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INEQUALITIES RELATING TO THE GAMMA FUNCTION

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ABSTRACT. For $x \in (0, 1)$, we have

$$\frac{x}{[\Gamma(x+1)]^{1/x}} < \left(1 + \frac{1}{x}\right)^x < \frac{x+1}{[\Gamma(x+1)]^{1/x}}.$$

For $x \geq 1$,

$$\left(1 + \frac{1}{x}\right)^x \geq \frac{x+1}{[\Gamma(x+1)]^{1/x}},$$

and equality occurs for $x = 1$.

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1. INTRODUCTION

It is well known that the classical gamma function $\Gamma(z)$ is defined for $\operatorname{Re} z > 0$ as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t. \tag{1.1}$$

The logarithmic derivative of the gamma function $\Gamma(x)$ for $x > 0$ can be expressed [15, p. 16] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, \mathrm{d}t, \tag{1.2}$$

where $\gamma = 0.57721566 \cdots$ denotes the Euler-Mascheroni constant, which is known in literature as psi function or digamma function. They are two of the most important functions in analysis and its applications. The history and the development of this function are described in detail in [8].

In 1995, "G. D. Anderson et al [5] proved that the function $f(x) = x(\ln x - \psi(x))$ is strictly decreasing and strictly convex on $(0, \infty)$" [4]. Moreover, they showed that

$$\lim_{x \to 0} f(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 1/2. \tag{1.3}$$

>From (1.3) and the monotonicity of f Alzer concludes

$$\frac{1}{2x} < \ln x - \psi(x) < \frac{1}{x}, \quad x > 0, \tag{1.4}$$

which extends a result of H. Minc and L. Sathre [16], who established (1.4) for $x > 1$ and used it to prove several discrete inequalities involving the geometric mean of the first n positive integers. Refinements of (1.4) were given by L. Gordon [10]. H. Alzer provided in [4] an extension of the result given by Anderson et al, and proved that f is not only strictly decreasing and strictly convex, but even strictly completely monotonic on $(0, \infty)$. In [20], the authors proved that the function $\frac{1}{x} \ln \Gamma(x+1) - \ln x + 1$ is strictly completely monotonic on $(0, \infty)$. A similar result was established in [29]: The function $1 + \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1)$ is strictly completely monotone on $(-1, \infty)$ and tends to 1 as $x \to -1$ and to 0 as $x \to \infty$.

In 1985, D. Kershaw and A. Laforgia [13] showed that the function $x[\Gamma(1 + \frac{1}{x})]^x$ is strictly increasing on $(0, \infty)$, which is equivalent to the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ being strictly decreasing on $(0, \infty)$. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1 + \frac{1}{x})^x]$ decreases for $0 < x < 1$, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1, \infty)$.

In [7, 21], it is proved that the function $f(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing and strictly logarithmically convex in $(0, \infty)$ and the function $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing and strictly logarithmically concave in $(0, \infty)$.

In [24, 25], among other things, some strictly completely monotonic properties of the functions $\frac{[\Gamma(x+1)]^{1/x}}{x}$ and $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ are obtained.

It is well-known that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}_{n \in \mathbb{N}}$ is increasing. Furthermore, the authors in [22] obtained some general results: The function $\left(1 + \frac{1}{x}\right)^{x+\alpha}$ increases with $x > 0$ if and only if $\alpha \le 0$ and decreases in $x > 0$ if and only if $\alpha \ge \frac{1}{2}$; The necessary and sufficient conditions of the sequence $a_n = \left(1 + \frac{1}{n}\right)^{n+\alpha}$ being decreasing or being increasing are $\alpha \ge \frac{1}{2}$ or $\alpha \le \frac{2\ln 3 - 3\ln 2}{2\ln 2 - \ln 3}$, respectively. Let $b_n = \left(1 + \frac{\alpha}{n}\right)^{n+\beta}$ for $\alpha > -1$ and $\alpha \ne 0$ and $F(x) = \left(1 + \frac{\alpha}{x}\right)^{x+\beta}$ for $x > \max\{0, -\alpha\}$ and $\alpha \ne 0$, then the function $F(x)$ increases if and only if $\alpha > 0$ and $\beta \le 0$, or $\alpha < 0$ and $\alpha \le 2\beta$; the function $F(x)$ decreases if and only if $2\beta \ge \alpha > 0$ or $\beta \le \alpha < 0$;

the sequence b_n increases if and only if $\alpha > 0$ and $\beta \le \frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$, or $-1 < \alpha < 0$ and $\alpha \le 2\beta$; the sequence b_n decreases if and only if $-1 < \alpha < \beta \le \frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$ and $\alpha < 0$, or $0 < \alpha \le 2\beta$.

In 1997, H. Alzer [4] proved that if $k \ge 1$ and $n \ge 0$ are two integers, then we have for all real $x > 0$

$$S_k(2n,x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1,x), \tag{1.5}$$

where

$$S_k(p,x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}},$$

B_i for $i = 0, 1, 2, \dots$ are Bernoulli's numbers. In particular, taking in (1.5) $k = 1$ and $n = 0$ we get

$$\frac{1}{x} + \frac{1}{2x^2} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}, \quad x > 0. \tag{1.6}$$

It is worth noting that inequality (1.6) is first obtained by Gordon [10]. Please also refer to [4, p. 384].

From

$$\frac{(n+1)^n}{n!} = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)^2\cdots\left(1+\frac{1}{n}\right)^n \le \left(1+\frac{1}{n}\right)^{n^2}, \quad n \ge 1,$$

we obtain that

$$\frac{n+1}{\sqrt[n]{n!}} \le \left(1+\frac{1}{n}\right)^n < e, \quad n \ge 1. \tag{1.7}$$

By using (1.7), G. H. Hardy [12] presented a proof of Carleman's inequality

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}a_n, \tag{1.8}$$

where $a_n \ge 0$ for $n = 1, 2, \dots$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$.

Since $\Gamma(n+1) = n!$, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and $\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}} = e$, hence it is natural to ask whether $\left(1+\frac{1}{n}\right)^n$ and $\frac{n}{\sqrt[n]{n!}}$ or $\frac{n+1}{\sqrt[n]{n!}}$ can be compared. This is indeed possible! The following results give affirmative answers.

Theorem 1. *Let $x > 0$, then we have*

$$\left(1+\frac{1}{x}\right)^x > \frac{x}{[\Gamma(x+1)]^{1/x}} \tag{1.9}$$

and

$$\lim_{x\to\infty}\frac{x}{[\Gamma(x+1)]^{1/x}} = e. \tag{1.10}$$

Theorem 2. *Let $x > 1$, then we have*

$$\left(1+\frac{1}{x}\right)^x > \frac{x+1}{[\Gamma(x+1)]^{1/x}}. \tag{1.11}$$

The inequality (1.11) *is reversed for* $0 < x < 1$.

Remark 1. Inequality (1.11) for $x > 1$ is already proved in [2, Theorem 3] with the same proof as in [2, p. 7].

Remark 2. Recently, the second author and B.-N. Guo further obtain the following and more general results: The function $\frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x$ is strictly logarithmically completely monotonic in $(0,\infty)$.

Remark 3. There exist a lot of literature investigating the behaviour of similar functions involving the gamma and incomplete functions, for example, [2, 3, 4, 6, 11, 17, 18, 19, 23, 26, 27, 28] and references therein.

2. PROOFS OF THEOREM 1 AND THEOREM 2

Proof of Theorem 1. Taking logarithm and rearranging shows that inequality (1.9) is equivalent to

$$f(x) \triangleq x^2\ln(x+1)-(x^2+x)\ln x+\ln\Gamma(x+1)>0, \quad x>0.$$

Differentiating $f(x)$ and applying the right hand side inequality of (1.4) yields

$$\begin{aligned} f'(x) &= 2x\ln\left(1+\frac{1}{x}\right)+\psi(x+1)-\ln x+\frac{1}{x+1}-2 \\ &> 2x\ln\left(1+\frac{1}{x}\right)+\ln(x+1)-\ln x-2 \\ &= (2x+1)\ln\left(1+\frac{1}{x}\right)-2. \end{aligned}$$

Using the following inequality in [14, p. 367]:

$$\ln\left(1+\frac{1}{x}\right)>\frac{2}{2x+1}, \quad x>0, \tag{2.1}$$

we obtain that $f'(x)>0$, and then $f(x)>\lim_{x\to 0}f(x)=0$ for $x>0$.

Using the asymptotic expansion [9]

$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x-x+\ln\sqrt{2\pi}+\frac{1}{12x}+O\left(\frac{1}{x^3}\right) \quad \text{as } x\to\infty \tag{2.2}$$

and from

$$\begin{aligned} \ln\frac{x}{[\Gamma(x+1)]^{1/x}} &= \frac{x+1}{x}-\ln\left(1+\frac{1}{x}\right)-\frac{\ln(x+1)}{2x} \\ &\quad -\frac{\ln\sqrt{2\pi}}{x}-\frac{1}{12x(x+1)}+\frac{1}{x}O\left(\frac{1}{x^3}\right)\to 1 \quad \text{as } x\to\infty, \end{aligned}$$

we conclude that formula (1.10) holds. The proof of is complete. □

Remark 4. Notice that, in Theorem 1, the proof of $\lim_{x\to\infty}\frac{x}{\sqrt[x]{\Gamma(x+1)}}=e$ follows immediately from the asymptotic formula 6.1.39 in [1, p. 257].

Proof of Theorem 2. Define for $x>0$

$$f(x)=x^2\ln\left(1+\frac{1}{x}\right)-x\ln(x+1)+\ln\Gamma(x+1).$$

Since $f(0)=f(1)=0$, in order to prove our theorem it is sufficient to show that $f'(x)>0$ for $x\ge 1$ and $f''(x)>0$ for $0<x\le 1$. In other words, the function f is strictly increasing on $[1,\infty)$ and strictly convex on $(0,1]$. Differentiation yields

$$f'(x)=2x\ln\left(1+\frac{1}{x}\right)-\frac{2x}{x+1}+\psi(x+1)-\ln(x+1),$$

$$f''(x) = 2\ln\left(1+\frac{1}{x}\right) - \frac{3}{x+1} - \frac{2}{(x+1)^2} + \psi'(x+1).$$

Using the inequalities in [10]

$$\psi(x) > \ln x - \frac{1}{2x} - \frac{1}{12x^2}, \quad x > 0 \tag{2.3}$$

and (2.1), we have

$$\begin{aligned} f'(x) &> \frac{4x}{2x+1} - \frac{2x}{x+1} - \frac{1}{2(x+1)} - \frac{1}{12(x+1)^2} \\ &= \frac{12x^2+4x-7}{12(x+1)^2(2x+1)} > 0, \quad x \geq 1. \end{aligned}$$

Using the left-hand inequality of (1.6) we have

$$\begin{aligned} f''(x) &> 2\ln\left(1+\frac{1}{x}\right) - \frac{3}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{2(x+1)^2} \\ &= 2\ln\left(1+\frac{1}{x}\right) - \frac{2}{x+1} - \frac{3}{2(x+1)^2} \triangleq \phi(x). \end{aligned}$$

A simple computation yields

$$\phi'(x) = \frac{x-2}{x(x+1)^3} < 0, \quad 0 < x \leq 1.$$

Hence, we have $\phi(x) > \phi(1) = 0.01129436\ldots$ for $0 < x \leq 1$, and then $f''(x) > 0$ for $0 < x \leq 1$. The proof is complete. □

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