AN APPLICATION OF QUASI POWER INCREASING SEQUENCES
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ABSTRACT. In this paper a result of Bor [2] has been proved under weaker conditions by using a $\beta$-quasi power increasing sequence instead of an almost increasing sequence.

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1. Introduction

Let \( \sum a_n \) be a given infinite series with partial sums \((s_n)\). We denote by \( t_n^\alpha \) n-th Cesàro mean of order \( \alpha \), with \( \alpha > -1 \), of the sequence \((na_n)\), i.e.,

\[
t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v,
\]

where

\[
A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.
\]

The series \( \sum a_n \) is said to be summable \( |C, \alpha; \delta|_k \) if \( k \geq 1 \) and \( \delta \geq 0 \), if (see [4])

\[
\sum_{n=1}^{\infty} n^{\delta k-1} | t_n^\alpha |^k < \infty.
\]

A positive sequence \((b_n)\) is said to be almost increasing if there exists a positive increasing sequence \(c_n\) and two positive constants A and B such that \( Ac_n \leq b_n \leq Bc_n\) (see [1]).

Quite recently Bor [2] has proved the following theorem.

**Theorem 1.1.** Let \((X_n)\) be an almost increasing sequence and the sequences \((\beta_n)\) and \((\lambda_n)\) such that

\[
| \Delta \lambda_n | \leq \beta_n
\]

\[
\beta_n \to 0 \quad \text{as} \quad n \to \infty
\]

\[
\sum_{n=1}^{\infty} n | \Delta \beta_n | X_n < \infty
\]

\[
| \lambda_n | X_n = O(1) \quad \text{as} \quad n \to \infty.
\]

If the sequence \((u_n^\alpha)\), defined by (see [6])

\[
u_n^\alpha = \begin{cases} 
| t_n^\alpha |, & \alpha = 1 \\
\max_{1 \leq v \leq n} | t_n^\alpha |, & 0 < \alpha < 1
\end{cases}
\]

satisfies the condition

\[
\sum_{n=1}^{m} n^{\delta k-1}(u_n^\alpha)^k = O(X_m) \quad \text{as} \quad m \to \infty,
\]

then the series \( \sum a_n \lambda_n \) is summable \( |C, \alpha; \delta|_k \) if \( k \geq 1 \) and \( 0 \leq \delta < \alpha \leq 1 \).

The aim of this paper is to prove Theorem 1.1 under weaker conditions, for this we need the concept of \( \beta \)-quasi power increasing sequence.

A positive sequence \((\gamma_n)\) is said to be quasi \( \beta \)-power increasing sequence if there exists a constant \( K = K(\beta, \gamma) \geq 1 \) such that

\[
Kn^\beta \gamma_n \geq m^\beta \gamma_m
\]

holds for all \( n \geq m \geq 1 \). It should be noted that every almost increasing sequence is quasi \( \beta \)-power increasing sequence for any nonnegative \( \beta \), but the converse need not be true as can be seen by taking the example, say \( \gamma_n = n^{-\beta} \) for \( \beta > 0 \). So we are weakening the hypotheses of
the theorem replacing an almost increasing sequence by a quasi \( \beta \)-power increasing sequence. Now, we shall prove the following theorem:

**Theorem 1.2.** Let \((X_n)\) be a quasi \( \beta \)-power increasing sequence for some \( 0 < \beta < 1 \). If all the conditions from 1.4 to 1.9 are satisfied, then the series \( \sum a_n \lambda_n \) is summable \( |C, \alpha; \delta|_k, k \geq 1 \) and \( 0 \leq \delta < \alpha \leq 1 \).

We need the following lemmas for the proof of our theorem.

**Lemma 1.3.** ([3]). If \( 0 < \alpha \leq 1 \) and \( 1 \leq v \leq n \), then

\[
| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p | \leq \max_{1 \leq m \leq v} | \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p |.
\]

**Lemma 1.4.** ([5]). Under the conditions on \((X_n)\), \((\beta_n)\) and \((\lambda_n)\) as taken in the statement of the theorem, the following conditions hold, when 1.6 is satisfied:

\[
(1.12) \quad n \beta_n X_n = O(1) \quad \text{as} \quad n \to \infty
\]

\[
(1.13) \quad \sum_{n=1}^{\infty} \beta_n X_n < \infty.
\]

2. **Proof of the Theorem**

Let \((T_n^\alpha)\) be the n-th \((C, \alpha)\), with \( 0 < \alpha \leq 1 \), mean of the sequence \((n a_n \lambda_n)\). Then, by 1.1, we have

\[
T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v.
\]

Applying Abel’s transformation, we get

\[
T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,
\]

so that making use of Lemma 1.3, we have

\[
| T_n^\alpha | \leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} | \Delta \lambda_v | \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p | + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v |
\]

\[
\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_{v}^{\alpha} u_v | \Delta \lambda_v | + | \lambda_n | v_n^\alpha
\]

\[
= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say}.
\]

Since

\[
| T_{n,1}^\alpha + T_{n,2}^\alpha |^k \leq 2^k (| T_{n,1}^\alpha |^k + | T_{n,2}^\alpha |^k),
\]

to complete the proof of the theorem, it is enough to show that

\[
\sum_{n=1}^{\infty} r_n^{\beta_k-1} | T_{n,r}^\alpha |^k < \infty \quad \text{for} \quad r = 1, 2, \quad \text{by} \quad 1.3
\]
Now, when $k > 1$, applying Hölder’s inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\sum_{n=2}^{m+1} n^{\delta k-1} \left| T_{n,1}^\alpha \right|^k \leq \sum_{n=2}^{m+1} n^{\delta k-1}(A_n^\alpha)^{-k} \left\{ \sum_{v=1}^{n-1} A_v^\alpha u_v^\alpha \beta_v \right\}^k$$

$$\leq \sum_{n=2}^{m+1} n^{\delta k-1}(A_n^\alpha)^{-k} \left\{ \sum_{v=1}^{n-1} (A_v^\alpha)^k (u_v^\alpha)^k \beta_v \right\} \times \left\{ \sum_{v=1}^{n-1} \beta_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} n^{\delta k-ak-1} \left\{ \sum_{v=1}^{n-1} v^{ak} (u_v^\alpha)^k \beta_v \right\}$$

$$= O(1) \sum_{v=1}^{m} v^{ak} (u_v^\alpha)^k \beta_v \sum_{n=\nu+1}^{m+1} \frac{1}{n^{1+ak-\delta k}}$$

$$= O(1) \sum_{v=1}^{m} v^{ak} (u_v^\alpha)^k \beta_v \int_{\nu}^{\infty} \frac{dx}{x^{1+ak-\delta k}}$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k} (u_v^\alpha)^k \beta_v = O(1) \sum_{v=1}^{m} v^{\delta k} v^{\delta k-1} (u_v^\alpha)^k$$

$$= O(1) \sum_{v=1}^{m} \Delta (v^\beta) \sum_{r=1}^{v} r^{\delta k-1} (u_r^\alpha)^k + O(1) m \beta_m \sum_{v=1}^{m} v^{\delta k-1} (u_v^\alpha)^k$$

$$= O(1) \sum_{v=1}^{m} \Delta (v^\beta) |X_v + O(1)m \beta_m X_m$$

$$= O(1) \sum_{v=1}^{m} v \Delta \beta_v |X_v + O(1) \sum_{v=1}^{m-1} \beta_{v+1} |X_{v+1} + O(1)m \beta_m X_m$$

$$= O(1) \text{ as } m \to \infty,$$

by virtue of the hypotheses of the Theorem 1.2 and Lemma 1.4.

Finally, since $|\lambda_n| = O\left(\frac{1}{\lambda_n}\right) = O(1)$, by 1.7 we have that

$$\sum_{n=1}^{m} n^{\delta k-1} \left| T_{n,2}^\alpha \right|^k = \sum_{n=1}^{m} \lambda_n |k-1| \lambda_n |n^{\delta k-1} (u_n^\alpha)^k$$

$$= O(1) \sum_{n=1}^{m} \lambda_n |n^{\delta k-1} (u_n^\alpha)^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} v^{\delta k-1} (u_v^\alpha)^k + O(1) |\lambda_m| \sum_{n=1}^{m} n^{\delta k-1} (u_n^\alpha)^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta \lambda_n |X_n + O(1) |\lambda_m| X_m$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \text{ as } m \to \infty,$$
by virtue of the hypotheses of the Theorem 1.2 and Lemma 1.4. Therefore, we get that

$$\sum_{n=1}^{m} \frac{1}{n} | T_{n,r}^{\alpha} |^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2.$$ 

This completes the proof of the Theorem 1.2.

**Remark 2.1.** It should be noted that if we take $\delta = 0$ (resp. $\alpha = 1$) in this theorem, then we get a new result for $|C, \alpha|_k$ (resp. $|C, 1; \delta|_k$) summability.

**REFERENCES**


