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AN APPLICATION OF QUASI POWER INCREASING SEQUENCES

HÜSEYİN BOR

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DEPARTMENT OF MATHEMATICS, ERCIYES UNIVERSITY, 38039 KAYSERI, TURKEY
bor@erciyes.edu.tr
URL: http://math.erciyes.edu.tr/hbor.htm

ABSTRACT. In this paper a result of Bor [2] has been proved under weaker conditions by using a β-quasi power increasing sequence instead of an almost increasing sequence.

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1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n^α n-th Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), i.e.,

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.1}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad and \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{1.2}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if (see [4])

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^\alpha \mid^k < \infty. \tag{1.3}$$

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]).

Quite recently Bor [2] has proved the following theorem.

Theorem 1.1. *Let (X_n) be an almost increasing sequence and the sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n \tag{1.4}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{1.5}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{1.6}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{1.7}$$

If the sequence (u_n^α), defined by (see [6])

$$u_n^\alpha = \begin{cases} \mid t_n^\alpha \mid, \alpha = 1 \\ \max_{1 \leq v \leq n} \mid t_n^\alpha \mid, 0 < \alpha < 1 \end{cases} \tag{1.8}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k-1} (u_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{1.9}$$

then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$ and $0 \leq \delta < \alpha \leq 1$.

The aim of this paper is to prove Theorem 1.1 under weaker conditions, for this we need the concept of β-quasi power increasing sequence.
A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \tag{1.10}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of the theorem replacing an almost increasing sequence by a quasi β-power increasing sequence. Now, we shall prove the following theorem:

Theorem 1.2. *Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If all the conditions from 1.4 to 1.9 are satisfied, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$ and $0 \leq \delta < \alpha \leq 1$.*

We need the following lemmas for the proof of our theorem.

Lemma 1.3. ([3]). *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \mid . \tag{1.11}$$

Lemma 1.4. ([5]). *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when 1.6 is satisfied:*

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty \tag{1.12}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{1.13}$$

2. PROOF OF THE THEOREM

Let (T_n^α) be the n-th (C, α), with $0 < \alpha \leq 1$, mean of the sequence $(na_n\lambda_n)$. Then, by 1.1, we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v.$$

Applying Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

so that making use of Lemma 1.3, we have

$$\begin{aligned} \mid T_n^\alpha \mid &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^\alpha} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \mid \\ &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha u_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid u_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad say. \end{aligned}$$

Since

$$\mid T_{n,1}^\alpha + T_{n,2}^\alpha \mid^k \leq 2^k (\mid T_{n,1}^\alpha \mid^k + \mid T_{n,2}^\alpha \mid^k),$$

to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\delta k - 1} \mid T_{n,r}^\alpha \mid^k < \infty \quad for \quad r = 1, 2, \quad by \quad 1.3.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} \mid T_{n,1}^{\alpha} \mid^{k} &\leq \sum_{n=2}^{m+1} n^{\delta k-1} (A_n^{\alpha})^{-k} \{\sum_{v=1}^{n-1} A_v^{\alpha} u_v^{\alpha} \beta_v\}^{k} \\
&\leq \sum_{n=2}^{m+1} n^{\delta k-1} (A_n^{\alpha})^{-k} \{\sum_{v=1}^{n-1} (A_v^{\alpha})^{k} (u_v^{\alpha})^{k} \beta_v\} \times \{\sum_{v=1}^{n-1} \beta_v\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k-\alpha k-1} \{\sum_{v=1}^{n-1} v^{\alpha k} (u_v^{\alpha})^{k} \beta_v\} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^{\alpha})^{k} \beta_v \sum_{n=v+1}^{m+1} \frac{1}{n^{1+\alpha k-\delta k}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^{\alpha})^{k} \beta_v \int_{v}^{\infty} \frac{dx}{x^{1+\alpha k-\delta k}} \\
&= O(1) \sum_{v=1}^{m} v^{\delta k} (u_v^{\alpha})^{k} \beta_v = O(1) \sum_{v=1}^{m} v \beta_v v^{\delta k-1} (u_v^{\alpha})^{k} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} r^{\delta k-1} (u_r^{\alpha})^{k} + O(1) m \beta_m \sum_{v=1}^{m} v^{\delta k-1} (u_v^{\alpha})^{k} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m \beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \mid \beta_{v+1} \mid X_{v+1} + O(1) m \beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem 1.2 and Lemma 1.4.
Finally, since $\mid \lambda_n \mid = O(\frac{1}{X_n}) = O(1)$, by 1.7, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1} \mid T_{n,2}^{\alpha} \mid^{k} &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\delta k-1} (u_n^{\alpha})^{k} \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid n^{\delta k-1} (u_n^{\alpha})^{k} \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\delta k-1} (u_v^{\alpha})^{k} + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\delta k-1} (u_n^{\alpha})^{k} \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem 1.2 and Lemma 1.4.
Therefore, we get that

$$\sum_{n=1}^{m} \frac{1}{n} \mid T_{n,r}^{\alpha} \mid^{k} = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2.$$

This completes the proof of the Theorem 1.2.

Remark 2.1. It should be noted that if we take $\delta = 0$ (resp. $\alpha = 1$) in this theorem, then we get a new result for $\mid C, \alpha \mid_k$ (resp. $\mid C, 1; \delta \mid_k$) summability.

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