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ON AN EXTENSION OF HILBERT'S INTEGRAL INEQUALITY WITH SOME PARAMETERS

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ABSTRACT. In this paper, by introducing some parameters and estimating the weight function, we give an extension of Hilbert's integral inequality with a best constant factor. As applications, we consider the equivalent form and some particular results.

Key words and phrases: Hilbert's integral inequality, Weight function, Beta function, $H\ddot{o}lder's$ inequality.

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1. INTRODUCTION

If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (cf. Hardy et al.[1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well know as Hilbert's integral inequality, which is important in analysis and its applications (cf. Mitrinovic et. al [2]). And the equivalent form is

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx \right)^2 dy < \pi^2 \int_0^\infty f^2(x)dx, \tag{1.2}$$

where the constant factor π^2 is still the best possible.

In 1925, Hardy and Riesz gave some classical extended results on (1.1) and (1.2), by introducing (p,q)-parameter as follows (see [3], [1]):

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g$ are real functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \\ < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}; \end{aligned} \tag{1.3}$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx \right)^p dy < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \int_0^\infty f^p(x)dx, \tag{1.4}$$

where the constant factors $\pi/\sin(\frac{\pi}{p})$ and $[\pi/\sin(\frac{\pi}{p})]^p$ are the best possible. Inequality (1.3) is named Hardy-Hilbert's integral inequality, which is equivalent to (1.4). For $p = q = 2$, inequality (1.3) reduces to (1.1), and (1.4) reduces to (1.2).

In 1998, by introducing a parameter $\lambda \in (0, 1]$ and the β function $B(u, v)$ as (cf. Wang et al. [4]),

$$B(u, v) := \int_0^\infty \frac{x^{-1+u}}{(1+x)^{u+v}}dx = B(v, u) \ (u, v > 0), \tag{1.5}$$

Yang [5] gave a generalization of (1.1). For an improvement of [5], Yang [6], [7] gave some generalizations of (1.3) and (1.4) as:

If $\lambda > 2 - \min\{p, q\}, f, g$ are non-negative functions such that

$$0 < \int_0^\infty x^{1-\lambda} f^p(x)dx < \infty \ and \ 0 < \int_0^\infty x^{1-\lambda} g^q(x)dx < \infty,$$

then the following two inequalities are equivalent:

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \\ < k_\lambda(p) \left\{ \int_0^\infty x^{1-\lambda} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} g^q(x)dx \right\}^{\frac{1}{q}}; \end{aligned} \tag{1.6}$$

$$\int_0^\infty y^{(p-1)(\lambda-1)}\left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]^p dy < [k_\lambda(p)]^p\int_0^\infty x^{1-\lambda}f^p(x)dx, \tag{1.7}$$

where the constant factors $k_\lambda(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ and $[k_\lambda(p)]^p$ are the best possible. And for a refinement of [8], [9] also gave some extensions of (1.3) and (1.4) as:

If $\lambda > 0, f, g$ are non-negative functions such that

$$0 < \int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty x^{(q-1)(1-\lambda)}g^q(x)dx < \infty,$$

then the following two inequalities are equivalent:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < \frac{\pi}{\lambda\sin(\frac{\pi}{p})}\left\{\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{(q-1)(1-\lambda)}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{1.8}$$

$$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{f(x)}{x^\lambda+y^\lambda}dx\right)^p dy < [\frac{\pi}{\lambda\sin(\frac{\pi}{p})}]^p\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx, \tag{1.9}$$

where the constant factors $\frac{\pi}{\lambda\sin(\pi/p)}$ and $[\frac{\pi}{\lambda\sin(\pi/p)}]^p$ are the best possible. For $\lambda = 1$, both (1.6) and (1.8) reduce to (1.3), and both (1.7) and (1.9) reduce to (1.4). Recently, [10], [11] considered some multiple extensions of (1.1) and (1.3). In 2003, Yang et al [12] provided an extensive account of the above results.

In this paper, by using the β function and obtaining the expression of the weight function, we give a new extension of (1.1) with some parameters, such that both (1.3) and (1.8) are its particular results. As applications, we also consider the equivalent form and some other particular results.

2. SOME LEMMAS

Lemma 2.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{s} = 1,$*and* $\lambda > 0$, *define the weight function* $\omega_\lambda(s,p,x)$ *as*

$$\omega_\lambda(s,p,x) := \int_0^\infty \frac{1}{x^\lambda+y^\lambda}\cdot\frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{s}}}dy, x\in(0,\infty). \tag{2.1}$$

Then we have

$$\omega_\lambda(s,p,x) = \frac{\pi}{\lambda\sin(\pi/r)}x^{p(1-\frac{\lambda}{r})-1}. \tag{2.2}$$

Proof. For fixed x, setting $u = y^\lambda/x^\lambda$ in the integral of (2.1), by (1.5) we find

$$\omega_\lambda(s,p,x) = x^{p(1-\frac{\lambda}{r})-1}\frac{1}{\lambda}\int_0^\infty \frac{1}{1+u}u^{-1+\frac{1}{s}}du = x^{p(1-\frac{\lambda}{r})-1}\frac{1}{\lambda}B(\frac{1}{s},\frac{1}{r}). \tag{2.3}$$

Since $B(\frac{1}{s},\frac{1}{r}) = \frac{\pi}{\sin(\pi/r)}$, (2.2) is valid and the lemma is proved. ∎

Note. By (2.3), we still have

$$\omega_\lambda(r,q,y) := \int_0^\infty \frac{1}{x^\lambda + y^\lambda} \cdot \frac{y^{(q-1)(1-\frac{\lambda}{s})}}{y^{1-\frac{\lambda}{r}}} dx = y^{q(1-\frac{\lambda}{s})-1} \frac{\pi}{\lambda \sin(\frac{\pi}{r})}. \tag{2.4}$$

Lemma 2.2. *On the assumption of Lemma 2.1, if* $\varepsilon > 0$ *is small enough* ($\varepsilon < \frac{p\lambda}{r}$), *then we have*

$$\begin{aligned} I &:= \int_1^\infty \left(\int_1^\infty \frac{x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}}{x^\lambda + y^\lambda} dx \right) y^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}} dy \\ &\geq \frac{1}{\varepsilon\lambda} B(\frac{1}{r} - \frac{\varepsilon}{p\lambda}, \frac{1}{s} + \frac{\varepsilon}{p\lambda}) - O(1) \ (\varepsilon \to 0^+). \end{aligned} \tag{2.5}$$

Proof. For fixed y, setting $u = x^\lambda / y^\lambda$ in the integral of expression I, by (1.5) we find

$$\begin{aligned} I &= \frac{1}{\lambda} \int_1^\infty y^{-1-\varepsilon} \left(\int_{1/y^\lambda}^\infty \frac{u^{-1+\frac{1}{r}-\frac{\varepsilon}{p\lambda}}}{1+u} du \right) dy \\ &= \frac{1}{\lambda} \int_1^\infty y^{-1-\varepsilon} \left(\int_0^\infty \frac{u^{-1+\frac{1}{r}-\frac{\varepsilon}{p\lambda}}}{1+u} du \right) dy \\ &- \frac{1}{\lambda} \int_1^\infty y^{-1-\varepsilon} \left(\int_0^{1/y^\lambda} \frac{u^{-1+\frac{1}{r}-\frac{\varepsilon}{p\lambda}}}{1+u} du \right) dy \\ &\geq \frac{1}{\varepsilon\lambda} B(\frac{1}{r} - \frac{\varepsilon}{p\lambda}, \frac{1}{s} + \frac{\varepsilon}{p\lambda}) - \frac{1}{\lambda} \int_1^\infty y^{-1} \left(\int_0^{1/y^\lambda} u^{-1+\frac{1}{r}-\frac{\varepsilon}{p\lambda}} du \right) dy \\ &= \frac{1}{\varepsilon\lambda} B(\frac{1}{r} - \frac{\varepsilon}{p\lambda}, \frac{1}{s} + \frac{\varepsilon}{p\lambda}) - \left(\frac{\lambda}{r} - \frac{\varepsilon}{p} \right)^{-2}. \end{aligned}$$

Hence , (2.5) is valid and the lemma is proved. ∎

3. MAIN RESULTS AND APPLICATIONS

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{s} = 1,$ *and* $\lambda > 0$, f, g *are non-negative functions such that* $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx < \infty$ *and* $0 < \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x) dx < \infty$ *, then we have*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy \\ &< \frac{\pi}{\lambda \sin(\frac{\pi}{r})} \left\{ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x) dx \right\}^{\frac{1}{q}}, \end{aligned} \tag{3.1}$$

where the constant factor $\frac{\pi}{\lambda \sin(\pi/r)}$ *is the best possible. In particular,*

(a) for $r = s = 2$, *we have*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy \\ &< \frac{\pi}{\lambda} \left\{ \int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x) dx \right\}^{\frac{1}{q}}; \end{aligned} \tag{3.2}$$

(b) for $\lambda = 1$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{r})}\left\{\int_0^\infty x^{\frac{p}{s}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{q}{r}-1}g^q(x)dx\right\}^{\frac{1}{q}}. \tag{3.3}$$

Proof. By $H\ddot{o}lder's$ inequality, we have .

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy &= \int_0^\infty \int_0^\infty [\frac{f(x)}{(x^\lambda+y^\lambda)^{1/p}}\frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}}][\frac{g(y)}{(x^\lambda+y^\lambda)^{1/q}}\frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}}]dxdy \\ &\leq \left\{\int_0^\infty [\int_0^\infty \frac{1}{x^\lambda+y^\lambda}\frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{s}}}dy]f^p(x)dx\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\int_0^\infty [\int_0^\infty \frac{1}{x^\lambda+y^\lambda}\frac{y^{(q-1)(1-\frac{\lambda}{s})}}{x^{1-\frac{\lambda}{r}}}dx]g^q(y)dy\right\}^{\frac{1}{q}}.\end{aligned} \tag{3.4}$$

If (3.4) takes the form of equality, then there exists constants A and B, such that they are not all zero and (see [13])

$$A\frac{f^p(x)}{x^\lambda+y^\lambda}\frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{s}}} = B\frac{g^q(y)}{x^\lambda+y^\lambda}\frac{y^{(q-1)(1-\frac{\lambda}{s})}}{x^{1-\frac{\lambda}{r}}}, a.e. \text{ in } (0,\infty)\times(0,\infty).$$

We find that $Ax^{p(1-\frac{\lambda}{r})-1}f^p(x) = By^{q(1-\frac{\lambda}{s})-1}g^q(y), a.e.$ in $(0,\infty)\times(0,\infty)$. Hence there exists a constant C, such that

$$Ax^{p(1-\frac{\lambda}{r})-1}f^p(x) = C = By^{q(1-\frac{\lambda}{s})-1}g^q(y), a.e. \, in (0,\infty).$$

Without losing the generality, suppose that $A \neq 0$, we may get $x^{p(1-\frac{\lambda}{r})}f^p(x) = C/(Ax), a.e.$ in $(0,\infty)$, which contradicts $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty$. Hence, by (2.1), we can rewrite (3.4) as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < \left\{\int_0^\infty \omega_\lambda(s,p,x)f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega_\lambda(r,q,y)g^q(y)dy\right\}^{\frac{1}{q}}, \tag{3.5}$$

and in view of (2.2) and (2.4), we have (3.1).

For $\varepsilon > 0$ small enough ($\varepsilon < \frac{p\lambda}{r}$), setting f_ε and g_ε as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0,1)$;

$$f_\varepsilon(x) = x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}, g_\varepsilon(x) = x^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}, \text{ for } x \in [1,\infty),$$

then we find

$$J := \left\{\int_0^\infty \omega_\lambda(s,p,x)f_\varepsilon^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega_\lambda(r,q,y)g_\varepsilon^q(y)dy\right\}^{\frac{1}{q}} = \frac{1}{\varepsilon}.$$

If the constant factor $\frac{\pi}{\lambda\sin(\pi/r)}$ in (3.1) is not the best possible, then there exists a positive constant K (with $K < \frac{\pi}{\lambda\sin(\pi/r)}$), such that (3.1) is still valid if we replace $\frac{\pi}{\lambda\sin(\pi/r)}$ by K. In particular, by (2.5), we have

$$\begin{aligned}&\frac{1}{\lambda}B(\frac{1}{r}-\frac{\varepsilon}{p\lambda},\frac{1}{s}+\frac{\varepsilon}{p\lambda})-O(1)\\ \leq\ &\varepsilon I=\int_0^\infty\int_0^\infty\frac{f_\varepsilon(x)g_\varepsilon(y)}{x^\lambda+y^\lambda}dxdy<\varepsilon KJ=K.\end{aligned}$$

For $\varepsilon\to 0^+$, it follows that $\frac{\pi}{\lambda\sin(\pi/r)}\leq K$, which contradicts the fact that $K<\frac{\pi}{\lambda\sin(\pi/r)}$. Hence the constant factor $\frac{\pi}{\lambda\sin(\pi/r)}$ in (3.1) is the best possible. The theorem is proved. ∎

Theorem 3.2. *If* $p>1,\frac{1}{p}+\frac{1}{q}=1,r>1,\frac{1}{r}+\frac{1}{s}=1,$ *and* $\lambda>0,\ f$ *is a non-negative function such that* $0<\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx<\infty$ *, then we have*

$$\begin{aligned}&\int_0^\infty y^{\frac{p\lambda}{s}-1}\left(\int_0^\infty\frac{f(x)}{x^\lambda+y^\lambda}dx\right)^p dy\\ <\ &\left[\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\right]^p\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx,\end{aligned}\tag{3.6}$$

where the constant factor $[\frac{\pi}{\lambda\sin(\pi/r)}]^p$ *is the best possible. Inequality (3.6) is equivalent to (3.1). In particular,*

(a) for $r=s=2,$ *we have*

$$\begin{aligned}&\int_0^\infty y^{\frac{p\lambda}{2}-1}\left(\int_0^\infty\frac{f(x)}{x^\lambda+y^\lambda}dx\right)^p dy\\ <\ &\left(\frac{\pi}{\lambda}\right)^p\int_0^\infty x^{p(1-\frac{\lambda}{2})-1}f^p(x)dx;\end{aligned}\tag{3.7}$$

(b) for $\lambda=1,$ *we have*

$$\int_0^\infty y^{\frac{p}{s}-1}\left(\int_0^\infty\frac{f(x)}{x+y}dx\right)^p dy<\left[\frac{\pi}{\sin(\frac{\pi}{s})}\right]^p\int_0^\infty x^{\frac{p}{s}-1}f^p(x)dx.\tag{3.8}$$

Proof. Setting a real function $g(y)$ as

$$g(y):=y^{\frac{p\lambda}{s}-1}\left(\int_0^\infty\frac{f(x)}{x^\lambda+y^\lambda}dx\right)^{p-1}dy,y\in(0,\infty),$$

then by (3.1), we find

$$\begin{aligned}&\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy\\ =\ &\int_0^\infty y^{\frac{p\lambda}{s}-1}\left(\int_0^\infty\frac{f(x)}{x^\lambda+y^\lambda}dx\right)^p dy=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy\\ \leq\ &\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1}g^q(x)dx\right\}^{\frac{1}{q}}.\end{aligned}\tag{3.9}$$

Hence we obtain

$$\begin{aligned}&0<\left\{\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy\right\}^{\frac{1}{p}}\\ \leq\ &\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\right\}^{\frac{1}{p}}<\infty.\end{aligned}\tag{3.10}$$

By (3.1), both (3.9) and (3.10) take the form of strict inequality, and we have (3.6).

On the other hand, suppose that (3.6) is valid. By $H\ddot{o}lder's$ inequality, we find

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy \\ = &\int_0^\infty (y^{\frac{\lambda}{s}-\frac{1}{p}} \int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx)(y^{-\frac{\lambda}{s}+\frac{1}{p}} g(y))dy \\ \leq &\left\{ \int_0^\infty y^{\frac{p\lambda}{s}-1} \left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx \right)^p dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right\}^{\frac{1}{q}}. \end{aligned} \tag{3.11}$$

Then by (3.6), we have (3.1). Hence (3.1) and (3.6) are equivalent.

If the constant factor $[\frac{\pi}{\lambda \sin(\pi/r)}]^p$ in (3.6) is not the best possible, by using (3.11), we may get a contradiction that the constant factor in (3.1) is not the best possible. Thus we complete the proof of the theorem. ∎

For $r = p, s = q$, by (3.1) and (3.6), we have

Corollary 3.3. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, *and* $\lambda > 0, f, g$ *are non-negative real functions, such that* $0 < \int_0^\infty x^{p-\lambda-1} f^p(x) dx < \infty$ *and* $0 < \int_0^\infty x^{q-\lambda-1} g^q(x) dx < \infty$ *, then we have the following two equivalent inequalities:*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy \\ < &\frac{\pi}{\lambda \sin(\frac{\pi}{p})} \left\{ \int_0^\infty x^{p-\lambda-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q-\lambda-1} g^q(x) dx \right\}^{\frac{1}{q}}; \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\int_0^\infty y^{(p-1)\lambda-1} \left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx \right)^p dy \\ < &\left[\frac{\pi}{\lambda \sin(\frac{\pi}{p})} \right]^p \int_0^\infty x^{p-\lambda-1} f^p(x) dx, \end{aligned} \tag{3.13}$$

where the constant factors $\frac{\pi}{\lambda \sin(\pi/p)}$ *and* $[\frac{\pi}{\lambda \sin(\pi/p)}]^p$ *are the best possible. In particular, for* $\lambda = 1$, *we have*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y} dxdy \\ < &\frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty x^{p-2} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q-2} g^q(x) dx \right\}^{\frac{1}{q}}; \end{aligned} \tag{3.14}$$

$$\int_0^\infty y^{p-2} \left(\int_0^\infty \frac{f(x)}{x + y} dx \right)^p dy < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \int_0^\infty x^{p-2} f^p(x) dx. \tag{3.15}$$

For $p = q = 2$, by (3.3) and (3.8), we have

Corollary 3.4. *If* $r > 1, \frac{1}{r} + \frac{1}{s} = 1$, f, g *are non-negative real functions, such that* $0 < \int_0^\infty x^{\frac{2}{s}-1} f^2(x) dx < \infty$ *and* $0 < \int_0^\infty x^{\frac{2}{r}-1} g^2(x) dx < \infty$ *, then we have the following two* *equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{s})}\left\{\int_0^\infty x^{\frac{2}{s}-1}f^2(x)dx \int_0^\infty x^{\frac{2}{r}-1}g^2(x)dx\right\}^{\frac{1}{2}}; \tag{3.16}$$

$$\int_0^\infty y^{\frac{2}{s}-1}\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^2 dy < \left[\frac{\pi}{\sin(\frac{\pi}{s})}\right]^2 \int_0^\infty x^{\frac{2}{s}-1}f^2(x)dx, \tag{3.17}$$

where the constant factors $\frac{\pi}{\sin(\pi/s)}$ *and* $[\frac{\pi}{\sin(\pi/s)}]^2$ *are the best possible.*

Remark 3.1. (a) For $r = q, s = p$, (3.1) reduces to (1.8), and (3.6) reduces to (1.9). Relating Corollary 3.3, it follows that (3.1) is a new extension of (1.8) and (3.12), and (3.6) is a new extension of (1.9) and (3.13).

(b) It is interesting that (1.8), (3.2) and (3.12) are deferent, although they are with the same parameters and possess the best constant factor.

(c) (3.16) is an extension of (1.1) with two parameters (r, s), and (3.7) is an extension of (1.2) with a single parameter $s > 1$.

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