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D-ITERATIVE METHOD FOR SOLVING A DELAY DIFFERENTIAL EQUATION AND A TWO-POINT SECOND-ORDER BOUNDARY VALUE PROBLEMS IN BANACH SPACES

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ABSTRACT. The purpose of this paper is to re-establish the convergence, stability and data dependence results established by [2] and [3] by removing the strong assumptions imposed on the sequences which were used to obtain their results. In addition, we introduced a modified approach using the D-iterative method to solve a two-point second-order boundary value problem, and also obtain the solution of a delay differential equations using the obtained results in this paper. The results presented in this paper do not only extend and improve the results obtained in [2, 3], it further extends and improve some existing results in the literature.

Key words and phrases: Iterative scheme; Fixed point; Delay differential equations, D-iteration.

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1. INTRODUCTION AND PRELIMINARIES

Let X be any arbitrary space, a point $x \in X$ is called a fixed point of a mapping $T : X \to X$ if

$$Tx = x, \tag{1.1}$$

that is, a point $x \in X$ which remains invariant under the action of the mapping T. Let C be a nonempty, closed and convex subset of a Banach space X and T a self map on C. We denote by $F(T) = \{x : x = Tx\}$ the fixed points of T. We recall that a mapping $T : C \to C$ is said to be a contraction if for all $x, y \in C$ there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{1.2}$$

For the past 70 years, researchers have paid a very good attention to finding an analytical solution to problem (1.1), but this have been almost practically impossible. In view of this, iterative method has been adopted in finding an approximate solution to (1.1). A good number of iterative processes (explicit, implicit, Jungck-type and so on) have been introduced and studied by many authors, (see [4, 5, 6, 7, **?**, 8, 9, 10] and the reference there in). Iterative methods can produce numerical solutions to certain classes of problems of nonlinear analysis, that can be thought in terms of fixed point theory, where analytical methods may fail. Developing a faster and more effective iterative techniques for approximating fixed points of nonlinear mappings is still an open problem in this area of research. In the light of this, Hussain et al. [2] introduced a new iterative method called the D-iteration. The D-iterative method is defined as follows

$$\begin{cases} x_0 \in C, \\ y_n = T((1-\alpha_n)x_n + \alpha_n Tx_n), \\ v_n = T((1-\beta_n)Tx_n + \beta_n Ty_n), \\ x_{n+1} = Tv_n, \ n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ for all $n \in \mathbb{N}$. They established that the D-iterative process is faster than M-iterative process in [11], M* iterative process in [12] and some other existing iterative process in the literature. Furthermore, Hussain et al. [3] presented the stability, data dependency and errors estimation results for D-iteration method. More so, they establish that the error in D-iterative process is controllable. However, the convergence, stability, and data dependence results were obtained under some strong assumptions imposed on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. For example, they established the following results.

Theorem 1.1 ([2, 3]). *Let C be a nonempty closed convex subset of a Banach space space X. and $T : C \to C$ be a contraction mapping. Assume $\{x_n\}$ to be an iterative sequence generated by* (1.3), *where $\{\alpha_n\}$, and $\{\beta_n\}$ are sequences in $[0, 1]$ are real sequences satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty = \sum_{n=1}^{\infty} \beta_n$. Then the sequence $\{x_n\}$ converges strongly to a unique fixed point of T.*

Remark 1.1. We claim that the assumption $\sum_{n=1}^{\infty} \alpha_n = \infty = \sum_{n=1}^{\infty} \beta_n$ is not relevant in achieving the above result. We shall establish our claim in the next section.

Theorem 1.2 ([3]). *Let C be a nonempty closed convex subset of a Banach space space X. and $T : C \to C$ be a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by* (1.3) *with sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ satisfying $\sum_{n=0}^{m} [\beta_n + k\alpha_n\beta_n] = \infty$ for all $n \in \mathbb{N}$. Then the iterative process* (2.9) *is T-stable.*

Remark 1.2. We claim that the assumption $\sum_{n=0}^{m} [\beta_n + k\alpha_n\beta_n] = \infty$ is not relevant in achieving the above result. We shall establish our claim in the next section.

Question: It is natural to ask if the D-iterative method (1.3) can be used to approximate the solution of a Two Point Second Order- Boundary Value Problem (TPSO-BVP)?

We recall the following results that will be relevant in the course of this study.

Definition 1.1. Let $T : C \to C$ be a mapping. Define an iterative method by

$$x_{n+1} = f(T, u_n) \tag{1.4}$$

such that $\{x_n\}$ converges to a fixed point x^* of T. Suppose that $\{u_n\}$ is an arbitrary sequence in C and set

$$\epsilon_n = \|u_{n+1} - f(T, u_n)\|$$

for all $n \in \mathbb{N}$. The iterative process (1.4) is said to be T stable or stable with respect to T if

$$\lim_{n\to\infty} \epsilon_n = 0$$

if and only if

$$\lim_{n\to\infty} u_n = x^*.$$

Definition 1.2. Let $T, \overline{T} : C \to C$ be two mappings. Then $\overline{T}$ is said to be an approximate operation of T if there exists $\epsilon > 0$ such that $\|Tx - \overline{T}x\| \le \epsilon$ for all $x \in C$.

Lemma 1.3. [13] *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be nonnegative real sequence satisfying the following inequalities*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \beta_n$$

where $\gamma_n \in (0, 1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} \gamma_n = \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{\gamma_n} = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.

The purpose of this paper is to provide an affirmative answer to the above question and to re-establish the results obtained in the above-mentioned works of (Hussain et al., [2] and Hussain et al., [3]) on convergence, stability and data dependence results by removing the strong assumptions used to obtain their results. Our approach modifies the existing results as well as improves and extends the results obtained in [2, 3] and in other literature.

2. MAIN RESULTS

In this section, we establish that the convergence, stability and data dependence results of the D-iterative method (1.3) for the contraction mappings are independent of the choice of the real sequences $\{\alpha_n\}$ and $\{\beta_n\}$. It is easy to see that the D-iteration can be re-written in the form

$$\begin{cases} x_0 \in C, \\ z_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n, \\ w_n = (1 - \beta_n)T x_n + \beta_n T y_n, \\ v_n = T w_n, \\ x_{n+1} = T v_n, \ n \ge 1, \end{cases} \tag{2.1}$$

Theorem 2.1. *Let C be a nonempty closed convex subset of a Banach space space X. and $T : C \to C$ be a contraction mapping. Assume $\{x_n\}$ to be an iterative sequence generated by* (2.1), *where $\{\alpha_n\}$, and $\{\beta_n\}$ are sequences in $[0, 1]$. Then the sequence $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. In the proof of the above result, the authors in [2, 3] arrived at the following inequality

$$\|x_{n+1} - p\| \leq k^{3(n+1)}\|x_0 - p\| \prod_{n=0}^{m}[1 - (\beta_n + k\alpha_n\beta_n)(1 - k)]. \tag{2.2}$$

It is easy to see that $1 - (\beta_n + k\alpha_n\beta_n)(1 - k) < 1$, since $k \in [0, 1), \{\alpha_n\}$ and $\beta_n \in [0, 1]$ for all $n \in \mathbb{N}$. Thus, we have

$$\prod_{n=0}^{m}[1 - (\beta_n + k\alpha_n\beta_n)(1 - k)] < 1,$$

as such we have (2.2) becomes

$$\|x_{n+1} - p\| \leq k^{3(n+1)}\|x_0 - p\|. \tag{2.3}$$

Taking the limit as $n \to \infty$ in (2.3), we have

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0.$$

Since $k \in (0, 1)$ and we know that $\lim\limits_{n\to\infty} k^{3(n+1)} = 0$. ∎

Remark 2.1. In the light of this development, we have provided an affirmative answer to Remark 1.1 that the condition $\sum_{n=1}^{\infty} \alpha_n = \infty = \sum_{n=1}^{\infty} \beta_n$ is not relevant in achieving the above result.

Theorem 2.2. *Let C be a nonempty closed convex subset of a Banach space space X. and $T : C \to C$ be a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by* (2.1) *with sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ for all $n \in \mathbb{N}$. Then the iterative process* (2.1) *is T-stable.*

Proof. Let $\{u_n\}$ be an arbitrary sequence in C and suppose that the sequence $\{\epsilon_n\}$ is defined as

$$\epsilon_n = \|u_{n+1} - Tq_n\|, \tag{2.4}$$

where $q_n = Tm_n,\ m_n = (1 - \beta_n)Tu_n + \beta_n Tl_n, l_n = Tk_n$ and $k_n = (1 - \beta_n)u_n + \beta_n Tu_n$ for all $n \in \mathbb{N}$. Suppose that $\lim\limits_{n\to\infty} \epsilon_n = 0$. We need to show that $\lim_{n\to\infty} u_n = p$. Using (2.4), (2.1) and (1.2), we obtain

$$\begin{aligned}
&\|u_{n+1}-p\|\\
&=\|u_{n+1}-x_{n+1}+x_{n+1}-p\|\\
&\le\|u_{n+1}-x_{n+1}\|+\|x_{n+1}-p\|\\
&\le\|u_{n+1}-Tq_n\|+\|Tq_n-x_{n+1}\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+\|Tq_n-Tv_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+k\|q_n-v_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+k^2\|m_n-w_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+k^2(1-\beta_n)\|Tu_n-Tx_n\|+k^3\beta_n\|y_n-l_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+k^3(1-\beta_n)\|u_n-x_n\|+k^4\beta_n\|k_n-z_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+k^3(1-\beta_n)\|u_n-x_n\|+k^4\beta_n[(1-\alpha_n)\|u_n-x_n\|+\alpha_nk\|u_n-x_n\|]+\|x_{n+1}-p\|\\
&\le\epsilon_n+k^3(1-\beta_n)\|u_n-x_n\|+k^4\beta_n(1-\alpha_n(1-k))\|u_n-x_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+k^3[1-\beta_n(1-k(1-\alpha_n(1-k)))]\|u_n-x_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+\|u_n-x_n\|+\|x_{n+1}-p\|\\
&\le\epsilon_n+\|u_n-p\|+\|x_n-p\|+\|x_{n+1}-p\|. \qquad (2.5)
\end{aligned}$$

It is easy to see that $k^3[1-\beta_n(1-k(1-\alpha_n(1-k)))]<1$, since $k\in[0,1),\{\alpha_n\},\{\beta_n\}$ are in $[0,1]$, we get $k^2(1-\alpha_n(1-k))<1$ and $1-\beta_n(1-k(1-\alpha_n(1-k)))<1$, thus, $k^3[1-\beta_n(1-k(1-\alpha_n(1-k)))]<1$. Then, we have

$$\|u_{n+1}-p\|-\|u_n-p\|\le\epsilon_n+\|x_n-p\|+\|x_{n+1}-p\|, \qquad (2.6)$$

using Theorem 2.1, we have that $\lim_{n\to\infty}\|x_{n+1}-p\|=0=\lim_{n\to\infty}\|x_n-p\|$ and our assumption that $\lim_{n\to\infty}\epsilon_n=0$, we have

$$\lim_{n\to\infty}[\|u_{n+1}-p\|-\|u_n-p\|]=0, \qquad (2.7)$$

as such, we have

$$\lim_{n\to\infty}u_n=p.$$

Conversely, using the fact that $\lim_{n\to\infty}u_n=p$ and (2.1), we have

$$\begin{aligned}
\epsilon_n&=\|u_{n+1}-Tq_n\|\\
&\le\|u_{n+1}-x_{n+1}\|+\|x_{n+1}-Tq_n\|\\
&=\|u_{n+1}-x_{n+1}\|+\|Tv_n-Tq_n\|\\
&\le\|u_{n+1}-x_{n+1}\|+k^3[1-\beta_n(1-k(1-\alpha_n(1-k)))]\|u_n-x_n\|\\
&\le\|u_{n+1}-p\|+\|x_{n+1}-p\|+k^3[1-\beta_n(1-k(1-\alpha_n(1-k)))]\|u_n-p\|\\
&\quad+k^3[1-\beta_n(1-k(1-\alpha_n(1-k)))]\|x_n-p\|. \qquad (2.8)
\end{aligned}$$

Using our assumption $\lim_{n\to\infty}\|u_n-p\|=0=\lim_{n\to\infty}\|u_{n+1}-p\|$ and Theorem 2.1 ($\lim_{n\to\infty}\|x_{n+1}-p\|=0\lim_{n\to\infty}\|x_n-p\|$). We have $\lim_{n\to\infty}\epsilon_n=0$. Hence, the sequence $\{x_n\}$ is T-Stable. ∎

Remark 2.2. In the light of this development, we have provided an affirmative answer to Remark 1.2 that the condition $\sum_{n=0}^{m}[\beta_n+k\alpha_n\beta_n]=\infty$ is not relevant in achieving the above result.

Theorem 2.3. *Let C be a nonempty closed convex subset of a Banach space X. and $T,: C \to C$ be a contraction mapping and $\overline{T}$ be an approximate mapping of T with maximum admissible error ϵ. Let $\{x_n\}$ be an iterative sequence generated by* (2.1) *and define an iterative sequence $\{\overline{x}\}$ as follows*

$$\begin{cases} \overline{z}_n = (1-\alpha_n)\overline{x}_n + \alpha_n \overline{T}\overline{x}_n, \\ \overline{y}_n = \overline{T}\overline{z}_n, \\ \overline{w}_n = (1-\beta_n)\overline{T}\overline{x}_n + \beta_n \overline{T}\overline{y}_n, \\ \overline{v}_n = \overline{T}\overline{w}_n, \\ \overline{x}_{n+1} = \overline{T}\overline{v}_n, \ n \geq 1, \end{cases} \tag{2.9}$$

with real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ for all $n \in \mathbb{N}$. If $Tp = p$ and $\overline{T}\overline{p} = \overline{p}$ such that $\lim_{n\to\infty} \overline{x} = \overline{p}$ then $\|p - \overline{p}\| \leq \left(\frac{3+2k}{1-k}\right)\epsilon$.

Proof. Using (2.1), (2.9) and (1.2), we have

$$\begin{aligned}
&\|x_{n+1} - \overline{x}_{n+1}\| \\
&= \|Tv_n - \overline{T}\overline{v}_n\| \\
&\leq \|Tv_n - T\overline{v}\| + \|T\overline{v} - \overline{T}\overline{v}_n\| \\
&\leq k\|v_n - \overline{v}\| + \epsilon \\
&= k\|Tw_n - \overline{T}\overline{w}_n\| + \epsilon \\
&\leq k\|Tw_n - T\overline{w}_n\| + k\|T\overline{w}_n - \overline{T}\overline{w}_n\| + \epsilon \\
&\leq k^2\|w_n - \overline{w}_n\| + k\epsilon + \epsilon \\
&= k^2\|(1-\beta_n)Tx_n + \beta_n Ty_n - (1-\beta_n)\overline{T}\overline{x}_n - \beta_n\overline{T}\overline{y}_n\| + (k+1)\epsilon \\
&\leq k^2(1-\beta_n)\|Tx_n - T\overline{x}\| + k^2(1-\beta_n)\|T\overline{x} - \overline{T}\overline{x}_n\| + \beta_n k^2\|Ty_n - T\overline{y}\| \\
&+ \beta_n k^2\|T\overline{x} - \overline{T}\overline{y}_n\| + (k+1)\epsilon \\
&\leq k^3(1-\beta_n)\|x_n - \overline{x}\| + \beta_n k^3\|y_n - \overline{y}\| + (k^2(1-\beta_n) + \beta_n k^2 + k + 1)\epsilon \\
&= k^3(1-\beta_n)\|x_n - \overline{x}_n\| + \beta_n k^3\|Tz_n - \overline{T}\overline{z}_n\| + (k^2(1-\beta_n) + \beta_n k^2 + k + 1)\epsilon \\
&= k^3(1-\beta_n)\|x_n - \overline{x}_n\| + \beta_n k^3\|Tz_n - T\overline{z}\| + \beta_n k^3\|T\overline{z} - \overline{T}\overline{z}_n\| + (k^2(1-\beta_n) \\
&+ \beta_n k^2 + k + 1)\epsilon \\
&= k^3(1-\beta_n)\|x_n - \overline{x}_n\| + \beta_n k^4\|z_n - \overline{z}\| + (\beta_n k^3 + k^2(1-\beta_n) + \beta_n k^2 + k + 1)\epsilon \\
&= k^3(1-\beta_n)\|x_n - \overline{x}_n\| + \beta_n k^4[\|(1-\alpha_n)x_n + \alpha_n Tx_n - (1-\alpha_n)\overline{x}_n - \alpha_n\overline{T}\overline{x}_n\|] \\
&+ (\beta_n k^3 + k^2(1-\beta_n) + \beta_n k^2 + k + 1)\epsilon \\
&= k^3(1-\beta_n)\|x_n - \overline{x}_n\| + \beta_n k^4(1-\alpha_n)\|x_n - \overline{x}\| + \beta_n k^4\alpha_n\|Tx_n - \overline{T}\overline{x}_n\|\| \\
&+ (\beta_n k^3 + k^2(1-\beta_n) + \beta_n k^2 + k + 1)\epsilon
\end{aligned}$$

(2.10)

$$\leq k^3[1-\beta_n(1-k(1-\alpha_n))]\|x_n-\overline{x}\|+\beta_n k^4\alpha_n\|Tx_n-T\overline{x}\|+\beta_n k^4\alpha_n\|T\overline{x}-\overline{T}\overline{x}_n\|\|$$
$$+(\beta_n k^3+k^2(1-\beta_n)+\beta_n k^2+k+1)\epsilon$$
$$\leq k^3[1-\beta_n(1-k(1-\alpha_n))]\|x_n-\overline{x}\|+\beta_n k^5\alpha_n\|x_n-\overline{x}\|$$
$$+(\beta_n k^4\alpha_n+\beta_n k^3+k^2(1-\beta_n)+\beta_n k^2+k+1)\epsilon$$

$$= k^3[1-\beta_n[1-k(1-\alpha_n(1-k))]]\|x_n-\overline{x}\|+[1+k(1+\beta_n k(1+k(1+k\alpha_n)))]\epsilon. \tag{2.11}$$

Since $\{\alpha_n\},\{\beta_n\}$ are in $[0,1]$ and $k\in[0,1)$, we have the following estimate

$$1-\beta_n[1-k(1-\alpha_n(1-k))]\leq 1 \Rightarrow k^3[1-\beta_n[1-k(1-\alpha_n(1-k))]]\leq k^3<k, \tag{2.12}$$
$$1+k\alpha_n\leq 2$$
$$k(1+k\alpha_n)\leq 2k<2$$
$$1+k(1+k\alpha_n)\leq 1+2k$$
$$\beta_n k(1+k(1+k\alpha_n))\leq (1+2k)\beta_n k<(1+2k)$$
$$1+\beta_n k(1+k(1+k\alpha_n))\leq 1+(1+2k)\beta_n k<(2+2k)=2(1+k)$$
$$k(1+\beta_n k(1+k(1+k\alpha_n)))\leq k(1+(1+2k)\beta_n k)<2k(1+k)<2(1+k)$$

$$[1+k(1+\beta_n k(1+k(1+k\alpha_n)))]\leq (1+k(1+(1+2k)\beta_n k))<3+2k. \tag{2.13}$$

Thus, using (2.12), (2.10), becomes

$$\|x_{n+1}-\overline{x}_{n+1}\|\leq k\|x_n-\overline{x}\|+(3+2k)\epsilon. \tag{2.14}$$

Now, by Theorem 2.1, we obtain $\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}x_n=p$ and by our assumption that $\lim_{n\to\infty}\overline{x}_{n+1}=\lim_{n\to\infty}\overline{x}_n=\overline{p}$. Taking the limit as $n\to\infty$ of (2.14), we have

$$\|p-\overline{p}\|\leq\left(\frac{3+2k}{1-k}\right)\epsilon. \tag{2.15}$$

∎

Remark 2.3. Our proof technique does not require the following assumption used in [3]. That is

(1) $\frac{1}{2}\leq\alpha_n+k\beta_n\alpha_n$.
(2) $\sum_{n=0}^{\infty}(\alpha_n+k\beta_n\alpha_n)=\infty$.

In addition, our estimate $\left(\frac{3+2k}{1-k}\right)\epsilon$ is better than the estimate $\left(\frac{7\epsilon}{1-k}\right)\epsilon$.

3. APPLICATION

3.1. **Application to a Two Point Second Order- Boundary Value Problem (TPSO-BVP).** We draw our inspiration from the work of Bello et al. [1]. The authors considered the following Two Point Second Order- Boundary Value Problem (TPSO-BVP).

$$x^{"}=f(t,x,x^{'}),\ 0\leq t\leq 1 \tag{3.1}$$

$$\begin{cases}\alpha_0x(0)+\beta_0x^{'}(0)=\gamma_0,\\ \alpha_1x(1)+\beta_1x^{'}(1)=\gamma_1,\end{cases} \tag{3.2}$$

where α_j, β_j and γ_j for $j = 0, 1$ and $\alpha_j^2 + \beta_j^2 > 0$. It was established in [1] that $x(t)$ is a solution of (3.1)-(3.2) if and only if $x(t)$ is a solution of the equivalent integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s), x^{'}(s)) ds + y(t) \tag{3.3}$$

on $[a, b]$ where

$$G(t,s) = \begin{cases} (t)(s-1), \ 0 \leq t \leq s \\ (t-1)(s), \ s \leq t \leq 1, \end{cases} \tag{3.4}$$

is the Green function associated to the TPSO-BVP (3.1)-(3.2),

$$x^{"} = 0, \ 0 \leq t \leq 1 \tag{3.5}$$

$$\begin{cases} \alpha_0 x(0) + \beta_0 x^{'}(0) = \gamma_0, \\ \alpha_1 x(1) + \beta_1 x^{'}(1) = \gamma_1, \end{cases} \tag{3.6}$$

and $y(t)$ is the solution (3.5)-(3.13).

In what follows, we introduce a new approach related to the iterative method (2.1) to solve TPSOâĂŞBVP (3.1)-(3.2). In the light of (3.2), we modified iterative process (2.1) as follows:

$$\begin{cases} z_n^{''} = (1-\alpha_n) x_n^{''} + \alpha_n f(t, x_n, x_n^{'}), \\ \alpha_0 z_n(0) + \beta_0 z_n^{'}(0) = \gamma_0, \ \alpha_1 z_n(1) + \beta_1 z_n^{'}(1) = \gamma_1 \\ \\ y_n^{''} = f(t, z_n, z_n^{'}), \\ \alpha_0 y_n(0) + \beta_0 y_n^{'}(0) = \gamma_0, \ \alpha_1 y_n(1) + \beta_1 y_n^{'}(1) = \gamma_1 \\ \\ w_n^{''} = (1-\alpha_n) f(t, x_n, x_n^{'}) + \alpha_n f(t, y_n, y_n^{'}), \\ \alpha_0 w_n(0) + \beta_0 w_n^{'}(0) = \gamma_0, \ \alpha_1 w_n(1) + \beta_1 w_n^{'}(1) = \gamma_1 \\ \\ v_n^{''} = f(t, w_n, w_n^{'}), \\ \alpha_0 v_n(0) + \beta_0 v_n^{'}(0) = \gamma_0, \ \alpha_1 v_n(1) + \beta_1 v_n^{'}(1) = \gamma_1 \\ \\ x_{n+1}^{''} = f(t, v_n, v_n^{'}), \\ \alpha_0 x_{n+1}(0) + \beta_0 x_{n+1}^{'}(0) = \gamma_0, \ \alpha_1 x_{n+1}(1) + \beta_1 x_{n+1}^{'}(1) = \gamma_1, \end{cases} \tag{3.7}$$

where $\{\alpha\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\alpha_j, \beta_j, \gamma_j$ are sequences in $\mathbb{R}$ with $\alpha_j^2 + \beta_j^2 > 0$ for $j = 0, 1$ and $x_0(t)$ is an initial function satisfying the boundary conditions in (3.2).

Theorem 3.1. *Suppose $f(t, x(t), x^{'}(t))$ be a function whose derivative is bounded with respect to x and that $\{x_n\}$ is an iterative sequence in $C^1[0, 1]$ generated by* (2.1) *with the sequences α_n, β_n in $[0, 1]$. Let $x_0(t)$ be an arbitrary function in $C^1[0, 1]$ that satisfies $x^{''} = 0$, as well as the boundary condition* (3.2) *and*

$$\Gamma = \max_{[0,1]\times\mathbb{R}^2} |\frac{\partial f}{\partial x}|,$$

where $\eta = \frac{3}{8}\Gamma < 1$. Then, (3.1)-(3.2) *has a unique solution in $C^1[a, b]$ and the iterative sequence $\{x_n\}$ converges uniquely to $x^*(t)$.*

Proof. To establish that (3.1)-(3.2) has a unique solution in $C^1[0,1]$ and that the iterative sequence $\{x_n\}$ converges uniquely to $x^*(t)$. It suffices to establish that

$$T(x(t)) = \int_0^1 G(t,s)f(s,x(s),x^{'}(s))ds + g(t)$$

is a contraction and then, apply the Banach principle. It is well-known that the absolute maximum value of the function $\int_0^1 G(t,s)ds = \frac{3}{8}$. Now, let $x(t), y(t) \in C^1[0,1]$ such that $x(t) \neq y(t)$, we have

$$\begin{aligned}|Tx(t) - Ty(t)| &= \left| \int_0^1 G(t,s)f(s,x(s),x^{'}(s))ds + g(t) - \left[\int_0^1 G(t,s)f(s,y(s),y^{'}(s))ds + g(t)\right]\right| \\ &\leq \int_0^1 |G(t,s)||f(s,x(s),x^{'}(s)) - f(s,y(s),y^{'}(s))|ds \\ &\leq \frac{3}{8}\int_0^1 |f(s,x(s),x^{'}(s)) - f(s,y(s),y^{'}(s))| \\ &\leq \frac{3}{8}\max_{s\in[0,1]} |f(s,x(s),x^{'}(s)) - f(s,y(s),y^{'}(s))| \\ &\leq \eta|x(t) - y(t)|.\end{aligned}$$

It is clear that $Tx(t)$ is a contraction mapping and hence by the Banach contraction mapping principle, it has a unique fixed point $x^*(t)$. In addition, using Theorem 2.1, the iterative sequence $\{x_n\}$ converges uniformly to $x^*(t)$. ∎

Example 3.1. *We consider the following two point second order boundary value problem*

$$\begin{cases} x^{''} = t^3 - tx(t) + 1 \\ x'(0) = x(0) \\ x'(1) + x(1) = 3. \end{cases} \tag{3.8}$$

Clearly $x^{''} = f(t,x,x') = t^3 - tx(t) + 1$, *we have* $\frac{\partial f}{\partial x} = -t$. *Thus, we have*

$$\Gamma = \max_{[0,1]\times\mathbb{R}^2} \left|\frac{\partial f}{\partial x}\right| = 1,$$

it follows that the derivate of $x^{''} = f(t,x,x')$ *with respect to* x *is bounded and* $\eta = \frac{3}{8} < 1$. *It is easy to see that the initial conditions are satisfied if we take* $x_0 = t+1$. *Clearly, we have* $x'(0) = x(0)$, *and* $x'(0) + x(1) = 3$. *It is easy to see that all the conditions in Theorem 3.1 are satisfied. Hence, the problem* (3.8) *has a unique solution* $x^*(t) = t^2 \in C^1[0,1]$.

In what follow, we compare the exact solution with the approximate solution using the iterative sequence (3.7) *with* $\alpha_n = \frac{1}{n+2}$ *and* $\beta_n = \frac{2n}{5n^2+70}$.

t	$x^*(t)$	$\{x_n\}$ *(Algorithm* (3.7)*)*
0	*0*	*0.00000006*
0 .1	*0.01*	*0.01000007*
0.2	*0.04*	*0.04000008*
0.3	*0.09*	*0.09000010*
0.4	*0.16*	*0.16000010*
0.5	*0.25*	*0.25000010*
0.6	*0.36*	*0.36000010*
0.7	*0.49*	*0.49000010*
0.8	*0.64*	*0.64000010*
0.9	*0.81*	*0.81000011*
1	*1*	*1.00000010*

Clearly, the approximate solution is equivalent to the exact solution.

3.2. **Application to a Delay Differential Equation.** Let $C(a,b)$ denote the space of all continuous real valued functions of a closed interval $[a,b]$ with the norm

$$\|x-y\|_\infty = \max_{t\in[a,b]} |x(t)-y(t)|.$$

It is well-known that $(C[a,b], \|\cdot\|_\infty)$ is a Banach space. In what follows, we apply our result to the following delay differential equation

$$x'(t) = f(t, x(t), x(t-\delta)),\ t \in [t_0, t_1], \tag{3.9}$$

with initial condition

$$x(t) = \phi(t)\ t \in [t_0-\delta, t_0]. \tag{3.10}$$

We suppose that the following conditions hold:

(1) $t_0, t_1 \in \mathbb{R}$ and $\delta > 0$;
(2) $f \in C([t_0,t_1]\times\mathbb{R}^2, \mathbb{R})$ such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \gamma(|x_1-y_1| + |x_2-y_2|), \tag{3.11}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \in [t_0, t_1]$ and $2\gamma(t_1-t_0) < 1$;
(3) $\phi \in C([t_0-\delta, t_1], \mathbb{R})$.

It is well-known that problems (3.9) and (3.10) can be formulated as follows;

$$x(t) = \begin{cases} \phi(t), t \in [t_0-\delta, t_0] \\ \phi(t_0) + \int_{t_0}^t f(s, x(s), x(s-\delta))ds, t \in [t_0, t_1]. \end{cases} \tag{3.12}$$

Theorem 3.2. *Suppose that the assumptions* $(1)-(3)$ *holds. Then the iterative process* (2.1) *converges strongly to the solution of problem* (3.9)-(3.10) *if* $\sum_{n=1}^{\infty}(\alpha_n+\beta_n) = \infty$.

Proof. Let $\{x_n\}$ be an iterative sequence (2.1) for an operator T defined by

$$Tx(t) = \begin{cases} \phi(t), t \in [t_0-\delta, t_0] \\ \phi(t_0) + \int_{t_0}^t f(s, x(s), x(s-\delta))ds, t \in [t_0, t_1]. \end{cases} \tag{3.13}$$

Let x^* be the fixed point of T. We need to show that $x_n \to x^*$ as $n \to \infty$. To see this, observe that

$$
\begin{aligned}
&\|z_n - x^*\|_\infty \\
&= \|(1-\alpha_n)x_n + \alpha_n T x_n - x^*\|_\infty \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \|Tx_n - Tx^*\|_\infty \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\delta,t_1]} |Tx_n(t) - Tx^*(t)| \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty \\
&+ \alpha_n \max_{t\in[t_0-\delta,t_1]} \left|\phi(t_0) + \int_{t_0}^t f(s, x_n(s), x_n(s-\delta))ds - (\phi(t_0) + \int_{t_0}^t f(s, x^*(s), x^*(s-\delta))ds)\right| \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\delta,t_1]} \int_{t_0}^t |f(s, x_n(s), x_n(s-\delta))ds - f(s, x^*(s), x^*(s-\delta))|ds \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty \\
&+ \alpha_n \gamma \int_{t_0}^t (\max_{t\in[t_0-\delta,t_1]} |x_n(s) - x^*(s)| + \max_{t\in[t_0-\delta,t_1]} |x_n(s-\delta) - x^*(s-\delta))|)ds \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \gamma \int_{t_0}^t (\|x_n - x^*\|_\infty + \|x_n - x^*\|_\infty)ds \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + 2\alpha_n \gamma \|x_n - x^*\|_\infty \int_{t_0}^t ds \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + 2\alpha_n \gamma \|x_n - x^*\|_\infty (t - t_0)ds \\
&\leq (1 - \alpha_n(1 - 2\gamma(t - t_0)))\|x_n - x^*\|_\infty.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\|y_n - x^*\|_\infty = \|Tz_n - Tx^*\|_\infty \\
&= \max_{t\in[t_0-\delta,t_1]} \left|\phi(t_0) + \int_{t_0}^t f(s, z_n(s), z_n(s-\delta))ds - (\phi(t_0) + \int_{t_0}^t f(s, x^*(s), x^*(s-\delta))ds)\right| \\
&\leq \max_{t\in[t_0-\delta,t_1]} \int_{t_0}^t |f(s, z_n(s), z_n(s-\delta))ds - f(s, x^*(s), x^*(s-\delta))|ds \\
&\leq \gamma \int_{t_0}^t (\max_{t\in[t_0-\delta,t_1]} |z_n(s) - x^*(s)| + \max_{t\in[t_0-\delta,t_1]} |z_n(s-\delta) - x^*(s-\delta))|)ds \\
&\leq \gamma \int_{t_0}^t (\|z_n - x^*\|_\infty + \|z_n - x^*\|_\infty)ds \\
&= 2\gamma \|z_n - x^*\|_\infty \int_{t_0}^t ds \\
&= 2\gamma \|z_n - x^*\|_\infty (t - t_0)ds \\
&\leq \|z_n - x^*\|_\infty \\
&\leq (1 - \alpha_n(1 - 2\gamma(t - t_0)))\|x_n - x^*\|_\infty.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
&\|w_n - x^*\|_\infty = \|(1-\beta_n)Tx_n + \beta_n Ty_n - x^*\|_\infty \\
&\leq (1-\beta_n)\|Tx_n - Tx^*\|_\infty + \alpha_n\|Ty_n - Tx^*\|_\infty \\
&= (1-\beta_n)\max_{t\in[t_0-\delta,t_1]}\bigg|\phi(t_0) + \int_{t_0}^{t} f(s, x_n(s), x_n(s-\delta))ds - (\phi(t_0) \\
&+ \int_{t_0}^{t} f(s, x^*(s), x^*(s-\delta))ds)\bigg| \\
&+ \beta_n \max_{t\in[t_0-\delta,t_1]}\bigg|\phi(t_0) + \int_{t_0}^{t} f(s, y_n(s), y_n(s-\delta))ds - (\phi(t_0) \\
&+ \int_{t_0}^{t} f(s, x^*(s), x^*(s-\delta))ds)\bigg| \\
&\leq (1-\beta_n)\gamma\int_{t_0}^{t}(\max_{t\in[t_0-\delta,t_1]}|x_n(s) - x^*(s)| + \max_{t\in[t_0-\delta,t_1]}|x_n(s-\delta) - x^*(s-\delta))|)ds \\
&+ \beta_n\gamma\int_{t_0}^{t}(\max_{t\in[t_0-\delta,t_1]}|y_n(s) - x^*(s)| + \max_{t\in[t_0-\delta,t_1]}|y_n(s-\delta) - x^*(s-\delta))|)ds \\
&\leq (1-\beta_n)\gamma\int_{t_0}^{t}(\|x_n - x^*\|_\infty + \|x_n - x^*\|_\infty)ds + \beta_n\gamma\int_{t_0}^{t}(\|y_n - x^*\|_\infty + \|y_n - x^*\|_\infty)ds \\
&= 2(1-\beta_n)\gamma\|x_n - x^*\|_\infty\int_{t_0}^{t}ds + 2\gamma\|y_n - x^*\|_\infty\int_{t_0}^{t}ds \\
&= 2(1-\beta_n)\gamma\|x_n - x^*\|_\infty(t-t_0) + 2\beta_n\gamma\|y_n - x^*\|_\infty(t-t_0) \\
&\leq (1-\beta_n\alpha_n(1-2\gamma(t-t_0)))\|x_n - x^*\|_\infty.
\end{aligned}
$$

Furthermore, using similar approach as the above, we obtain

$$
\begin{aligned}
&\|v_n - x^*\|_\infty = \|Tw_n - Tx^*\|_\infty \\
&\leq 2\gamma(t-t_0)\|w_n - x^*\|_\infty \\
&\leq \|w_n - x^*\|_\infty \\
&\leq (1-\beta_n\alpha_n(1-2\gamma(t-t_0)))\|x_n - x^*\|_\infty.
\end{aligned}
$$

Finally, using similar approach, we obtain

$$
\begin{aligned}
&\|x_{n+1} - x^*\|_\infty = \|Tv_n - Tx^*\|_\infty \\
&\leq 2\gamma(t-t_0)\|v_n - x^*\|_\infty \\
&\leq \|v_n - x^*\|_\infty \\
&\leq (1-\beta_n\alpha_n(1-2\gamma(t-t_0)))\|x_n - x^*\|_\infty.
\end{aligned}
$$

Having

$$\|x_{n+1} - x^*\|_\infty \leq (1-\beta_n\alpha_n(1-2\gamma(t-t_0)))\|x_n - x^*\|_\infty,$$

we suppose that $\zeta_n = \beta_n\alpha_n(1-2\gamma(t-t_0)) < 1$, thus, $\zeta_n \in [0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$ and $\Gamma_n = \|x_n - x^*\|_\infty$. Hence, we have

$$\Gamma_{n+1} \leq (1-\zeta_n)\Gamma_n.$$

It is easy to see that the conditions in Lemma 1.3 are satisfied. Hence, applying Lemma 1.3, we have that $\lim_{n\to\infty}\|x_n - x^*\|_\infty = 0$. ∎

Example 3.2. *Consider the following first order delay differential equation*

$$x'(t) = \frac{1}{16}(x(t) - x(t-1))\ t \in [0, 4], \tag{3.14}$$

with initial condition

$$x(t) = \phi(t) = e^t\ t \in [-1-\delta, 0]. \tag{3.15}$$

It is easy to see that the conditions $(1)-(3)$ *above are satisfied. We have*

(1) $t_0 = 0, t_1 = 4$ *and* $\delta = 1$;
(2) $f : [0,4] \times \mathbb{R} \to \mathbb{R}$ *is continuous and*

$$f(t, x(t), x(t-\delta)) = \frac{1}{16}(x(t) - x(t-1)),\ t \in [0,4]$$

and for any $x_1, x_2, y_1, y_2 \in \mathbb{R}, t \in [0,4]$, *we have*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| = \frac{1}{16}[|x_1 - y_1| + |x_2 - y_2|].$$

It is clear that $\gamma = \frac{1}{16}$, *thus, we have* $2\gamma(t_1 - t_0) = 2 \times \frac{1}{16} \times 4 = \frac{1}{2} < 1$.

The problems (3.14) *and* (3.15) *can be reformulated as the following integral equation*

$$x(t) = \begin{cases} e^t, t \in [t_0 - \delta, t_0] \\ \phi(t_0) + \frac{1}{16}\int_{t_0}^{t}(x(s) - x(s-1))ds, t \in [0,4]. \end{cases} \tag{3.16}$$

Thus, the exact solution of the problems (3.14) *and* (3.15) *is*

$$x(t) = \begin{cases} e^t, t \in [t_0 - \delta, t_0] \\ 1 + \frac{1}{16}[e^t - 1 - e^{t-1} + e^{-1}], t \in [0,4]. \end{cases} \tag{3.17}$$

4. CONCLUSION

In this article, using the D-iterative method we re-establish the convergence, stability and data dependence results obtained by the authors in [2, 3]. We use this approach to solve a two-point second-order boundary value problem and for the solution of a delay differential equations which was presented in our numerical examples. Our result shows our approximate solution is equivalent to the exact solution with a fewer time step which proves that D-iterative process has a better approximation rate than existing iteration processes.

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