm 3 in the following form:

Algorithm 7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
u_{n+1} &= (1 - \rho_n)u_n + \rho_n\{u_n - g(u_n) + P_K[g(w_n) - \rho_n \eta_n]\} \\
\eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)) \\
w_n &= (1 - \beta_n)u_n + \beta_n\{w_n - g(w_n) + P_K[g(y_n) - \beta_n \xi_n]\} \\
\xi_n \in T(y_n) &: \|\xi_{n+1} - \xi_n\| \leq M(T(y_{n+1}), T(y_n)) \\
y_n &= (1 - \mu_n)u_n + \mu_n\{y_n - g(y_n) + P_K[g(y_n) - \mu_n \nu_n]\} \\
\nu_n \in T(u_n) &: \|\nu_{n+1} - \nu_n\| \leq M(T(u_{n+1}), T(u_n)),
\end{aligned}$$

where the sequences $\{\rho_n\}, \{\beta_n\}, \{\mu_n\}$ satisfy some certian conditions.

Algorithm 7 is also known as three-step (Noor) iteration process. Clearly Ishikawa and Mann iterations are special cases of Noor (three-step) iterations.

Clearly for $K = H$ and a single-valued operator T with $g = I$, the identity operator, Algorithm 7 collapses to the following three-step iterative method for solving nonlinear equation $Tu = 0$ which has been studied in the Banach spaces setting.

Algorithm 8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n T w_n \\
w_n &= (1 - \beta_n)u_n + \beta_n T y_n \\
y_n &= (1 - \mu_n)u_n + \mu_n T u_n, \quad n = 0, 1, 2, \ldots
\end{aligned}$$

Algorithm 8 is well known three-step (Noor iteration) iterative method which has been studied extensively in recent years. It is obvious that the three-step iterative method includes Ishikawa-Mann iterations as special cases.

For a suitable choice of the operators and the space H, one can obtain various new and known methods for solving equilibrium, variational inequality and complementarity problems.

For the convergence analysis of Algorithm 1, we need the following result.

Theorem 3.1. *Let $u \in H$ be the exact solution of 2.1 and u_{n+1} be the approximate solution obtained from Algorithm 1. If the bifunction $F(.,.,.)$ is a partially relaxed strongly g-monotone operator with constant $\alpha > 0$, then*

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(w_n) - g(u)\|^2 - (1-2\rho\alpha)\|g(u_{n+1}) - g(w_n)\|^2 \tag{3.8}$$

$$\|g(w_n) - g(u)\|^2 \leq \|g(y_n) - g(u)\|^2 - (1-2\alpha\beta)\|g(y_n) - g(w_n)\|^2 \tag{3.9}$$

$$\|g(y_n) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - (1-2\alpha\mu)\|g(y_n) - g(u_n)\|^2. \tag{3.10}$$

Proof. Let $u \in H, \quad \nu \in T(u)$ be solution of 2.1. Then

$$\rho F(u,\nu,g(v)) \geq 0, \quad \forall g(v) \in K \tag{3.11}$$

$$\beta F(u,\nu,g(v)) \geq 0, \quad \forall g(v) \in K \tag{3.12}$$

$$\mu F(u,\nu,g(v)) \geq 0, \quad \forall g(v) \in K, \tag{3.13}$$

where $\rho > 0$, $\beta > 0$ and $\mu > 0$ are constants.

Now taking $v = u_{n+1}$ in 3.11 and $v = u$ in 3.2, we have

$$\rho F(u,\nu,g(u_{n+1})) \geq 0 \tag{3.14}$$

and

$$\rho F(w_n,\eta_n,g(u)) + \langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle \geq 0. \tag{3.15}$$

Adding 3.14 and 3.15, we have

$$\begin{aligned}\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle &\geq -\rho\{F(w_n,\eta_n,g(u)) + F(u,\nu,g(u_{n+1}))\} \\ &\geq -\alpha\rho\|g(u_{n+1}) - g(w_n)\|^2,\end{aligned} \tag{3.16}$$

where we have used the fact that $F(.,.,.)$ is partially relaxed strongly g-monotone with constant $\alpha > 0$.

Setting $u = g(u) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(w_n)$ in 2.8, we obtain

$$\begin{aligned}\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle &= \frac{1}{2}\{\|g(u) - g(w_n)\|^2 - \|g(u) - g(u_{n+1})\|^2 \\ &- \|g(u_{n+1}) - g(w_n)\|^2\}.\end{aligned} \tag{3.17}$$

Combining 3.16 and 3.17, we have

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(w_n) - g(u)\|^2 - (1-2\alpha\rho)\|g(u_{n+1}) - g(w_n)\|^2,$$

the required 3.8.

Taking $v = u$ in 3.4 and $v = w_n$ in 3.12, we have

$$\beta F(u,\nu,g(w_n)) \geq 0 \tag{3.18}$$

and

$$\beta F(y_n,\xi_n,g(u)) + \langle g(w_n) - g(y_n), g(u) - g(w_n)\rangle \geq 0. \tag{3.19}$$

Adding 3.18 and 3.19 and rearranging the terms, we have

$$\begin{aligned}\langle g(w_n) - g(y_n), g(u) - g(w_n)\rangle &\geq -\beta\{F(y_n,\xi_n,g(u)) + F(u,\nu,g(w_n))\} \\ &\geq -\beta\alpha\|g(y_n) - g(w_n)\|^2,\end{aligned} \tag{3.20}$$

since $F(.,.,.)$ is a partially relaxed strongly g-monotone operator with constant $\alpha > 0$.

Now taking $v = g(w_n) - g(y_n)$ and $u = g(u) - g(w_n)$ in 2.8, 3.20 can be written as

$$\|g(u) - g(w_n)\|^2 \leq \|g(u) - g(y_n)\|^2 - (1 - 2\beta\alpha)\|g(y_n) - g(w_n)\|^2,$$

the required 3.9.

Similarly, by taking $v = u$ in 3.6 and $v = u_{n+1}$ in 3.13 and using the partially relaxed strongly g-monotonicity of the operator $F(.,.,.)$, we have

$$\langle g(y_n) - g(u_n), g(u) - g(y_n)\rangle \geq -\mu\alpha\|g(y_n) - g(u_n)\|^2. \tag{3.21}$$

Letting $v = y_n - u_n$, and $u = u - y_n$ in 2.8, and combining the resultant with 3.21, we have

$$\|g(y_n) - g(u)\|^2 \leq \|g(u) - g(u_n)\|^2 - (1 - 2\mu\alpha)\|g(y_n) - g(u_n)\|^2,$$

the required 3.10. ∎

Theorem 3.2. *Let H be a finite dimensional space. Let $g : H \longrightarrow H$ be injective and $0 < \rho < \frac{1}{2\alpha}$, $0 < \beta < \frac{1}{2\alpha}$, $0 < \mu < \frac{1}{2\alpha}$. Let $T : H \longrightarrow C(H)$ be M-Lipschitz continuous operator. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 1 converges to a solution u of 2.1.*

Proof. Let $u \in H$ be a solution of 2.1. Since $0 < \rho < \frac{1}{2\alpha}, 0 < \beta < \frac{1}{2\alpha}$, $0 < \mu < \frac{1}{2\alpha}$, from 3.8-3.10, it follows that the sequences $\{\|g(u) - g(u_n)\|\}, \{\|g(u) - g(y_n)\|\}$, $\{g(u) - g(w_n)\|\}$ are nonincreasing and consequently $\{u_n\}$, $\{y_n\}$ and $\{w_n\}$ are bounded under the assumptions on the operator g. Furthermore, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|g(w_n) - g(u_n)\|^2 &\leq \|g(u) - g(w_0)\|^2 \\
\sum_{n=0}^{\infty}(1 - 2\alpha\beta)\|g(y_n) - g(w_n)\|^2 &\leq \|g(u) - g(y_0)\|^2 \\
\sum_{n=0}^{\infty}(1 - 2\alpha\mu)\|g(y_n) - g(u_n)\|^2 &\leq \|g(u) - g(u_0)\|^2
\end{aligned}$$

which implies that

$$\begin{aligned}
\lim_{n\to\infty}\|g(w_n) - g(u_n)\| &= 0 \\
\lim_{n\to\infty}\|g(y_n) - g(w_n)\| &= 0 \\
\lim_{n\to\infty}\|g(y_n) - g(u_n)\| &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty}\|g(u_{n+1}) - g(u_n)\| &= \lim_{n\to\infty}\|g(u_{n+1}) - g(w_n)\| + \lim_{n\to\infty}\|g(y_n) - g(w_n)\| \\
&= \lim_{n\to\infty}\|g(y_n) - g(u_n)\| = 0.
\end{aligned} \tag{3.22}$$

Let $\hat{u}$ be the limlit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n and y_n by u_{n_j} in 3.2,3.4 and 3.6, taking the limit $n_j \longrightarrow \infty$ and using 3.22, we have

$$F(\hat{u}, \hat{\nu}, g(v)) \geq 0, \quad \forall g(v) \in K,$$

which implies that $\hat{u}$ solves the multivalued equilibrium problems 2.1 and

$$\|g(u_{n+1}) - g(\hat{u})\|^2 \leq \|g(u_n) - g(\hat{u})\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} g(u_n) = g(\hat{u}).$$

Since g is injective, thus

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

It remains to show that $\nu \in T(u)$. From 3.7 and using the M-Lipschitz continuity of T, we have

$$||\nu_n - \nu|| \leq M(T(u_n), T(u)) \leq \delta||u_n - u||,$$

which implies that $\nu_n \longrightarrow \nu$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned} d(\nu, T(u)) &\leq \|\nu - \nu_n\| + d(\nu, T(u)) \\ &\leq \|\nu - \nu_n\| + M(T(u_n), T(u)) \\ &\leq \|\nu - \nu_n\| + \delta\|u_n - u\| \quad \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

where $d(\nu, T(u)) = inf\{\|\nu - z\| : z \in T(u)\}$ and $\delta > 0$ is the M-Lipschitz continuity constant of the operator T. From the above inequality, it follows that $d(\nu, T(u)) = 0$. This implies that $\nu \in T(u)$, since $T(u) \in C(H)$. This completes the proof. ∎

We now use the auxiliary principle technique to suggest an inertial proximal method for solving multi-valued equilibrium problems, which were studied and considered by Noor [14] for solving multivalued equilibrium problems 2.3. We remark that the inertial proximal method includes the proximal method as a special case.

For a given $u \in H, \quad g(u) \in K$, consider the auxiliary problem of finding $w \in H, g(w) \in K, \quad \eta \in T(w)$ such that

$$\rho F(w, \eta, g(v)) + \langle g(w) - g(u) - \alpha(g(u) - g(u)), g(v) - g(w)\rangle \geq 0, \quad \forall g(v) \in K, \tag{3.23}$$

where $\rho > 0$ and $\alpha > 0$ are constants. Note that if $w = u$, then w is a solution of 2.1. We use this fact to suggest the following iterative method for solving 2.1.

Algorithm 9. *For a given $u_0 \in H$, compute the approximate solution by the iterative schemes:*

$$\rho F(w_{n+1}, \eta_{n+1}, g(v)) + \langle g(u_{n+1}) - g(u_n) - \alpha_n(g(u_n) - g(u_{n-1})), g(v) - g(u_{n+1})\rangle \geq 0, \quad \forall g(v) \in K,$$
$$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leq M(T(w_{N+1}), T(w_n)),$$

where $\rho > 0$ and $\alpha_n > 0$ are constants.

Algorithm 9 is known as the inertial proximal method. Note that for $\alpha_n = 0$, Algorithm 9 reduces to:

Algorithm 10. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(w_{n+1}, \eta_{n+1}, g(v)) &+ \langle g(u_{n+1}) - g(u_n), g(v) - g(u_n)\rangle \geq 0, \quad \forall g(v) \in K \\ \eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)), \end{aligned}$$

which is called the proximal method for solving multivalued equilibrium problem 2.1.

If $F(u, \nu, g(v)) = \langle \nu, g(v) - g(u)\rangle$, then Algorithm 9 reduces to:

Algorithm 11. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\langle \rho\eta_{n+1} + g(u_{n+1}) - g(u_n) - \alpha_n(g(u_n) - g(u_{n-1})), g(v) - g(u_{n+1})\rangle \geq 0, \quad \forall g(v) \in K,$$
$$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leq M(T(w_{N+1}), T(w_n)),$$

which can be written as

$$\begin{aligned} g(u_{n+1}) &= P_K[g(u_n) - \rho\eta_{n+1} + \alpha_n(g(u_n) - g(u_{n-1})], \\ \eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \le M(T(w_{N+1}), T(w_n)), \end{aligned}$$

which is known as an inertial proximal method for solving the multivalued variational inequalities and appears to be a new one. Note for $\alpha_n = 0$, Algorithm 11 reduces to the well known proximal method for solving multivalued variational inequalities 2.4. In a similar way, for suitable and appropriate choices of the trifunction $F(.,.,.), T, g$ and the space $H,$ one can obtain a number of new and known iterative methods for solving equilibrium and variational inequality problems. Using the techniques and ideas of Noor [14],[16],[17], one can study the convergence analysis of Algorithm 9 for pseudomonotone trifunction $F(.,.,.)$.

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