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INTRODUCING THE PICARD-S3 ITERATION FOR FIXED POINTS

PRAVIN SINGH, VIRATH SINGH, AND SHIVANI SINGH

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UNIVERSITY OF KWAZULU-NATAL, SCHOOL OF MATHEMATICS STATISTICS AND COMPUTER SCIENCES, PRIVATE BAG X54001, DURBAN, 4000, SOUTH AFRICA.

UNISA, DEPARTMENT OF DECISION SCIENCES, PO BOX 392, PRETORIA, 0003, SOUTH AFRICA.
singhprook@gmail.com
singhv@ukzn.ac.za
shivstarsingh@gmail.com

ABSTRACT. In this paper we introduce a three step iteration method and show that it can be used to approximate the fixed point of a weak contraction mapping. Furthermore, we prove that this scheme is equivalent to the Mann iterative scheme. A comparison is made with other three step iterative methods by examining the speed of convergence. Results are presented in tables to support our conclusion.

Key words and phrases: Mann iteration; Weak contraction; Picard method; Fixed point.

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1. INTRODUCTION

Let X be a Banach space, and C be a nonempty, closed, convex subset of X. Let T be a mapping from a set C to itself. An element $x^\star$ of C is called a fixed point of T if $Tx^\star = x^\star$. The iterative approximation of a fixed point is crucial in fixed point theory and has dominated this field to a large extent. Many iterative methods have been proposed and studied. Numerous authors have claimed/proved that their methods are faster than others and substantiated this with examples. Here we show that such claims are not always true. Also some have compared third order methods with first and second unnecessary in our setting to compare different order methods, as one is in most cases order methods, and recently one has compared a fourth order method with third order methods and claimed superiority of the latter [1]. It is stressed here that it is always almost guaranteed that a higher order method will dominate a lower order method, although this may sometimes not be the case. For example when comparing a first order method like Picard's iteration to a third order method it would only be fair to compare every third iterate of Picard's with successive iterates of the third order method. This takes into account equal numerical effort across different methods. Unfortunately this latter effort has not been taken into account before and ignores the numerical effort/flops (floating point operations). Hence we restrict ourselves to comparing the speed of only three step or third order methods, we prefer to use the word order so as to avoid confusion. All these methods exploit the convexity of the space C to ensure that the iterates get closer to the fixed point. Firstly we list some third order methods and their proposers. In what follows $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ subject to some restrictions.

2. THEOREMS

$$\begin{cases} x_1 \in C \\ z_n = Ty_n \\ y_n = Tz_n \\ x_{n+1} = Ty_n \end{cases} \tag{2.1}$$

or

$$x_{n+1} = T^3 x_n \tag{2.2}$$

We shall call this the Picard-T3 method and denote it by PT3.

$$\begin{cases} x_1 \in C \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ y_n = (1-\beta_n)x_n + \beta_n T z_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases} \tag{2.3}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T((1-\beta_n)x_n + \beta_n T((1-\gamma_n)x_n + \gamma_n T x_n)), \tag{2.4}$$

proposed in 2000 by Noor [8], called the Noor scheme and denoted by NOO here.

$$\begin{cases} x_1 \in C \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ y_n = (1-\beta_n)z_n + \beta_n T z_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases} \tag{2.5}$$

$$x_{n+1} = (1-\alpha_n)((1-\beta_n)((1-\gamma_n)x_n + \gamma_n Tx_n) + \beta_n T((1-\gamma_n)x_n + \gamma_n Tx_n)) + \alpha_n T((1-\beta_n)((1-\gamma_n)x_n + \gamma_n Tx_n) + \beta_n T((1-\gamma_n)x_n + \gamma_n Tx_n)), \tag{2.6}$$

proposed by Phuengrattana and Suanti [10] in 2011 called the SP iteration.

$$\begin{cases} x_1 \in C \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n \end{cases} \tag{2.7}$$

$$x_{n+1} = (1-\alpha_n)((1-\beta_n)Tx_n + \beta_n T((1-\gamma_n)x_n + \gamma_n Tx_n)) + \alpha_n T((1-\beta_n)Tx_n + \beta_n T((1-\gamma_n)x_n + \gamma_n Tx_n)) \tag{2.8}$$

proposed by Chugh etal [5] in 2012 called the CR iteration.

$$\begin{cases} x_1 \in C \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n \\ y_n = (1-\alpha_n)Tx_n + \alpha_n Tz_n \\ x_{n+1} = Ty_n \end{cases} \tag{2.9}$$

equivalently

$$x_{n+1} = T((1-\alpha_n)Tx_n + \alpha_n T((1-\beta_n)x_n + \beta_n Tx_n)), \tag{2.10}$$

proposed by Gursoy and Karakaya [6] in 2014 called the Picard-S iterative process denoted by PS.

$$\begin{cases} x_1 \in C \\ z_n = Tx_n \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n \\ x_{n+1} = Ty_n \end{cases} \tag{2.11}$$

$$x_{n+1} = T((1-\alpha_n)Tx_n + \alpha_n T^2 x_n)), \tag{2.12}$$

proposed by Karakaya etal [7] in 2017 called the Karakaya scheme denoted by KA.

$$\begin{cases} x_1 \in C \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n \\ y_n = (1-\alpha_n)x_n + \alpha_n Tz_n \\ x_{n+1} = Ty_n \end{cases} \tag{2.13}$$

$$x_{n+1} = T((1-\alpha_n)x_n + \alpha_n T((1-\beta_n)x_n + \beta_n Tx_n)) \tag{2.14}$$

proposed by Okeke [9] in 2019 called the Picard-Ishikawa iteration denoted by PIK.

$$
\begin{cases} x_1 \in C \\ y_n = (1-\alpha_n)x_n + \alpha_n T^2 x_n \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n \end{cases} \tag{2.15}
$$

or

$$
x_{n+1} = (1-\beta_n)((1-\alpha_n)x_n + \alpha_n T^2 x_n) + \beta_n T((1-\alpha_n)x_n + \alpha_n T^2 x_n), \tag{2.16}
$$

proposed by us referred to as the Picard-S3 scheme denoted by PS3.

Lemma 2.1. [11] *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative sequences satisfying the condition*

$$
a_{n+1} \le (1-\mu_n)a_n + b_n, \tag{2.17}
$$

where $\mu_n \in (0,1)$ for all $n \ge n_0$,$\sum_{n=1}^{\infty} \mu_n = \infty$ and $\frac{b_n}{\mu_n} \to \infty$ as $n \to \infty$, then $\lim_{n\to\infty} a_n = 0$

Definition 2.1. [2] The self-map $T : C \to C$ is called a weak-contraction if there exist $\delta \in (0,1)$ and $L_1 \ge 0$ such that

$$
\|Tx - Ty\| \le \delta\|x-y\| + L_1\|y - Tx\|
$$

Definition 2.2. [4] Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real convergent sequences with limits a and b respectively. Then, $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$ if

$$
\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0 \tag{2.18}
$$

Definition 2.3. [3] Let $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be two fixed point iterative processes, both converging to fixed point $x^\star$ of a given operator T. Suppose that the error estimates

$$
\begin{aligned} \|u_n - x^\star\| &\le a_n \\ \|x_n - x^\star\| &\le b_n, \end{aligned} \tag{2.19}
$$

for all $n \in N$ are available, where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive numbers converging to 0. If $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$, then $\{u_n\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$ to $x^\star$.

Remark 2.1. Let $T : x \to \frac{x}{5}$, $x \in [-2,2]$, choose $x_1 = 1$ and consider the Picard iteration $x_{n+1} = Tx_n$. It is easily verified that

$$
\begin{aligned} x_n = \left(\frac{1}{5}\right)^n &\le \left(\frac{4}{5}\right)^n \\ &= b_n. \end{aligned} \tag{2.20}
$$

Also consider the Mann iteration

$$
u_{n+1} = \alpha u_n + (1-\alpha)Tu_n, \tag{2.21}
$$

with $u_n = 1$ and $\alpha = \frac{1}{2}$. Then $u_{n+1} = \frac{3}{5}u_n$ which implies that

$$
\begin{aligned} u_n = \left(\frac{3}{5}\right)^n &\le \left(\frac{3}{5}\right)^n \\ &= a_n. \end{aligned} \tag{2.22}
$$

Now by Definition 2.3 $\{a_n\}_{n=1}^{\infty}$ converges to zero faster than $\{b_n\}_{n=1}^{\infty}$, so we should expect $\{u_n\}_{n=1}^{\infty}$ to converge to zero faster than $\{x_n\}_{n=1}^{\infty}$, but this is clearly false as per Definition 2.2. The shortcoming in Definition 2.3 is that it should refer to the least upper bounds. However for an arbitrary operator T which may be non linear, it may be very difficult or indeed impossible to found such a bound. Unfortunately Definition 2.3 has been used to claim that some methods are faster than others.

Theorem 2.2. [2] *Let X be a Banach space and $T : X \to X$ be a weak contraction, then*

$$F(T) = \{x \in X : Tx = x\} \neq \emptyset \tag{2.23}$$

Theorem 2.3. [2] *Let X be a Banach space and $T : X \to X$ be a weak contraction for which there exist $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\| \tag{2.24}$$

Then, T has a unique fixed point.

Theorem 2.4. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak-contraction map satisfying the additional condition* (2.24)*. Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by with a real sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \in (0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point $x^\star$ of T.*

Proof. The existence of a fixed point $x^\star$ is guaranteed by Theorem 2.2. The uniqueness follows from Theorem 2.3 as is shown by using (2.24). Suppose that $x^\star = Tx^\star$ and $x^{\star\star} = Tx^{\star\star}$ are two fixed points then

$$\|x^\star - x^{\star\star}\| \leq \delta\|x^\star - x^{\star\star}\| + L\|x^\star - Tx^\star\|. \tag{2.25}$$

If $x^\star \neq x^{\star\star}$, then $\delta \geq 1$ is a contradiction which ensures uniqueness.

$$\begin{aligned}
\|y_n - x^\star\| &= \|(1 - \alpha_n)x_n + \alpha_n T^2 x_n - (1 - \alpha_n)x^\star - \alpha_n T^2 x^\star\| \\
&\leq (1 - \alpha_n)\|x_n - x^\star\| + \alpha_n \delta^2 \|x_n - x^\star\| \\
&= (1 - \alpha_n + \alpha_n \delta^2)\|x_n - x^\star\| \\
&= (1 - \alpha_n(1 - \delta^2))\|x_n - x^\star\| \qquad (2.26)
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - x^\star\| &= \|(1 - \beta_n)y_n + \beta_n T y_n - (1 - \beta_n)x^\star - \beta_n T x^\star\| \\
&\leq (1 - \beta_n)\|y_n - x^\star\| + \beta_n \delta\|y_n - x^\star\| \\
&= (1 - \beta_n + \beta_n \delta)\|y_n - x^\star\| \\
&= (1 - \beta_n(1 - \delta))\|y_n - x^\star\| \qquad (2.27)
\end{aligned}$$

Substituting (2.26) into (2.27) we obtain

$$\begin{aligned}
\|x_{n+1} - x^\star\| &= (1 - \beta_n(1 - \delta))(1 - \alpha_n(1 - \delta^2))\|x_n - x^\star\| \\
&\leq (1 - \beta_n(1 - \delta))(1 - \beta_{n-1}(1 - \delta))(1 - \alpha_n(1 - \delta^2)) \\
&\quad\ (1 - \alpha_{n-1}(1 - \delta^2))\|x_{n-1} - x^\star\| \\
&\leq \prod_{i=1}^{n}(1 - \beta_i(1 - \delta))(1 - \alpha_i(1 - \delta^2)\|x_1 - x^\star\| \qquad (2.28)
\end{aligned}$$

Using $1 - x \leq e^{-x}$ for $x \in (0,1)$ in (2.28) we simplify

$$\begin{aligned}\|x_{n+1} - x^\star\| &\leq \prod_{i=1}^{n} e^{-\beta_i(1-\delta)} e^{-\alpha_i(1-\delta^2)}\\ &= \prod_{i=1}^{n} e^{-(1-\delta)\sum_{i=1}^{n}\beta_i} e^{-(1-\delta^2)\sum_{i=1}^{n}\alpha_i} \end{aligned} \tag{2.29}$$

Now since $\sum_{i=1}^{n} \beta_i \to \infty, \sum_{i=1}^{n} \alpha_i \to \infty$ as $n \to \infty$, it follows that $x_n \to x^\star$. ∎

Theorem 2.5. *Let X be a Banach space, C be a nonempty, closed, convex subset of X and $T : C \to C$ be a weak-contraction map satisfying condition* (2.24) *with a fixed point $x^\star$. Let $\{u_n\}_{n=1}^{\infty}$ be the Mann iteration process defined in* (2.21) *with $u_1 \in C$ and $\{x_n\}_{n=1}^{\infty}$ be defined by* (2.15) *with $x_1 \in C$ with real sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \in (0,1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then the following assertions are equivalent:*

(a) *Mann's iteration converges to $x^\star$.*
(b) *The new iteration method* (2.15) *converges to $x^\star$.*

Proof. We write Mann's iteration as $u_{n+1} = (1-\beta_n)u_n + \beta_n T u_n$ and first show that $(a) \implies (b)$

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &= \|(1-\beta_n)u_n + \beta_n T u_n - (1-\beta_n)y_n - \beta_n T y_n\|\\ &\leq (1-\beta_n)\|u_n - y_n\| + \beta_n\|Tu_n - Ty_n\|\\ &\leq (1-\beta_n)\|u_n - y_n\| + \beta_n(\delta\|u_n - y_n\| + L\|u_n - Tu_n\|)\\ &= (1-\beta(1-\delta))\|u_n - y_n\| + \beta_n L\|u_n - Tu_n\| \end{aligned} \tag{2.30}$$

Now

$$\begin{aligned}\|u_n - y_n\| &= \|(1-\alpha_n)u_n + \alpha_n u_n - (1-\alpha_n)x_n - \alpha_n T^2 x_n\|\\ &\leq (1-\alpha_n)\|u_n - x_n\| + \alpha_n\|u_n - T^2 x_n\| \end{aligned} \tag{2.31}$$

Also

$$\begin{aligned}\|u_n - T^2 x_n\| &\leq \|u_n - Tu_n\| + \|Tu_n - T^2 x_n\|\\ &\leq \|u_n - Tu_n\| + \delta\|u_n - Tx_n\| + L\|u_n - Tu_n\|\\ &\leq (1+L)\|u_n - Tu_n\| + \delta\|u_n - Tu_n\| + \delta\|Tu_n - Tx_n\|\\ &\leq (1+L+\delta)\|u_n - Tu_n\| + \delta(\delta\|u_n - x_n\| + L\|u_n - Tu_n\|)\\ &\leq (2L+2)\|u_n - Tu_n\| + \delta^2\|u_n - x_n\|. \end{aligned} \tag{2.32}$$

Substituting (2.32) into (2.31) and simplifying we obtain

$$\|u_n - y_n\| \leq (1 - \alpha_n + \alpha_n\delta^2)\|u_n - x_n\| + (2L+2)\|u_n - Tu_n\| \tag{2.33}$$

∎

Further substituting (2.33) into (2.30) yields

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &\leq (1 - \beta_n(1-\delta)(1-\alpha_n(1-\delta^2))\|u_n - x_n\|\\ &\quad + (2L + 2 + \beta_n L)\|u_n - Tu_n\|\\ &\leq (1-\beta_n(1-\delta))\|u_n - x_n\| + (3L+2)\|u_n - Tu_n\| \end{aligned} \tag{2.34}$$

Since $u_n \to x^\star$ it follows that

$$\begin{aligned} \|u_n - Tu_n\| &\leq \|u_n - x^\star\| + \|Tx^\star - Tu_n\| \\ &\leq \|u_n - x^\star\| + \delta\|x^\star - u_n\| \\ &= (1+\delta)\|u_n - x^\star\| \\ &\leq 2\|u_n - x^\star\| \end{aligned} \tag{2.35}$$

Finally (2.34) is simplified by (2.35) yielding

$$\|u_{n+1} - x_{n+1}\| \leq (1 - \beta_n(1-\delta))\|u_n - x_n\| + 2(3L+2)\|u_n - x^\star\| \tag{2.36}$$

Let $a_n = \|u_n - x_n\|$,$b_n = 2(3L+2)\|u_n - x^\star\|$ and $\mu_n = \beta_n(1-\delta)$ and apply Lemma 2.1 to obtain $\|u_n - x_n\| \to 0$. Hence

$$\|x_n - x^\star\| \leq \|x_n - u_n\| + \|u_n - x^\star\| \tag{2.37}$$

proving that $x_n \to x^\star$ since $u_n \to x^\star$.

We now show that $(b) \implies (a)$

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|(1-\beta_n)y_n + \beta_n Ty_n - (1-\beta_n)u_n - \beta_n Tu_n\| \\ &\leq (1-\beta_n)\|y_n - u_n\| + \beta_n\|Ty_n - Tu_n\| \\ &\leq (1-\beta_n)\|y_n - u_n\| + \beta_n(\delta\|y_n - u_n\| + L\|y_n - Ty_n\|) \\ &= (1-\beta_n(1-\delta))\|y_n - u_n\| + \beta_n L\|y_n - Ty_n\| \\ &\leq (1-\beta_n(1-\delta))(\|y_n - x_n\| + \|x_n - u_n\|) + \beta_n L\|y_n - Ty_n\| \\ &\leq (1-\beta_n(1-\delta))\|x_n - u_n\| + \|y_n - x_n\| + L\|y_n - Ty_n\| \end{aligned} \tag{2.38}$$

Now

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\alpha_n)x_n + \alpha_n T^2x_n - x_n\| \\ &= \alpha_n\|T^2x_n - x_n\| \\ &\leq \alpha_n(\|T^2x_n - Tx_n\| + \|Tx_n - x_n\|) \\ &\leq \alpha_n(\delta\|Tx_n - x_n\| + L\|Tx_n - x_n\| + \|Tx_n - x_n\|) \\ &\leq \alpha_n(\delta + L + 1)\|Tx_n - x_n\| \\ &\leq (L+2)\|Tx_n - x_n\| \end{aligned} \tag{2.39}$$

Also

$$\begin{aligned} \|Ty_n - y_n\| &\leq \|(1-\alpha_n)Ty_n + \alpha_n Ty_n - (1-\alpha_n)x_n - \alpha_n T^2x_n\| \\ &\leq (1-\alpha_n)\|Ty_n - x_n\| + \alpha_n\|Ty_n - T^2x_n\| \\ &\leq \|Ty_n - x_n\| + \|Ty_n - T^2x_n\| \end{aligned} \tag{2.40}$$

But

$$\begin{aligned} \|Ty_n - x_n\| &\leq \|Ty_n - Tx_n\| + \|Tx_n - x_n\| \\ &\leq \delta\|y_n - x_n\| + L\|x_n - Tx_n\| + \|Tx_n - x_n\| \\ &\leq \|y_n - x_n\| + (L+1)\|x_n - Tx_n\| \end{aligned} \tag{2.41}$$

Substituting (2.39) into (2.41) yields

$$\|Ty_n - x_n\| \leq (2L+3)\|x_n - Tx_n\| \tag{2.42}$$

Also

$$\begin{aligned}\|Ty_n - T^2x_n\| &\le \delta\|y_n - Tx_n\| + L\|Tx_n - T^2x_n\| \\ &\le \|y_n - Tx_n\| + L\|Tx_n - T^2x_n\| \\ &\le \|y_n - Tx_n\| + L(\delta\|x_n - Tx_n\| + L\|x_n - Tx_n\|) \\ &\le \|y_n - Tx_n\| + L(L+1)\|x_n - Tx_n\| \end{aligned} \tag{2.43}$$

A further simplification shows that

$$\begin{aligned}\|y_n - Tx_n\| &= \|(1-\alpha_n)x_n + \alpha_n T^2x_n - (1-\alpha_n)Tx_n - \alpha_n Tx_n\| \\ &\le (1-\alpha_n)\|x_n - Tx_n\| + \alpha_n\|T^2x_n - Tx_n\| \\ &\le (1-\alpha_n)\|x_n - Tx_n\| + \alpha_n\delta\|x_n - Tx_n\| + \alpha_n L\|x_n - Tx_n\| \\ &\le (L+2)\|x_n - Tx_n\| \end{aligned} \tag{2.44}$$

Substituting (2.44) into (2.43) yields

$$\|Ty_n - T^2x_n\| \le (L^2 + 2L + 2)\|x_n - Tx_n\| \tag{2.45}$$

Eventually substituting (2.42) and (2.45) into (2.40) we arrive at

$$\|Ty_n - y_n\| \le (L^2 + 4L + 5)\|x_n - Tx_n\| \tag{2.46}$$

Since $x_n \to x^\star$ it follows that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x^\star\| + \|Tx^\star - Tx_n\| \\ &\le \|x_n - x^\star\| + \delta\|x^\star - u_n\| \\ &= (1+\delta)\|x_n - x^\star\| \\ &\le 2\|x_n - x^\star\| \end{aligned} \tag{2.47}$$

With (2.47), (2.46) becomes

$$\|Ty_n - y_n\| \le 2(L^2 + 4L + 5)\|x_n - x^\star\| \tag{2.48}$$

Finally substitute (2.39) and (2.48) into (2.38) and simplify to obtain

$$\|x_{n+1} - u_{n+1}\| \le (1 - \beta_n(1-\delta))\|x_n - u_n\| + 2(L^3 + 4L^2 + 6L + 2)\|x_n - x^\star\| \tag{2.49}$$

Let $a_n = \|x_n - u_n\|$, $b_n = 2(L^3 + 4L^2 + 6L + 2)\|x_n - x^\star\|$ and $\mu_n = \beta_n(1-\delta)$ and apply Lemma 1 to obtain $\|x_n - u_n\| \to 0$. Hence

$$\|u_n - x^\star\| \le \|u_n - x_n\| + \|x_n - x^\star\| \tag{2.50}$$

proving that $u_n \to x^\star$ since $x_n \to x^\star$.

3. EXAMPLES

Example 3.1. *$T : [0,1] \to [0,1]$ be defined by $Tx = \sqrt{1-x^3}$ with $x_0 = 0.8$. The exact solution is given by*

$$x^\star = \frac{\sqrt[3]{25+\sqrt{621}} + \sqrt[3]{25-\sqrt{621}}}{3\sqrt[3]{2}} - \frac{1}{3}$$

Example 3.2. *$T : [0,6] \to [0,6]$ be defined by $Tx = \sqrt[3]{2x+4}$ with $x_0 = 5.0$. The exact solution is given by $x^\star = 2$*

Example 3.3. *$T : [1,2] \to [1,2]$ be defined by $Tx = \frac{3}{4}(1+\frac{1}{x})$ with $x_0 = 1.0$. The exact solution is given by $x^\star = \sqrt{3}$*

Example 3.4. $T : [0, 2] \to [0, 2]$ *be defined by* $Tx = \frac{1}{1+x^2}$ *with* $x_0 = 0.5$. *The exact solution is given by*

$$x^\star = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{31}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{31}{108}}}$$

Example 3.5. $T : [0, 2] \to [0, 2]$ *be defined by* $Tx = \frac{x^2+9}{10}$ *with* $x_0 = 2.0$. *The exact solution is given by* $x^\star = 1$

Example 3.6. $T : [0, 2] \to [0, 2]$ *be defined by* $Tx = \frac{-x^2+10}{9}$ *with* $x_0 = 2.0$. *The exact solution is given by* $x^\star = 1$

Example 3.7. $T : [1.5, 2] \to [1.5, 2]$ *be defined by* $Tx = 2 \sin x$ *with* $x_0 = 2.0$. *The exact solution is given by* $x^\star = 1.895494267033$ *to twelve decimal digits.*

Example 3.8. $T : [0, 0.5] \to [0, 0.5]$ *be defined by* $Tx = \frac{(1-x)^7}{10}$ *with* $x_0 = 0.5$. *The exact solution is given by* $x^\star = 0.063280205813$ *to twelve decimal digits.*

The number of iterations to converge to within 10^{-12} of $x^\star$ is summarized in the tables 4.1-4.11, here X denotes oscillation between two fixed iterates and hence non convergence. We have chosen constant sequences $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ as parameters. For example 3.1 as T is not a contraction in any neighbourhood of the fixed point, it follows that PT3 cannot converge. For certain choices of the parameters other methods, like PS and PIK mimic PT3 and do not converge. For examples 3.2,3.3,3.5 and 3.6 PT3 does very well (recall it is independent of parameters). Indeed it is quite obvious that where T is a contraction then PT3 will outperform all third order methods for increasing functions $f(x) = Tx$, from elementary fixed point theory (monotonic convergence). Hence it is possible for SP to beat PT3 for example 3.6 in table 4.3 as in example 3.6, $f(x)$ is decreasing. Overall the SP iteration is quite attractive as it performs exceptionally well with no optimization of parameters. The CR and PS3 iterations perform reasonabily well. Also in table 4.11 it is illustrated that there are parameters for which PS3 performs the best for example 3.1.

4. TABLES

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	40	**7**	27	X	50	70	37
2	**7**	118	30	15	9	9	16	58
3	**14**	175	50	32	20	19	33	81
4	21	59	**11**	23	27	20	31	39
5	**7**	119	31	16	10	10	17	57
6	**8**	82	18	15	11	10	17	45
7	20	57	**11**	23	27	19	31	38
8	**13**	68	14	20	18	15	24	41

Table 4.1: $\alpha = 0.25,\ \beta = 0.25,\ \gamma = 0.5$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	61	**8**	12	96	50	X	57
2	**7**	108	18	13	9	9	15	24
3	**14**	136	32	27	20	19	32	41
4	21	82	**8**	15	22	20	39	17
5	**7**	106	18	14	10	10	17	25
6	**8**	93	9	13	10	10	17	11
7	20	80	**8**	15	22	19	38	17
8	13	87	**8**	15	16	15	27	12

Table 4.2: $\alpha = 0.25,\ \beta = 0.75,\ \gamma = 0.5$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	41	15	**12**	96	50	X	57
2	**7**	107	15	12	9	9	15	24
3	**14**	131	27	25	20	19	32	41
4	21	72	**9**	**9**	22	20	39	17
5	**7**	105	15	13	10	10	17	25
6	8	92	**2**	11	10	10	17	11
7	20	70	**9**	**9**	22	19	38	17
8	13	82	**7**	10	16	15	27	12

Table 4.3: $\alpha = 0.25,\ \beta = 0.75,\ \gamma = 0.75$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	26	**5**	10	49	11	51	11
2	**7**	49	23	13	9	8	13	25
3	**14**	72	40	27	19	17	27	37
4	21	30	**8**	12	20	11	22	14
5	**7**	49	24	14	10	9	14	25
6	**8**	36	14	12	10	9	14	19
7	20	29	**8**	12	19	11	22	14
8	13	32	**10**	13	15	10	18	16

Table 4.4: $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.25$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	47	**8**	9	20	11	483	66
2	**7**	47	18	13	9	8	13	19
3	**14**	63	32	26	18	17	26	30
4	21	40	**8**	12	15	11	27	16
5	**7**	46	18	14	9	9	14	19
6	**8**	40	**8**	12	9	9	14	10
7	20	39	**8**	12	15	11	27	15
8	13	39	**8**	12	13	10	20	11

Table 4.5: $\alpha = 0.5,\ \beta = 0.75,\ \gamma = 0.25$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	23	**6**	7	20	11	483	66
2	**7**	46	15	12	9	8	13	19
3	**14**	61	27	24	18	17	26	30
4	21	34	**7**	10	15	11	27	16
5	**7**	46	16	13	9	9	14	19
6	**8**	39	**8**	11	9	9	14	10
7	20	33	**7**	10	15	11	27	15
8	13	36	**7**	11	13	10	20	11

Table 4.6: $\alpha = 0.5,\ \beta = 0.75,\ \gamma = 0.5$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	10	8	**7**	20	11	483	66
2	**7**	46	13	11	9	8	13	19
3	**14**	59	23	22	18	17	26	30
4	21	29	**7**	**7**	15	11	27	16
5	**7**	45	13	12	9	9	14	19
6	8	38	**2**	9	9	9	14	10
7	20	28	**7**	**7**	15	11	27	15
8	13	34	**6**	8	13	10	20	11

Table 4.7: $\alpha = 0.5,\ \beta = 0.75,\ \gamma = 0.75$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	10	**8**	37	19	94	25	11
2	**7**	27	18	11	9	8	11	17
3	**14**	43	32	23	18	16	22	26
4	21	13	**8**	13	15	12	13	13
5	**7**	27	18	12	9	8	12	17
6	8	18	9	8	9	**7**	10	14
7	20	13	**8**	13	15	12	13	12
8	13	15	**8**	10	13	**8**	12	14

Table 4.8: $\alpha = 0.75,\ \beta = 0.5,\ \gamma = 0.25$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	26	**15**	29	20	94	164	78
2	**7**	25	15	11	8	8	11	14
3	**14**	37	27	23	17	16	21	22
4	21	21	**9**	13	**9**	12	19	14
5	**7**	25	15	12	9	8	12	14
6	8	20	**6**	8	8	7	11	9
7	20	21	**9**	13	**9**	12	19	14
8	13	20	**7**	9	10	8	15	9

Table 4.9: $\alpha = 0.75,\ \beta = 0.75,\ \gamma = 0.25$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	**7**	8	14	20	94	164	78
2	**7**	25	13	11	8	8	11	14
3	**14**	35	23	21	17	16	21	22
4	21	16	**7**	10	9	12	19	14
5	**7**	25	13	11	9	8	12	14
6	8	20	**6**	8	8	7	11	9
7	20	16	**7**	10	9	12	19	14
8	13	18	**6**	9	10	8	15	9

Table 4.10: $\alpha = 0.75,\ \beta = 0.75,\ \gamma = 0.5$

Ex	PT3	NOO	SP	CR	PS	KK	PIK	PS3
1	X	12	6	8	86	11	50	**2**
2	**7**	58	16	12	9	9	14	30
3	**14**	81	28	25	19	18	28	43
4	21	28	**8**	11	22	13	23	18
5	**7**	58	16	13	10	9	15	30
6	**8**	42	7	11	10	9	14	22
7	20	27	**8**	11	21	13	23	17
8	13	34	**8**	12	16	11	19	20

Table 4.11: $\alpha = 0.441946314,\ \beta = 0.441946314,\ \gamma = 0.8$

5. CONCLUSION

We have succeeded in showing that the speed of convergence for third order methods are problem and parameter dependent. Also it is alarming that PT3 has been ignored in comparison to other third order methods, when T is a contraction. Also we have constructed a method PS3 which we have shown can be superior in some cases. This however holds true for judicious choices of parameters for other third order methods as well.

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