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INTEGRABILITY OF SINE AND COSINE SERIES HAVING COEFFICIENTS OF A NEW CLASS

L. LEINDLER

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BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, ARADI VÉRTANÚK TERE 1, H-6720 SZEGED, HUNGARY
leindler@math.u-szeged.hu

ABSTRACT. Some integrability theorems or only their sufficient part are generalized such that the coefficients of the sine and cosine series belong to a new class of sequences being wider than the class of sequences of rest bounded variation, which itself is a generalization of the monotone decreasing sequences, but a subclass of the almost monotone decreasing sequences. It is also verified that the new class of sequences and the class of almost monotone decreasing sequences are not comparable.

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1. INTRODUCTION

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers tending to zero is called of rest bounded variation, and briefly denoted by $\mathbf{c} \in \mathrm{RBVS}$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(\mathbf{c}) c_m \tag{1.1}$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

In view of (1.1) it is clear that if $\mathbf{c} \in \mathrm{RBVS}$ then it is also *almost monotonic*, that is, for all $n \ge m$

$$c_n \le K(\mathbf{c}) c_m \tag{1.2}$$

stays, but (1.2) does not imply (1.1) if $K(\mathbf{c}) > 1$. If $\mathbf{c}$ satisfies (1.2) we write $\mathbf{c} \in \mathrm{AMS}$. If $K(\mathbf{c}) = 1$, we denote $\mathbf{c} \in \mathrm{MS}$.

Now we generalize the definition (1.1) with a view to broaden the class of RBVS. Our definition is the following.

Definition 1.3. *Let $\boldsymbol{\gamma} := \{\gamma_n\}$ be a fixed sequence of positive numbers. We say that a nonnegative null-sequence $\mathbf{c}$ belongs to the class $\boldsymbol{\gamma}\mathrm{RBVS}$ if*

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(\mathbf{c}) \gamma_m, \quad (c_n \to 0), \tag{1.4}$$

holds for all $m \in \mathbb{N}$. The notation $\mathbf{c} \in \boldsymbol{\gamma}\mathrm{RBVS}$ will denote the property (1.4)*. We mention that the conditions $\gamma_n \ge \gamma_{n+1}$ and $\mathbf{c} \in \boldsymbol{\gamma}\mathrm{RBVS}$ jointly do not even imply that $\mathbf{c} \in \mathrm{RBVS}$.*

In order to emphasize the requirement $c_n \ge 0$ it would be more precise to use the notation $\boldsymbol{\gamma}\mathrm{RBVS}_+$ instead of $\boldsymbol{\gamma}\mathrm{RBVS}$, but for the simplicity we use the latter one.

It is clear that (1.4) implies that $c_n \le K(\mathbf{c})\gamma_m$ $(n \ge m)$, but it does not exhibit that $\mathbf{c} \in \mathrm{AMS}$, see our Remark 2.14. Furthermore if one $c_k = 0$, then (1.4) does not imply that all of the terms with index $n > k$ are zero, too, while (1.1) claims this. By all means (1.4) gives much greater freedom for the sequence $\mathbf{c}$ than (1.1) does, consequently, in general, $\boldsymbol{\gamma}\mathrm{RBVS}$ is a larger class than RBVS; they are identical only if $\gamma_n = Kc_n$.

In the sequel the capital letters $K, K_1, \dots$ and $K(\cdot)$ will denote positive constants which are either absolute constants or constants depending on certain sequence, and not necessarily the same at each occurrences.

Originally we have intended to generalize the theorems having conditions with sequences belonging to the classes MS, AMS or RBVS to the classes $\boldsymbol{\gamma}\mathrm{RBVS}$ with suitable sequences $\boldsymbol{\gamma}$, but it turned out that the plenty of rope of the sequences $\mathbf{c} \in \boldsymbol{\gamma}\mathrm{RBVS}$ does not make it possible. We can verify generalization of this type only for the sufficient part of the known theorems. See e.g. the papers [8] and [9], where we considered theorems pertaining to the space of continuous functions.

In the present paper we verify generalization of some theorems relating to the spaces L^p, $p \ge 1$.

Among others we generalize the following theorems, and the sufficient part of some other results.

Theorem 1.5. *Let $\boldsymbol{\lambda} := \{\lambda_n\} \in \mathrm{MS}$ be such that for a fixed p $(1 < p < \infty)$*

$$\sum_{n=1}^{\infty} n^{p-2} \lambda_n^p < \infty. \tag{1.6}$$

If φ is the sum of either of the series

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \quad \textit{or} \quad \sum_{n=1}^{\infty} \lambda_n \sin nx, \tag{1.7}$$

then

$$\omega_p(\varphi, n^{-1}) \leq K_1 n^{-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \lambda_\nu^p \right\}^{1/p} + K_2 \left\{ \sum_{\nu=n}^{\infty} \nu^{p-2} \lambda_\nu^p \right\}^{1/p}, \tag{1.8}$$

where $\omega_p(f, \delta)$ denotes the modulus of continuity of f in L^p.

This is a sharp result of S. ALJANČIĆ [1] which is improved in [7] such that the condition $\boldsymbol{\lambda} \in \mathrm{MS}$ is replaced by $\boldsymbol{\lambda} \in \mathrm{RBVS}$.

Theorem 1.9 ([6]). *Let $\mathbf{b} := \{b_n\} \in \mathrm{RBVS}$.*

(1) *If $0 < \gamma \leq 1$ and*

$$\sum_{n=1}^{\infty} n^{\gamma-1} b_n < \infty, \tag{1.10}$$

then $x^{-\gamma} g(x) \in L(0, \pi)$, where $g(x) := \sum_{n=1}^{\infty} b_n \sin nx$.

(2) *If $0 < \gamma < 1$ and* (1.10) *holds, then $x^{-\gamma} f(x) \in L(0, \pi)$, where $f(x) := \sum_{n=1}^{\infty} b_n \cos nx$.*

(3) *If* (1.10) *is convergent for $\gamma = 0$, then both $g(x)$ and $f(x)$ are integrable.*

We note that Theorem 1.9 with classical quasi-monotone $\mathbf{b}$ $(b_{n+1} \leq b_n(1 + \frac{\alpha}{n}))$ was proved by S.M. SHAH [10]. For further similar theorems we refer to the well-known monograph of R.P. BOAS, JR. [4].

2. THEOREMS

Our results read as follows.

Theorem 2.1. *Let $1 < p < \infty$ and $\boldsymbol{\lambda} := \{\lambda_n\} \in \boldsymbol{\gamma}\mathrm{RBVS}$ with the additional condition*

$$\sum_{n=1}^{\infty} n^{p-2} \gamma_n^p < \infty. \tag{2.2}$$

If φ is the sum is either of the series (1.7)*, then*

$$\omega_p(\varphi, n^{-1}) \leq K_1 n^{-1} \left(\sum_{\nu=1}^{n-1} \nu^{2p-2} \gamma_\nu^p \right)^{1/p} + K_2 \left(\sum_{\nu=n}^{\infty} \nu^{p-2} \gamma_\nu^p \right)^{1/p}. \tag{2.3}$$

It is clear that if $\gamma_n = \lambda_n$ and $\{\lambda_n\} \in \mathrm{MS}$ or $\{\lambda_n\} \in \mathrm{RBVS}$, then Theorem 2.1 as special case reduces to Theorem 1.5 or Theorem 1 of [7], respectively.

Theorem 2.4. *Theorem 1.9 can be improved such that the conditions $\mathbf{b} := \{b_n\} \in \mathrm{RBVS}$ and* (1.10) *are replaced by the assumption $\mathbf{b} \in \boldsymbol{\gamma}\mathrm{RBVS}$ with the additional condition*

$$\sum_{n=1}^{\infty} n^{\gamma-1} \gamma_n < \infty. \tag{2.5}$$

Next we establish sufficient conditions for $x^{-\gamma}\varphi(x)$ to belong to $L^p := L^p(0, \pi)$.

Theorem 2.6. *If $1 < p < \infty$, $(1/p) - 1 < \gamma < 1/p$, and $\boldsymbol{\lambda} \in \boldsymbol{\gamma}\mathrm{RBVS}$ with the additional condition*

$$\sum_{n=1}^{\infty} n^{p\gamma+p-2}\gamma_n^p < \infty, \tag{2.7}$$

then $x^{-\gamma}\varphi(x) \in L^p$, where φ is the sum of either of (1.7).

Theorem 2.8. *If $1 < p < \infty$ and $\boldsymbol{\lambda} \in \boldsymbol{\gamma}\mathrm{RBVS}$ with the additional condition*

$$\sum_{n=1}^{\infty} n^{2p-2}\gamma_n^p < \infty, \tag{2.9}$$

then

$$\omega_p(\varphi, h) = O(h), \tag{2.10}$$

where φ is the sum of either of the series of (1.7).

Corollary 2.11. *If $1 < p < \infty$ and $\boldsymbol{\lambda} \in \boldsymbol{\gamma}\mathrm{RBVS}$ with $\gamma_n = O(n^{-2+1/p})$, then*

$$\omega_p(\varphi, h) = O(h|\log h|^{1/p}), \tag{2.12}$$

where φ is the sum of either of (1.7).

Observation 2.13. *If we consider only one fixed coefficient sequence $\boldsymbol{\lambda}$ then clearly the best way to choose the sequence $\boldsymbol{\gamma}$ with the terms $\gamma_n := \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}|$ (in Theorem 2.4 $\gamma_n := \sum_{k=n}^{\infty} |b_n - b_{k+1}|$).*

Remark 2.14. *The classes* AMS *and* $\boldsymbol{\gamma}\mathrm{RBVS}$ *are not comparable.*

1. *There exists a sequence $\mathbf{c} := \{c_n\}$ which belongs to* AMS*, but does not to* $\mathbf{c}\mathrm{RBVS} \equiv \mathrm{RBVS}$.
2. *There exists a sequence $\mathbf{d} := \{d_n\}$ which belongs to* $\boldsymbol{\gamma}\mathrm{RBVS}$*, but does not to* AMS.

3. LEMMAS

Lemma 3.1 ([3]). *Let $g(x)$ and $f(x)$ denote the functions defined in Theorem 1.9. If $b_n \to 0$ and*

$$\sum_{n=2}^{\infty} |b_{n-1} - b_{n+1}| n^{\gamma} < \infty,$$

then $x^{-\gamma}g(x) \in L$ if $0 < \gamma \le 1$, and $x^{-\gamma}f(x) \in L$ if $0 < \gamma < 1$.

Furthermore, if

$$\sum_{n=2}^{\infty} |b_{n-1} - b_{n+1}| \log n < \infty,$$

then both g and f are integrable.

The following lemma can be found implicitely in [4, p. 37] (see also [2]).

Lemma 3.2. *If $\varphi(x)$ is the sum of either of* (1.7)*, $\lambda_n \ge 0$, $1 < p < \infty$, and $(1/p) - 1 < \gamma < 1/p$, then*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}\Big(\sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+2}|\Big)^p < \infty \tag{3.3}$$

implies that $x^{-\gamma}\varphi(x) \in L^p$.

Lemma 3.4 ([5], Theorem 1). *If $p \geq 1$ and $\alpha_n \geq 0$, then for any sequence $\{\kappa_m\}$ of positive numbers*

$$\sum_{m=1}^{\infty} \kappa_m \Big(\sum_{k=1}^{m} \alpha_k\Big)^p \leq p^p \sum_{m=1}^{\infty} \kappa_m^{1-p} \Big(\sum_{k=m}^{\infty} \kappa_k\Big)^p \alpha_m^p \tag{3.5}$$

and

$$\sum_{m=1}^{\infty} \kappa_m \Big(\sum_{k=m}^{\infty} \alpha_k\Big)^p \leq p^p \sum_{m=1}^{\infty} \kappa_m^{1-p} \Big(\sum_{k=1}^{m} \kappa_k\Big)^p \alpha_m^p \tag{3.6}$$

hold.

Lemma 3.7. *If $\lambda_k \geq 0$ then*

$$\Big|\sum_{k=m}^{n} \lambda_k \cos kx \sin \frac{x}{2}\Big| \leq \frac{1}{2}\Big(\lambda_m + \sum_{k=m}^{n-1} |\lambda_k - \lambda_{k+1}| + \lambda_n\Big).$$

The assertion is trivial using Abel rearrangement.

Lemma 3.8. *If $\mathbf{c} := \{c_n\} \in \boldsymbol{\gamma}\mathrm{RBVS}$ then there exists a nonincreasing sequence $\boldsymbol{\gamma}^* := \{\gamma_n^*\}$ such that $\mathbf{c} \in \boldsymbol{\gamma}^*\mathrm{RBVS}$ also holds and for all n $\gamma_n^* \leq \gamma_n$.*

Proof. Let $\gamma_1^* := \gamma_1$ and $\gamma_n^* := \min(\gamma_{n-1}^*, \gamma_n)$ for $n \geq 2$. It is obvious that $\gamma_n^* \leq \gamma_n$, furthermore (1.4) holds with γ_1^*, too.

Let us assume that (1.4) is verified for $m \geq 1$ with γ_m^* in place of γ_m. If $\gamma_{m+1} \leq \gamma_m^*$ then $\gamma_{m+1}^* = \gamma_{m+1}$ thus (1.4) clearly stays with γ_{m+1}^* as well. If $\gamma_{m+1} > \gamma_m^*$ then $\gamma_{m+1}^* = \gamma_m^*$ and then the following inequalities

$$\sum_{n=m+1}^{\infty} |c_n - c_{n+1}| \leq \sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\mathbf{c})\gamma_m^* = K(\mathbf{c})\gamma_{m+1}^*$$

convey the inequality (1.4) with γ_{m+1}^* in place of γ_{m+1}.

The proof is complete. ∎

4. PROOFS

First we verify Theorem 2.6 because its result will be used in the proof of Theorem 2.1.

Proof of Theorem 2.6. The conditions $\boldsymbol{\lambda} \in \boldsymbol{\gamma}\mathrm{RBVS}$ and (2.7) imply that (3.3) holds, thus Lemma 3.2 conveys that $x^{-\gamma}\varphi(x) \in L^p$, and this completes the proof. ∎

Proof of Theorem 2.1. The special case $\gamma = 0$ of Theorem 2.6 shows that the conditions $\boldsymbol{\lambda} \in \boldsymbol{\gamma}\mathrm{RBVS}$ and (2.2) imply that $\varphi \in L^p$. We verify (2.3) only for cosine series, the sine case runs similarly. We assume that $h = \pi/2n$. It is clear that

$$\omega_p(\varphi, h) \leq K \sup_{0<t\leq h}\Bigg(\Big(\int_0^{\pi/n} |\varphi(x \pm t) - \varphi(x)|^p \, dx\Big)^{1/p} + \Big(\int_{\pi/n}^{\pi} |\varphi(x \pm t) - \varphi(x)|^p \, dx\Big)^{1/p}\Bigg) = K \sup_{0<t\leq h} (I_1 + I_2), \quad \text{say.}$$

By Minkowski's inequality

$$\frac{1}{2}I_1 \le \Big(\int_0^{\pi/n}\Big|\sum_{\nu=1}^{n-1}\lambda_\nu \sin\frac{1}{2}\nu t \sin\nu\Big(x\pm\frac{t}{2}\Big)\Big|^p\,dx\Big)^{1/p}$$
$$+\Big(\int_0^{\pi/n}\Big|\sum_{\nu=n}^{\infty}\lambda_\nu\big(\cos\nu(x\pm t)-\cos\nu x\big)\Big|^p\,dx\Big)^{1/p} =: I_{11}+I_{12}.$$

Since

$$I_{11}\le t\Big(\int_0^{\pi/n}\Big(\sum_{\nu=1}^{n-1}\nu\lambda_\nu\Big)^p dx\Big)^{1/p},$$

and if we apply Hölder's inequality to $\sum_{\nu=1}^{n-1}\nu\lambda_\nu$, we obtain that

$$I_{11}\le Kn^{-1}\Big(\sum_{\nu=1}^{n-1}\nu^{2p-2}\lambda_\nu^p\Big)^{1/p}.$$

As in [1] we get that

$$I_{12}\le K\Big(\sum_{m=n}^{\infty}\int_{3\pi/2(m+1)}^{3\pi/2m}\Big|\sum_{\nu=n}^{\infty}\lambda_\nu\cos\nu x\Big|^p\,dx\Big)^{1/p}.$$

Here using Lemma 3.7 and $\boldsymbol{\lambda}\in\boldsymbol{\gamma}\mathrm{RBVS}$, we get that

$$\Big|\sum_{\nu=n}^{\infty}\lambda_\nu\cos\nu x\Big|\le\sum_{\nu=n}^{m}\lambda_\nu+K(m+1)\gamma_{m+1}.$$

Thus

$$I_{12}^p\le K_1\sum_{m=n}^{\infty}m^{-2}\Big(\sum_{\nu=n}^{m}\gamma_\nu\Big)^p+K_2\sum_{m=n}^{\infty}m^{p-2}\gamma_m^p. \tag{4.1}$$

Now using (3.5) with $\kappa_m=m^{-2}$ and

$$\alpha_k:=\begin{cases}0 & \text{if } k<n,\\ \gamma_k & \text{if } k\ge n,\end{cases}$$

we obtain that

$$\sum_{m=1}^{\infty}m^{-2}\Big(\sum_{k=1}^{m}\alpha_k\Big)^p\le K\sum_{m=1}^{\infty}m^{p-2}\alpha_m^p.$$

This inequality shows that the first sum in (4.1) is majorized by the second one. Consequently

$$I_{12}\le K\Big(\sum_{m=n}^{\infty}m^{p-2}\gamma_m^p\Big)^{1/p}.$$

If $D_\nu(x)$ denotes the Dirichlet kernel, an Abel transformation combined with Minkowski's inequality gives

$$I_2\le\Big(\int_{\pi/n}^{\pi}\Big|\sum_{\nu=1}^{n}\Delta\lambda_\nu\big(D_\nu(x\pm t)-D_\nu(x)\big)\Big|^p\,dx\Big)^{1/p}$$
$$+\Big(\int_{\pi/n}^{\pi}\Big|\sum_{\nu=n+1}^{\infty}\Delta\lambda_\nu\big(D_\nu(x\pm t)-D_\nu(t)\big)\Big|^p\,dx\Big)^{1/p} =: I_{21}+I_{22}.$$

Following the discussing of the proof of Theorem 1.5 given in [1] we obtain that

$$
\begin{aligned}
I_{21}^p &\le K \sum_{m=1}^{n-1} \int_{\pi/(m+1)}^{\pi/m} \Big| \Delta\lambda_\nu \big(D_\nu(x \pm t) - D_\nu(x)\big)\Big|^p \, dx \\
&\le K_1 t^p \Bigg(\sum_{m=1}^{n-1} m^{-2} \Big(\sum_{\nu=1}^{m} \nu^2 |\Delta\lambda_\nu| \Big)^p + \sum_{m=1}^{n-1} m^{p-2} \Big(\sum_{\nu=m+1}^{n} \nu |\Delta\lambda_\nu| \Big)^p \Bigg).
\end{aligned} \tag{4.2}
$$

In the following steps we shall assume that the sequence $\boldsymbol{\gamma}$ is nonincreasing, by Lemma 3.8 we can do this without loss of generality. Since $\boldsymbol{\lambda} \in \boldsymbol{\gamma}\mathrm{RBVS}$ we clearly get that

$$
\begin{aligned}
\sum_{\nu=1}^{m} \nu^2 |\Delta\lambda_\nu| &\le \sum_{k=1}^{\log m} \sum_{\nu=2^{k-1}}^{2^k} \nu^2 |\Delta\lambda_\nu| \le K \sum_{k=1}^{\log m} 2^{2k} \gamma_{2^{k-1}} \\
&\le 4K \sum_{k=1}^{\log m} 2^k \sum_{\nu=2^{k-1}}^{2^k} \gamma_\nu \le 8K \sum_{\nu=1}^{m} \nu \gamma_\nu,
\end{aligned}
$$

and similarly

$$
\sum_{\nu=m+1}^{n} \nu |\Delta\lambda_\nu| \le K_1 \Big(\sum_{\nu=m+1}^{n} \gamma_\nu + m\gamma_m \Big).
$$

These inequalities and (4.2) imply that

$$
\begin{aligned}
I_{21}^p \le K t^p \Bigg(&\sum_{m=1}^{n-1} m^{-2} \Big(\sum_{\nu=1}^{m} \nu \gamma_\nu \Big)^p \\
&+ \sum_{m=1}^{n-1} m^{p-2} \Big(\sum_{\nu=m}^{n} \gamma_\nu \Big)^p + \sum_{m=1}^{n-1} m^{2p-2} \gamma_m^p \Bigg).
\end{aligned} \tag{4.3}
$$

We estimate the first sum in (4.3) using the assertion (3.5) of Lemma 3.4 with $\kappa_m = m^{-2}$, plus $\alpha_k = k\gamma_k$ for $k < n$ and $\alpha_k = 0$ for $k \ge n$, thus we can see that it is majorized by the third sum multiplied by K. The same holds for the second sum if we use (3.6) with $\kappa_m = m^{p-2}$, furthermore with $\alpha_k = \gamma_k$ for $k < n$ and $\alpha_k = 0$ for $k \ge n$. Thus, in view of $t \le h = \pi/2n$, we obtain that

$$
I_{21} \le K n^{-1} \Big(\sum_{\nu=1}^{n-1} \nu^{2p-2} \gamma_\nu^p \Big)^{1/p}.
$$

Since

$$
\begin{aligned}
I_{22} &\le 2 \Bigg(\int_{\pi/2n}^{\pi+\pi/2n} \Big(\sum_{\nu=n+1}^{\infty} |\Delta\lambda_\nu| |D_\nu(x)| \Big)^p \, dx \Bigg)^{1/p} \\
&\le K \gamma_n \Big(\int_{\pi/2n}^{\infty} x^{-p} \, dx \Big)^{1/p} \le K_1 n^{1-1/p} \gamma_n,
\end{aligned}
$$

and

$$
\gamma_n \le K n^{1/p-2} \Big(\sum_{\nu=1}^{n-1} \nu^{2p-2} \gamma_\nu^p \Big)^{1/p}
$$

clearly holds, namely we assumed that γ_n is nonincreasing, thus we have that

$$
I_{22} \le K n^{-1} \Big(\sum_{\nu=1}^{n-1} \nu^{2p-2} \gamma_\nu^p \Big)^{1/p}
$$

holds.

Lastly, collecting our partial estimates, we see that (2.3) is verified, and herewith the proof of Theorem 2.1 is finished. ∎

Proof of Theorem 2.4. Since $\mathbf{b} \in \boldsymbol{\gamma}\mathrm{RBVS}$ thus $b_n \to 0$, furthermore (2.5) implies that

$$\sum_{n=1}^{\infty} n^{\gamma-1} \sum_{k=n}^{\infty} |b_k - b_{k+1}| < \infty,$$

therefore if $\gamma > 0$ then

$$\sum_{k=1}^{\infty} |b_k - b_{k+1}| k^{\gamma} < \infty,$$

and if $\gamma = 0$ then

$$\sum_{k=2}^{\infty} |b_k - b_{k+1}| \log k < \infty$$

also hold. Thus, by Lemma 3.1, we immediately obtain all the assertions of Theorem 2.4. ∎

Proof of Theorem 2.8. To the proof we shall use Theorem 2.1. By (2.9) the first term in (2.3) is clearly $O(1/n)$, and the second one is also $O(1/n)$, namely (2.9) and the obvious inequality

$$\sum_{\nu=n}^{\infty} \nu^{p-2} \gamma_{\nu}^{p} = \sum_{\nu=n}^{\infty} \nu^{-p} \nu^{2p-2} \gamma_{\nu}^{p} \le K n^{-p}$$

verify this. Hence (2.10) plainly follows. ∎

Proof of Corollary 2.11. The assumptions of Corollary 2.11 imply that the condition (2.2) holds, therefore Theorem 2.1 can be applied again. An easy calculation gives that in (2.3) the first term is $O(n^{-1} \log n)$ and the second one $O(n^{-1})$, whence (2.12) obviously follows. ∎

Proof of Remark 2.14. First we verify the assertion (1). Let $\mathbf{c} := \{c_n\}$, where

$$c_n := 2^{-m} + (-1)^n 2^{-m-1}, \quad \text{if} \quad 2^m \le n < 2^{m+1}.$$

It is clear that (1.2) with $K(\mathbf{c}) = 8$ holds, thus $\mathbf{c} \in \mathrm{AMS}$. Since

$$\sum_{n=2^m}^{2^{m+1}} |c_n - c_{n+1}| \ge \frac{1}{2} \quad \text{for any } m,$$

thus (1.1) does not hold, consequently $\mathbf{c} \notin \mathrm{RBVS} \equiv \mathbf{c}\mathrm{RBVS}$.

To prove the statement (2) we define the following sequence $\mathbf{d} := \{d_n\}$: Let $d_1 = d_2 = 1$ and for $n \ge 3$

$$d_n := \begin{cases} 2^{-m} & \text{if } 2^m < n < 2^{m+1}, \\ m2^{-m} & \text{if } n = 2^{m+1}, \end{cases} \qquad m = 1, 2, \ldots$$

Furthermore we define the sequence $\boldsymbol{\gamma} := \{\gamma_n\}$ as follows:

$$\gamma_1 = \gamma_2 = 1, \quad \text{and } \gamma_n := m2^{-m}, \text{ if } 2^m < n \le 2^{m+1}, m = 1, 2, \ldots$$

Then it is easy to see that $\mathbf{d}$ does not belong to AMS, namely $\sup_n d_{n+1}/d_n = \infty$, but (1.4) holds with $K(\mathbf{d}) = 8$, thus $\mathbf{d} \in \boldsymbol{\gamma}\mathrm{RBVS}$.

The proof is complete. ∎

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