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FEKETE-SZEGÖ INEQUALITY FOR SAKAGUCHI TYPE OF FUNCTIONS IN PETAL SHAPED DOMAIN

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ABSTRACT. In this paper, we estimate coefficient bounds,$|a_2|$, $|a_3|$ and $|a_4|$, Fekete-Szegö inequality $|a_3 - \gamma a_2^2|$ and Toeplitz determinant $\mathcal{T}_2(2)$ and $\mathcal{T}_3(1)$ for functions belonging to the following class

$$\frac{(1-t)[\rho z^2 f''(z) + zf'(z)]}{\rho z[f'(z) - tf'(tz)] + (1-\rho)[f(z) - f(tz)]} \prec \tilde{\Lambda}(z)$$

the function being holomorphic, we expand using Taylor series and obtain several corollaries and consequences for the main result.

Key words and phrases: Analytic function; subordination; Petal shaped domain; Fekete-Szegö inequality.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of all functions f which are holomorphic in the region $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. Therefore, for $f \in \mathcal{A}$, one has

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{D}). \tag{1.1}$$

We write $g_1 \prec g_2$, if there is an analytic function ν in $\mathbb{D}$, with limitations $\nu(0) = 0$ and $|\nu(z)| < 1$, such that $g_1(z) = g_2(\nu(z)),\ (z \in \mathbb{D})$. In case of univalency of g_2 in $\mathbb{D}$, the following relation holds:

$$g_1(z) \prec g_2(z),\ (z \in \mathbb{D}) \iff g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{D}) \subset g_2(\mathbb{D}).$$

In geometric function theory, the most basic and important subfamilies of the set $\mathcal{S}$ are the family $\mathcal{S}^*$ and $\mathcal{C}$ which are starlike and convex functions respectively.
By varying the subordination condition, we arrive at various geometrical sense. For example,

$$q(z) = 1 + sinh^{-1} z \tag{1.2}$$

then the class $\mathcal{S}_q^* := \mathcal{S}^*(1 + sinh^{-1} z)$ was provided by Kumar and Arora [8]. Clearly, the function $q(z)$ is a multivalued function and has the branch cuts about the line segments $(-i\infty, -i) \cup (i, i\infty)$, on imaginary axis and hence, it is holomorphic in $\mathbb{D}$. In a geometric point of view, the function $q(z)$ maps the unit disc $\mathbb{D}$ onto a petal-shaped region Ω_p,

$$\Omega_p = \{\omega \in \mathcal{C} : |sinh(\omega - 1)| < 1\} \tag{1.3}$$

Using this idea, we now consider a subfamily $\mathcal{SC}^{t,\rho}$ of analytic functions as

Definition 1.1. The function $f \in \mathcal{A}$ is in the class $\mathcal{SC}^{t,\rho}$ if

$$\frac{(1-t)[\rho z^2 f''(z) + z f'(z)]}{\rho z[f'(z) - t f'(tz)] + (1-\rho)[f(z) - f(tz)]} \prec \tilde{\Lambda}(z), \tag{1.4}$$

where $\tilde{\Lambda}(z)$ is given by (1.3), with $|t| \le 1$, $t \ne 1$, and $0 \le \rho \le 1$.

Remark 1.1. (i) For $\rho = 0$, we get the following new class $\mathcal{SC}^t$,

$$\mathcal{SC}^t = \left\{ f \in \mathcal{A} : \frac{(1-t) z f'(z)}{f(z) - f(tz)} \prec \tilde{\Lambda}(z),\ z \in \mathbb{D} \right\}.$$

(ii) For $\rho = 1$, we get the following new class $\mathcal{SC}^{t,1}$,

$$\mathcal{SC}^{t,1} = \left\{ f \in \mathcal{A} : \frac{(1-t) z f''(z) + f'(z)}{f'(z) - t f'(tz)} \prec \tilde{\Lambda}(z),\ z \in \mathbb{D} \right\}.$$

2. A SET OF LEMMAS

Let $\mathcal{P}$ be the family of functions p that are holomorphic in $\mathbb{D}$ with $\Re\,(p(z)) > 0$ and the power series form as follows:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in \mathbb{D}). \tag{2.1}$$

Lemma 2.1. [6, 9] *If* $p \in \mathcal{P}$ *be expressed in series expansion* (2.1)*, then*

$$|c_k| \leq 2 \qquad \textit{for} \quad k \geq 1, \tag{2.2}$$

and for complex number γ*, we have*

$$\left|c_2 - \gamma c_1^2\right| \leq 2\ \max\{1, |2\gamma - 1|\}, \tag{2.3}$$

Lemma 2.2. [1] *Let* $p \in \mathcal{P}$ *has power series expansion* (2.1)*, then*

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \leq 2\,|J| + 2\,|K - 2J| + 2\,|J - K + L|\,.$$

3. COEFFICIENTS ESTIMATES FOR THE CLASS $\mathcal{SC}^{t,\rho}$

Some recent work on coefficient problems includes Barukab et al.[3]. In this section, we obtain the initial coefficient bounds a_2, a_3 and a_4 for the function defined in the class $\mathcal{SC}^{t,\rho}$

Theorem 3.1. *If the function* f *of the form* (1.1) *belongs to* $\mathcal{SC}^{t,\rho}$*, then*

$$|a_2| \leq \frac{H_1U_2}{u_2},$$

$$|a_3| \leq \frac{H_2U_3}{H_1U_2u_3}\ max(1, U_2),$$

$$|a_4| \leq \frac{H_3U_4}{4H_2U_3u_4}\left\{\left|\frac{5}{6} - U_2 - \frac{U_3}{U_2} + U_3\right| + \left|\frac{7}{3} - 2U_3\right| + \left|\frac{5}{6} + U_2 + \frac{U_3}{U_2} + U_3\right|\right\}$$

where

$$u_n = \frac{1 - t^n}{1 - t},\ U_n = \prod \frac{u_n}{n - u_n}\ , n = 2, 3, \cdots \textit{ and } H_n = \prod \frac{1}{1 + n\rho}\ , n = 1, 2, \cdots$$

Proof. Let $f \in \mathcal{SC}^{t,\rho}$. Then, (1.4) can be written in the form of the Schwarz function as

$$\frac{(1 - t)[\rho z^2 f''(z) + zf'(z)]}{\rho z[f'(z) - tf'(tz)] + (1 - \rho)[f(z) - f(tz)]} = 1 + sinh^{-1}(w(z))\ (z \in \mathbb{D}) \tag{3.1}$$

Now, if $p \in \mathcal{P}$, then it may be written in terms of the Schwarz function w by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots\ ,$$

equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1z + c_2z^2 + c_3z^3 + c_4z^4 \ldots}{2 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 \cdots} \tag{3.2}$$

Using (1.1) in (1.4), we get

$$\frac{(1 - t)[\rho z^2 f''(z) + zf'(z)]}{\rho z[f'(z) - tf'(tz)] + (1 - \rho)[f(z) - f(tz)]}$$

$$= \frac{1 + 2a_2(1 + \rho)z + 3a_3(1 + 2\rho)z^2 + 4a_4(1 + 3\rho)z^3 + \cdots}{1 + a_2u_2(1 + \rho)z + a_3u_3(1 + 2\rho)z^2 + a_4u_4(1 + 3\rho)z^3 + \cdots} \tag{3.3}$$

By simplification and using the series expansion (3.2), we obtain

$$\begin{aligned} 1 + sinh^{-1}(w(z)) \quad &= \quad 1 + \frac{1}{2}c_1z + \left(-\frac{c_1^2}{4} + \frac{c_2}{2}\right)z^2 + \left(\frac{5c_1^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right)z^3 \\ &\quad + \left(-\frac{1}{32}c_1^4 + \frac{5}{16}c_1^2c_2 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2 + \frac{1}{2}c_4\right)z^4 + \cdots\,. \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we get

$$a_2 = \frac{H_1U_2}{2u_2}c_1 \tag{3.5}$$

$$a_3 = \frac{H_2U_3}{H_1U_2u_3}\left(\frac{1}{2}c_2 - \left(\frac{1}{4} - \frac{U_2}{4}\right)c_1^2\right) \tag{3.6}$$

$$a_4 = \frac{H_3U_4}{H_2U_3u_4}\left\{\frac{1}{2}c_3 + \left(\frac{5}{48} - \frac{U_2}{8} - \frac{U_3}{8U_2} + \frac{U_3}{8}\right)c_1^3 - \left(\frac{1}{2} - \frac{U_2}{4} - \frac{U_3}{4U_2}\right)c_1c_2\right\} \tag{3.7}$$

Now implementing (2.2) in (3.5), we obtain

$$|a_2| \leq \frac{H_1U_2}{u_2}. \tag{3.8}$$

Now implementing (2.3) in (3.5), we obtain

$$|a_3| \leq \frac{H_2U_3}{H_1U_2u_3}\, max(1, U_2). \tag{3.9}$$

Implementation of triangle inequality and Lemma 2.2 in (3.7), leads us to

$$|a_4| \leq \frac{H_3U_4}{4H_2U_3u_4}\left\{\left|\frac{5}{6} - U_2 - \frac{U_3}{U_2} + U_3\right| + \left|\frac{7}{3} - 2U_3\right| + \left|\frac{5}{6} + U_2 + \frac{U_3}{U_2} + U_3\right|\right\}. \tag{3.10}$$

This completes the proof ∎

Letting $\rho = 0$ in Theorem 3.1, we get the following result.

Corollary 3.2. *If the function f of the form (1.1) belongs to $\mathcal{SC}^t$, then*

$$|a_2| \leq \frac{U_2}{u_2},$$

$$|a_3| \leq \frac{U_3}{U_2u_3}\, max(1, U_2),$$

$$|a_4| \leq \frac{U_4}{4U_3u_4}\left\{\left|\frac{5}{6} - U_2 - \frac{U_3}{U_2} + U_3\right| + \left|\frac{7}{3} - 2U_3\right| + \left|\frac{5}{6} + U_2 + \frac{U_3}{U_2} + U_3\right|\right\}.$$

Putting $\rho = 1$ in Theorem 3.1, we get the following result.

Corollary 3.3. *If the function f of the form (1.1) belongs to $\mathcal{SC}^{t,1}$, then*

$$|a_2| \leq \frac{U_2}{2u_2},$$

$$|a_3| \leq \frac{U_3}{3U_2u_3}\, max(1, U_2),$$

$$|a_4| \leq \frac{U_4}{16U_3u_4}\left\{\left|\frac{5}{6} - U_2 - \frac{U_3}{U_2} + U_3\right| + \left|\frac{7}{3} - 2U_3\right| + \left|\frac{5}{6} + U_2 + \frac{U_3}{U_2} + U_3\right|\right\}.$$

4. FEKETE-SZEGÖ INEQUALITY FOR THE CLASS $\mathcal{SC}^{t,\rho}$

In this section, we determine the bound for the Fekete-Szegö inequality [4] *for the function defined in the class* $\mathcal{SC}^{t,\rho}$

Theorem 4.1. *If the function f of the form (1.1) belonging to the class $\mathcal{SC}^{t,\rho}$, then for any complex number γ*

$$|a_3 - \gamma a_2^2| \leq \frac{H_2 U_3}{H_1 U_2 u_3}\, max\left\{1, \left|\frac{\gamma H_1^3 U_2^3 u_3}{H_2 U_3 u_2^2} - U_2\right|\right\}$$

Proof. Employing (3.5) and (3.6), we may write

$$|a_3 - \gamma a_2^2| = \left|\frac{H_2 U_3}{2H_1 U_2 u_3} c_2 + \frac{H_2 U_3}{4H_1 u_3} c_1^2 - \frac{H_2 U_3}{4H_1 U_2 u_3} c_1^2 - \gamma \frac{H_1^2 U_2^2}{4u_2^2}\right|$$

By rearranging, it yields

$$|a_3 - \gamma a_2^2| = \frac{H_2 U_3}{2H_1 U_2 u_3}\left|c_2 - \left(\frac{1}{2} - \frac{U_2}{2} + \frac{\gamma H_1^3 U_2^3 u_3}{H_2 U_3 u_2^2}\right) c_1^2\right|$$

Application of (2.3) leads us to

$$|a_3 - \gamma a_2^2| \leq \frac{H_2 U_3}{2H_1 U_2 u_3}\, 2\, max\left(1, \left|2\left(\frac{1}{2} - \frac{U_2}{2} + \frac{\gamma H_1^3 U_2^3 u_3}{H_2 U_3 u_2^2}\right) - 1\right|\right)$$

After the simplification, we obtain

$$|a_3 - \gamma a_2^2| \leq \frac{H_2 U_3}{H_1 U_2 u_3}\, max\left\{1, \left|\frac{\gamma H_1^3 U_2^3 u_3}{H_2 U_3 u_2^2} - U_2\right|\right\} \tag{4.1}$$

∎

Letting $\rho = 0$ in Theorem 4.1, we get the following consequence.

Corollary 4.2. *If the function f of the form (1.1) belongs to $\mathcal{SC}^t$, then for any complex number γ*

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{U_3}{U_2 u_3}\, max\left\{1, \left|\frac{\gamma U_2^3 u_3}{U_3 u_2^2} - U_2\right|\right\}$$

For $\rho = 1$ in Theorem 4.1, we obtain the following result.

Corollary 4.3. *If the function f of the form (1.1) belonging to the class $\mathcal{SC}^{t,1}$, then for any complex number γ*

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{U_3}{3U_2 u_3}\, max\left\{1, \left|\frac{3\gamma U_2^3 u_3}{4U_3 u_2^2} - U_2\right|\right\}$$

5. TOEPLITZ DETERMINANT $\mathcal{T}_2(2)$ AND $\mathcal{T}_3(1)$

Toeplitz matrices and their determinants play an important role in several branches of mathematics and have many applications Toeplitz [13]. For information on applications of Toeplitz matrices to several areas of pure and applied mathematics, we refer to the survey article by Ye and Lim [7]. We recall that Toeplitz symmetric matrices have constant entries along the diagonal. For the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{D})$, we associate a determinant $T_q(n)$ defined by

$$\mathcal{T}_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_n \end{vmatrix}$$

In 2017, Ali et al.[10] studied Toeplitz determinants $\mathcal{T}_q(n)$ for initial values of n and q, where the entries of $\mathcal{T}_q(n)$ are the coefficients of the functions that are starlike, convex and close to convex. Motivated by Ali et al.[10], some researchers in the last three years studied $\mathcal{T}_q(n)$ for low values of n and q, where entries are the coefficients of functions in several subclasses of analytic functions. Some recent work on coefficient problems includes Cho et al.[12]; Cudna et al.[5]; Lecko et al.[11]; O. P. Ahuja et al.[14].
In this section, we obtain estimates for Toeplitz determinants $\mathcal{T}_2(2)$ and $\mathcal{T}_3(1)$ for function belonging to the class $\mathcal{SC}^{t,\rho}$

Theorem 5.1. *If $f \in \mathcal{SC}^{t,\rho}$, then the Toeplitz determinant $\mathcal{T}_2(2)$ is given by*

$$|\mathcal{T}_2(2)| \leq \left[\frac{H_2U_3}{H_1U_2u_3}\, max(1, U_2)\right]^2 + \frac{H_1^2U_2^2}{u_2^2}.$$

Proof. Since,
$\mathcal{T}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix} = a_3^2 - a_2^2,$
The bound of $\mathcal{T}_2(2)$ is denoted by

$$|\mathcal{T}_2(2)| = |a_3^2 - a_2^2| \tag{5.1}$$

Applying triangle inequality to the above equation, we arrive at

$$|\mathcal{T}_2(2)| \leq |a_3^2| + |a_2^2| \tag{5.2}$$

Now, substituting equations (3.8) and (($3.9)$ to (5.2), we obtain,

$$|\mathcal{T}_2(2)| \leq \left[\frac{H_2U_3}{H_1U_2u_3}\, max(1, U_2)\right]^2 + \frac{H_1^2U_2^2}{u_2^2}. \tag{5.3}$$

∎

Taking $\rho = 0$ in Theorem 5.1, we get the following consequence.

Corollary 5.2. *If the function f of the form (1.1) belongs to $\mathcal{SC}^t$, then*

$$|\mathcal{T}_2(2)| \leq \left[\frac{U_3}{U_2u_3}\, max(1, U_2)\right]^2 + \frac{U_2^2}{u_2^2}$$

For $\rho = 1$ in Theorem 5.1, we obtain the following result.

Corollary 5.3. *If the function f of the form (1.1) belongs to $\mathcal{SC}^{t,1}$, then*

$$|\mathcal{T}_2(2)| \leq \left[\frac{U_3}{3U_2u_3}\, max(1, U_2)\right]^2 + \frac{U_2^2}{4u_2^2}$$

Theorem 5.4. *If $f \in \mathcal{SC}^{t,\rho}$, then the Toeplitz determinant $\mathcal{T}_3(1)$ is given by,*

$$|\mathcal{T}_3(1)| \leq 1 + \frac{2H_1^2U_2^2}{u_2^2} + \frac{H_2^2U_3^2}{H_1^2U_2^2u_3^2}\, max(1, U_2) max\left\{1, \left|\frac{2H_1^3U_2^3u_3}{H_2U_3u_2^2} - U_2\right|\right\}$$

Proof. Since,

$$\mathcal{T}_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{vmatrix} = 1 - 2a_2^2 - a_3(a_3 - 2a_2^2)$$

The bound of $\mathcal{T}_3(1)$ is denoted by

$$|\mathcal{T}_3(1)| = |1 - 2a_2^2 - a_3(a_3 - 2a_2^2)| \tag{5.4}$$

Applying triangle inequality to the above equation, we obtain

$$|\mathcal{T}_3(1)| \leq 1 + 2|a_2^2| + |a_3||a_3 - 2a_2^2| \tag{5.5}$$

Substituting equation (3.8),(3.9) and (4.1) to (5.5), we get

$$|\mathcal{T}_3(1)| \leq 1 + \frac{2H_1^2U_2^2}{u_2^2} + \frac{H_2^2U_3^2}{H_1^2U_2^2u_3^2}\, max(1, U_2)max\left\{1, \left|\frac{2H_1^3U_2^3u_3}{H_2U_3u_2^2} - U_2\right|\right\} \tag{5.6}$$

∎

Putting $\rho = 0$ in Theorem 5.4, we get the following consequence.

Corollary 5.5. *If the function f of the form (1.1) belongs to $\mathcal{SC}^t$, then*

$$|\mathcal{T}_3(1)| \leq 1 + \frac{2U_2^2}{u_2^2} + \frac{U_3^2}{U_2^2u_3^2}\, max(1, U_2)max\left\{1, \left|\frac{2U_2^3u_3}{U_3u_2^2} - U_2\right|\right\}$$

For $\rho = 1$ in Theorem 5.4, we obtain the following result.

Corollary 5.6. *If the function f of the form (1.1) belongs to $\mathcal{SC}^{t,1}$, then*

$$|\mathcal{T}_3(1)| \leq 1 + \frac{U_2^2}{2u_2^2} + \frac{U_3^2}{9U_2^2u_3^2}\, max(1, U_2)max\left\{1, \left|\frac{3U_2^3u_3}{2U_3u_2^2} - U_2\right|\right\}$$

6. CONCLUSION

In our present work, we have deïňĄned new subclass of Sakaguchi type of functions. Further, we have discussed some geometric properties like Coefficients estimates, Fekete-Szegö inequality, Toeplitz Determinant $\mathcal{T}_2(2)$ and $\mathcal{T}_3(1)$ for this newly deïňĄned class.

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