Aust. J. Math. Anal. Appl.
Vol. **19** (2022), No. 2, Art. 14, 12 pp.
AJMAA

ON STATISTICALLY ϕ-CONVERGENCE

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Received 26 January, 2022; accepted 9 November, 2022; published 30 November, 2022.

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ABSTRACT. The idea of statistical convergence was introduced by Antoni Zygmund in 1935. Based on in the idea of Zygmund, Henry Fast and Hugo Steinhaus independently introduced a concept of statistical convergence as a generalization of an ordinary convergence in the same year 1951. In this paper, by using the Orlicz function, we introduce a concept of statistical ϕ-convergence, as a generalization of the statistical convergence. Further, we observe some basic properties and some topological properties of the statistical ϕ-convergent sequences.

Key words and phrases: Orlicz function, ϕ-convergence, density, statistical ϕ-convergence, statistical ϕ-topology.

2010 *Mathematics Subject Classification.* Primary 40A05; Secondary 40C05.

ISSN (electronic): 1449-5910

The author is grateful to the anonymous referees for their valuable comments that improve the quality of the paper.

1. INTRODUCTION

Convergence of sequences plays crucial and important roles and has applications in many areas, such as approximation theory, measure theory, theory of probability, and trigonometric series [3, 7, 8, 10, 11, 16]. Even though, it does not mean that a divergent sequence is useless. We know that a divergent sequence might be has a convergent subsequence. However, the size of the set of subsequence's indexes in the natural numbers system $\mathbb{N}$ is varying. It depends on each sequence.

The main idea of the statistical convergence of a sequence (a_n) is that the existence of subsequence (a_{n_k}) of (a_n) with "majority" number of elements that still converges, no matter what happened to the other elements. The idea was proposed by Antoni Zygmund in 1935. In his monograph entitled *Trigonometric Series* [16], Zygmund proposed a term *almost convergence* instead of *statistical convergence* [6]. Based on the fact mentioned in previous paragraph and idea of Zygmund, Fast [6] and Steinhaus [13] independently in the same year 1951 defined a concept of statistical convergence [1, 4, 10, 14].

Although a concept of statistical convergence was introduced over 60 years ago, however it become an area of active research just in the beginning of 2000. In this era, a lot of works have been focused on the topic of statistical convergence (See for e.g. [1, 2, 4, 5, 8, 9, 15]). Altinoky and Memet [1], Cakali [4], and Kaya et.al [10] have observed some basic properties of statistical convergence. Meanwhile, Tabib [14] constructed a concept of topology based on statistical convergence. In [14], Tabib also showed that the topology induced by statistical convergence are identically to regular topology, i.e. the topology induced by convergence of sequence.

In many literatures, statistical convergence of any real or complex valued sequence is defined relatively to the absolute value or modulus [4]. While, we know that the absolute value on the real numbers system and modulus on complex numbers system are both a special case of an Orlicz function [12], i.e. a function $\phi : \mathbb{R} \to \mathbb{R}$ such that it is even, nondecreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\phi(x) = 0 \Leftrightarrow x = 0 \text{ and } \lim_{x\to\infty} \phi(x) = \infty.$$

Generally, any Orlicz function does not satisfy the triangular inequality. Because of this fact, we need a certain property of the Orlicz function, so called the Δ_2-condition, so that the absence of the triangle inequality property can be ignored. An Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ ia said to satisfy the Δ_2-condition, if there exists an $M > 0$ such that $\phi(2x) \leq M\phi(x)$ for every $x \in \mathbb{R}^+$.

Example 1.1. (i) *A function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(x) = |x|^2$, is an Orlicz function satisfying the Δ_2-condition.*
(ii) *A function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the Δ_2-condition.*

Orlicz functions, as described in [12], have important roles and applications in many areas, such as economics, finance, stochastic problems, etc. In connection with the fact, we are going to introduce a concept of statistical ϕ-convergence, as a generalization of the statistical convergence, by using an Orlicz function. We will also observe some basic properties, the monotonicity, and some topological properties of the statistical ϕ-convergence.

2. STATISTICAL ϕ-CONVERGENCE

For any set A, the symbol of $|A|$ denotes a cardinality of A. Let $K \subset \mathbb{N}$. For any $n \in \mathbb{N}$, we define

$$K(n) = \{k \in K : \ k \leq n\}$$

A density of K is defined as follows

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|K(n)|$$

Example 2.1. (i) *The set* $K = \{2n : n \in \mathbb{N}\}$ *has a density* $\frac{1}{2}$.
(ii) *The set* $P = \{n : n = k^2, k \in \mathbb{N}\}$ *has a density 0.*

It is easy to check that for any $K \subset \mathbb{N}$, $0 \leq \delta(K) \leq 1$.

Theorem 2.1. *Let* $K, P \subset \mathbb{N}$, *then* $\delta(\mathbb{N} - K) = 1 - \delta(K)$ *and* $\delta(K \cup P) \leq \delta(K) + \delta(P)$.

Definition 2.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence of real numbers (x_n) is said to be statically ϕ-convergent to some $x \in \mathbb{R}$ if for every real number $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : \phi(x_k - x) \geq \epsilon\}$ has a density zero. In this case, x is called an st-ϕ-limit of the sequence (x_n), and denoted by

$$st - \phi - \lim x_n = x$$

Example 2.2. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function with* $\phi(x) = x^2$. *The function* ϕ *satisfies the* Δ_2*-condition. If for any* $n \in \mathbb{N}$,

$$x_n = \begin{cases} \sqrt{n} & , n = k^2 \\ \frac{1}{\sqrt{n}} & , \text{otherwise} \end{cases}$$

then the sequence $\{x_n\}$ *is statistically* ϕ*-convergent to 0. It can also be shown that* $\{x_n\}$ *is not convergent.*

Example 2.3. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function with* $\phi(x) = |x|$. *The sequence* (a_n) *where* $a_n = n^2$ *for every* $n \in \mathbb{N}$ *is not statistically* ϕ*-convergent.*

Proof. Take any $a \in \mathbb{R}$, then $a \leq 0$ or $a > 0$. If $a \leq 0$, choose $\epsilon = \frac{1}{2}$, then for every $n \in \mathbb{N}$,

$$K_\epsilon = \{n : |a_n - a| \geq \epsilon\} = \mathbb{N}$$

Hence, $\delta(K_\epsilon) = 1$. If $a > 0$, then there exists an $N \in \mathbb{N}$ such that

$$a_{N-1} \leq a < a_N.$$

In this case, if $a < 1$, we define $a_{N-1} = 0$. Further, by choosing $\epsilon = \frac{1}{2}\min\{a - a_{N-1}, a_N - a\}$, then

$$K_\epsilon = \{n : |a_n - a| \geq \epsilon\} = \mathbb{N}$$

So, $\delta(K_\epsilon) = 1$. ∎

As we mention before, an absolute value on $\mathbb{R}$ is a special case on an Orlicz function. Moreover, by choosing the Orlicz function $\phi(x) = |x|$ on $\mathbb{R}$, we can show that every convergent sequence is statistically ϕ-convergence. However, the converse is not true. See Example 2.2. Thus, statistiscally ϕ-convergence is a generalization of an ordinary convergence.

In fact, there are some properties for convergent sequence that do not hold for statistically ϕ-convergent sequence. The following example shows there exists a statistically ϕ-convergent sequence which has a subsequence that is not statistically ϕ-convergent.

Example 2.4. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function with* $\phi(x) = |x|$ *and* (a_n) *a sequence with*

$$a_n = \begin{cases} n & , n = k^2 \\ \frac{1}{n} & , \text{otherwise} \end{cases}$$

Analogous to Example 2.2, then the sequence $\{a_n\}$ *is statistically* ϕ*-convergent to 0. However,* $\{a_n\}$ *has a subsequence that is not statistically* ϕ*-convergent, i.e. a sequence as given in Example 2.3.*

Theorem 2.2. *Let ϕ be a convex Orlicz function. If the sequence (x_n) is statistically ϕ-convergent, then its st-ϕ-limit is unique.*

Proof. Suppose $st-\phi-\lim x_n = a$ and $st-\phi-\lim x_n = b$. For any $\epsilon > 0$, we define

$$K'_\epsilon = \{n: \ |x_n - a| \geq \frac{\epsilon}{2}\} \text{ and } K''_\epsilon = \{n: \ |x_n - b| \geq \frac{\epsilon}{2}\}$$

Since $st-\phi-\lim x_n = a$ and $st-\phi-\lim x_n = b$, then $\delta(K'_\epsilon) = \delta(K''_\epsilon) = 0$. Let $K_\epsilon = K'_\epsilon \cup K''_\epsilon$, then $\delta(K_\epsilon) = 0$. Hence, $\delta(\mathbb{N} - K_\epsilon) = 1$. Choose $N \in \mathbb{N} - K_\epsilon$, then the convexity of ϕ implies

$$\begin{aligned} \phi(\frac{1}{2}(a-b)) &= \phi(\frac{1}{2}(a - x_N + x_N - b)) \\ &\leq \phi(a - x_N) + \phi(x_N - b) < \epsilon \end{aligned}$$

that is $a = b$. ∎

In general, any Orlicz function is not homogenous. However, in many cases the Δ_2-condition can take over some roles of the homogenous properties. Based on this fact and Theorem 2.2, for the rest discussion we always assume that any Orlicz function is convex and satisfying the Δ_2-condition, unless otherwise stated. Further, we observe some basic properties of statistical ϕ-convergence as given in the following theorems.

Theorem 2.3. *If (x_n) and (y_n) are statistically ϕ-convergent with respect to the Orlicz function ϕ and α any real constant, then*

(i) $(x_n + y_n)$ *is statistically ϕ-convergent, and*

$$st-\phi-\lim(x_n + y_n) = st-\phi-\lim x_n + st-\phi-\lim y_n$$

(ii) (αx_n) *is statistically ϕ-convergent, and*

$$st-\phi-\lim(\alpha x_n) = \alpha st-\phi-\lim x_n$$

Proof. Since ϕ satisfies the Δ_2-condition, then there exists an $M > 0$ such that $\phi(2x) \leq M\phi(x)$ for every $x \in \mathbb{R}$.
(i) Let (x_n) and (y_n) be statistically ϕ-convergent, respectively. Say

$$st-\phi-\lim x_n = x \text{ dan } st-\phi-\lim y_n = y$$

For any $\epsilon > 0$, the sets

$$K'_\epsilon = \{n: \ |x_n - x| \geq \frac{\epsilon}{2M}\} \text{ and } K''_\epsilon = \{n: \ |y_n - y| \geq \frac{\epsilon}{2M}\}$$

have densities zero, respectively, i.e. $\delta(K'_\epsilon) = \delta(K''_\epsilon) = 0$. Let $K_\epsilon = K'_\epsilon \cup K''_\epsilon$, then $\delta(K_\epsilon) = 0$. Since, $\delta(\mathbb{N} - K_\epsilon) = 1$, then $\mathbb{N} - K_\epsilon \neq \emptyset$. Further, for any $n \in \mathbb{N} - K_\epsilon$, we have

$$\begin{aligned} \phi((x_n + y_n) - (x + y)) &\leq \phi(2(x_n - x)) + \phi(2(y_n - y)) \\ &\leq M\phi(x_n - x) + M\phi(y_n - y) < \epsilon. \end{aligned}$$

This implies $(x_n + y_n)$ is statisticalley ϕ-convergent to $x + y$.
(ii) Choose $p \in \mathbb{N}$ such that $|\alpha| \leq 2^p$. Let (x_n) be statistically ϕ-convergent to x, then for any $\epsilon > 0$, the set

$$K = \{n: \ \phi(x_n - x) \geq \frac{\epsilon}{M^p}\}$$

has a density 0. Since $\delta(\mathbb{N} - K) = 1$, then $\mathbb{N} - K \neq \emptyset$. Since for every $n \in \mathbb{N} - K$,

$$\phi(\alpha(x_n - x)) = \phi(|\alpha|(x_n - x)) \leq \phi(2^p(x_n - x)) \leq M^p\phi(x_n - x) < \epsilon$$

then (αx_n) is statistically ϕ-convergent to αx. ∎

Theorem 2.4. *A sequence (x_n) is statistically ϕ-convergent to x if and only if there exists a set $K = \{n_k : \ n_k \in \mathbb{N}, n_k < n_{k+1}, \ k = 1, 2, 3, \ldots\}$ with a density 1 such that the sequence $(\phi(x_{n_k} - x))$ converges to 0.*

Proof. Let (x_n) be statistically ϕ-convergent to x, then there exists a set

$$K = \{n_k : \ n_k \in \mathbb{N}, n_k < n_{k+1}, \ k = 1, 2, 3, \ldots\}$$

with a density 1 such that $(\phi(x_{n_k} - x))$ converges to 0.

For the converse, let

$$K = \{n_k : \ n_k \in \mathbb{N}, n_k < n_{k+1}, \ k = 1, 2, 3, \ldots\}$$

be a set with a density 1 such that $(\phi(x_{n_k} - x))$ converges to 0. Then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $k \geq N$ we have

$$\phi(x_{n_k} - x) < \epsilon$$

Let

$$K_\epsilon = \{n : \ \phi(x_n - x) < \epsilon\} \text{ and } K_0 = \{n_k \in K : \ k \geq N\}$$

then $K_0 \subset K_\epsilon$. Since

$$\delta(K_0) = \delta(K - \{n_1, n_2, \ldots, n_N\}) = 1 - 0 = 1$$

then $\delta(K_\epsilon) = 1$ or $\delta(\{n : \ \phi(x_n - x) \geq \epsilon\}) = 0$. This means (x_n) is statistically ϕ-convergent to x. ∎

Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlics function. A sequence (x_n) is said to be ϕ-convergent to some $x \in \mathbb{R}$ if the sequence $(\phi(x_n - x))$ converges to 0. In this case, x is called the ϕ-limit of (x_n), and denoted by

$$x = \phi - \lim x_n$$

The definition of a ϕ-Cauchy sequence is defined analogous to the definition of a ϕ-convergent sequence. A sequence (x_n) is called a ϕ-Cauchy sequence if for any $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\phi(x_n - x_m)) < \epsilon,$$

for every natural numbers $m, n \geq n_0$. It is easy to check, if (x_n) is ϕ-convergent to x, then its any subsequence is ϕ-convergent to x as well. Note that, in case the Orlicz function $\phi = |.|$, then the ϕ-convergence becomes the ordinary convergence.

Following the definition of ϕ-convergent sequence, then we can reformulate the Theorem 2.4 to be the following theorem.

Theorem 2.5. *A sequence (x_n) is statistically ϕ-convergent to x if and only if there exists a set $K = \{n_k : \ n_k \in \mathbb{N}, n_k < n_{k+1}, \ k = 1, 2, 3, \ldots\}$ with a density 1 such that the sequence (x_{n_k}) is ϕ-convergent to x.*

Moreover, we can easily prove the following theorem.

Theorem 2.6. *If (x_n) ϕ-converges to x, then it is statistically ϕ-convergent to x.*

Note that the converse of the Teorema 2.6 is not true. Lets consider the following example.

Example 2.5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function with $\phi(x) = x^2$. The function satisfies the Δ_2-condition. Then sequence $n \in \mathbb{N}$, where*

$$x_n = \begin{cases} \sqrt{n} & , n = k^2 \\ \frac{1}{\sqrt{n}} & , \textit{otherwise} \end{cases}$$

is statistically ϕ-convergent to 0, but it does not ϕ-converge to 0.

A sequence (x_n) is said to be ϕ-bounded with respect to an Orlicz function ϕ, if there exists $K > 0$ such that $\phi(x_n) \leq K$ for every $n \in \mathbb{N}$.

Theorem 2.7. *If (x_n) is ϕ-bounded and statistically ϕ-convergent to x, relative with respect to an Orlicz function ϕ, then* $\lim \frac{1}{n}\sum_{k=1}^{n}\phi(x_k - x) = 0$.

Proof. Let $\epsilon > 0$ be an arbitrary. Since $st - \phi - \lim x_n = x$, then the set

$$L_\epsilon = \{k \in \mathbb{N} :\ \phi(x_k - x) \geq \epsilon\}$$

has a density 0, i.e. $\delta(L_\epsilon) = 0$. By the hypothesis, there exists $K > 0$ such that $\phi(x_n) \leq K$ for every $n \in \mathbb{N}$. Since ϕ is convex and satisfies the Δ_2-condition, then there exists $M > 0$ such that

$$\begin{aligned} \phi(x_n - x) &= \phi(\frac{1}{2}2x_n + \frac{1}{2}(-2x)) \\ &\leq M(K + \phi(x)) \end{aligned}$$

for every $n \in \mathbb{N}$. By choosing $M(K + \phi(x)) = M'$, then for every $n \in \mathbb{N}$

$$\phi(x_n - x) \leq M'$$

For any $n \in \mathbb{N}$, lets define

$$L_\epsilon(n) = \{k \leq n :\ \phi(x_k - x) \geq \epsilon\}$$

If $u_n = |L_\epsilon(n)|$ for every $n \in \mathbb{N}$, then $\lim \frac{u_n}{n} = \delta(L_\epsilon) = 0$. So, we can choose $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\frac{u_n}{n} = |\frac{u_n}{n}| < \epsilon$$

Moreover,

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\phi(x_k - x) &\leq \frac{1}{n}\Big\{M'u_n + (n - u_n)\epsilon\Big\} \\ &\leq \frac{1}{n}\Big\{M'u_n + n\epsilon\Big\} = \epsilon + M'\frac{u_n}{n} \\ &< \epsilon(1 + M') \end{aligned}$$

for every $n \geq n_0$. Thus $\lim \frac{1}{n}\sum_{k=1}^{n}\phi(x_k - x)$. ∎

Definition 2.2. A sequence (x_n) is called a statistically ϕ-Cauchy sequence if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} :\ \phi(x_n - x_N) \geq \epsilon\}) = 0$$

It is easy to prove that every ϕ-Cauchy sequence is a statistically ϕ-Cauchy sequence.

Theorem 2.8. *If (x_n) is statistically ϕ-convergent, then it is a statistically ϕ-Cauchy sequence.*

Proof. Let (x_n) is statistically ϕ-convergent to x. For any $\epsilon > 0$, let

$$A(\epsilon) = \{n \in \mathbb{N} :\ \phi(2(x_n - x)) \geq \epsilon\}$$

Since $x = st - \phi - \lim x_n$, then $\delta(A(\epsilon)) = 0$. This implies that

$$\delta(\mathbb{N} - A(\epsilon)) = \delta(\{n \in \mathbb{N} :\ \phi(2(x_n - x)) < \epsilon\}) = 1$$

Let $m, n \in \mathbb{N} - A(\epsilon)$, then

$$\phi(x_n - x_m) \leq \frac{1}{2}\phi(2(x_m - x)) + \frac{1}{2}\phi(2(x - x_n) < \epsilon$$

Choose $N \in \mathbb{N} - A(\epsilon)$ and let

$$B(\epsilon) = \{n \in \mathbb{N} : \ \phi(x_n - x_N) < \epsilon\}$$

then $\mathbb{N} - A(\epsilon) \subset B(\epsilon)$. Since

$$1 = \delta(\mathbb{N} - A(\epsilon)) \leq \delta(B(\epsilon)) \leq 1$$

then $\delta(B(\epsilon)) = 1$ or $\delta(\mathbb{N} - B(\epsilon)) = 0$. This means (x_n) is a statistically ϕ-Cauchy sequence. ∎

Definition 2.3. A sequence (x_n) is said to be

(i) statistically ϕ-monotone increasing if there exists a set $H \subset \mathbb{N}$ with $\delta(H) = 1$ such that $\phi(x_n) \leq \phi(x_{n+1})$ for every $n \in H$.
(ii) statistically ϕ-monotone decreasing if there exists a set $H \subset \mathbb{N}$ with $\delta(H) = 1$ such that $\phi(x_n) \geq \phi(x_{n+1})$ for every $n \in H$.
(iii) statistically ϕ-monotone if it is either statistically ϕ-monotone increasing or statistically ϕ-monotone decreasing.

The following theorem states that the statistically ϕ-monotonicity and ϕ-boundedness of the sequence (x_n) is a sufficient condition of the statistically convergence of the sequence $(\phi(x_n))$.

Theorem 2.9. *If (x_n) is statistically ϕ-monotone and ϕ-bounded, then the sequence $(\phi(x_n))$ is statistically convergent.*

Proof. We prove the theorem for the case (x_n) is statistically ϕ-monotone increasing only. Let (x_n) be statistically ϕ-monotone increasing, then there exists a set $H \subset \mathbb{N}$ with $\delta(H) = 1$ such that

$$\phi(x_n) \leq \phi(x_{n+1}),$$

for every $n \in H$. Since (x_n) is ϕ-bounded, then there is a constant K such that

$$\phi(x_n) \leq K$$

for every $n \in H$. By completeness of $\mathbb{R}$, then

$$y = \sup \phi(x_n)$$

exists. Let $\epsilon > 0$ be an arbitrary, then there exists an $N \in H$ such that for every $n \in H$ with $n \geq N$,

$$y - \epsilon < \phi(x_N) \leq \phi(x_n) \leq y < y + \epsilon$$

This means $(\phi(x_n))$ is statistically convergent to y. ∎

3. SOME TOPOLOGICAL PROPERTIES

In this section, we are going to discuss about a concept of statistically ϕ-topology. We begin by the following definition.

Definition 3.1. A point $x \in \mathbb{R}$ is called an st-ϕ-closure point of a set $F \subset \mathbb{R}$ if there exists a sequence (x_n) in F that statistically ϕ-converges to x, or

$$x = st - \phi - \lim x_n$$

A collection of all st-ϕ-closure points of F will be denoted by $\bar{F}^{\phi}_{st}$.

It is clear that $F \subset \bar{F}^{\phi}_{st}$ for every $F \subset \mathbb{R}$. However, the converse is not true. This fact leads us to the following definition.

Definition 3.2. Any subset $F \subset \mathbb{R}$ is said to be st-ϕ-closed if $\bar{F}^{\phi}_{st} = F$.

We have $\emptyset$ and $\mathbb{R}$ are st-ϕ-closed. Further investigation needs some definitions related to the ϕ-convergence. A point $x \in \mathbb{R}$ is called a ϕ-closure point of a subset $F \subset \mathbb{R}$ if there exists a sequence (x_n) in F which ϕ-converges to x, i.e.

$$x = \phi - \lim x_n$$

A collection of all ϕ-closure point of F will be denoted by $\bar{F}^\phi$. Moreover, we have $F \subset \bar{F}^\phi$ for every $F \subset \mathbb{R}$. A subset $F \subset \mathbb{R}$ is said to be ϕ-closed if $\bar{F}^\phi = F$. A set $E \subset \mathbb{R}$ is said to be ϕ-open if E^C is ϕ-closed.

Following Theorem 2.5 and Theorem 2.6, we have the following property.

Theorem 3.1. *Any $F \subset \mathbb{R}$ is st-ϕ-closed if and only if F is ϕ-closed.*

Further, we also observe the following theorem.

Theorem 3.2. *If F_i is st-ϕ-closed for every $i \in \mathbb{N}$, then*

(i) *$F_1 \cap F_2$ is st-ϕ-closed.*
(ii) *$\cap_{i=1}^{\infty} F_i$ is st-ϕ-closed.*
(iii) *$F_1 \cup F_2$ is st-ϕ-closed.*

Proof. Part (i) and (ii) are easy.
(iii) Take any $x \in \overline{F_1 \cup F_2}^{\phi}_{st}$, then there exists (x_n) in $F_1 \cup F_2$ such that

$$x = st - \phi - \lim x_n$$

Following Theorem 2.5 there exists a set

$$K = \{n_k : \; n_k \in \mathbb{N}, \; n_k < n_{k+1}, \; k \in \mathbb{N}\}$$

with $\delta(K) = 1$ such that (x_{n_k}) ϕ-converges to x. Let

$$P = \{n_k : \; x_{n_k} \in F_1\} \text{ and } Q = \{n_k : \; x_{n_k} \in F_2 - F_1\}$$

then $P \cup Q = K$. Since $\delta(P) + \delta(Q) = \delta(K) = 1$, then at least P or Q is an infinite set. Assume P is an infinite set, then

$$P = \{\psi(n) : \; n \in \mathbb{N}\}$$

for some increasing $\psi : \mathbb{N} \to \mathbb{N}$. Since $(x_{\psi(n)}) \subset (x_{n_k})$, then by Theorem 2.5,

$$\lim \phi(x_{\psi(n)}) = x$$

Finally by Theorem 3.1, $x \in F_1 \subset F_1 \cup F_2$. ∎

Any set $U \subset \mathbb{R}$ is said to be st-ϕ-open if U^C is st-ϕ-closed. Moreover, Theorem 3.1 implies the following theorem.

Theorem 3.3. *Any set $E \subset \mathbb{R}$ is st-ϕ-open if and only if E ϕ-open.*

Also, following Theorem 3.2, we have the following theorem.

Theorem 3.4. *If E_i is st-ϕ-open for every $i \in \mathbb{N}$, then*

(i) *$E_1 \cup E_2$ is st-ϕ-open.*
(ii) *$\cup_{i=1}^{\infty} E_i$ is st-ϕ-open.*
(iii) *$E_1 \cap E_2$ is st-ϕ-open.*

Recall that a set $K \subset \mathbb{N}$ is said to be statistically dense if $\delta(K) = 1$. It can be easily checked that the set $K = \{n \in \mathbb{N} : \; n \text{ is not a square } \}$ is statistically dense, meanwhile the set $F = \{3n : \; n \in \mathbb{N}\}$ is not statistically dense. Further, in [14] Talib prove the following useful lemma.

Lemma 3.5. *The following statements are true.*

(i) *If $K \subset \mathbb{N}$ is statistically dense, then any set $F \subset K$ with $\delta(K - F) = 0$ is statistically dense.*

(ii) *If $K, M \subset \mathbb{N}$ are statistically dense, then $K \cap M$ is statistically dense.*

Based on the definition of a statistically dense subset $K \subset \mathbb{N}$ and Lemma 3.5, we define a statistically and a statistically ϕ-dense subsequence of any sequence.

Definition 3.3. A subsequence (x_{n_k}) of the sequence (x_n) is said to be

(i) statistically dense in (x_n) if the set $K = \{n_k : \ k \in \mathbb{N}\}$ is statistically dense.

(i) statistically ϕ-dense in (x_n) if $(\phi(x_{n_k}))$ is statistically dense in $(\phi(x_n))$.

In the following theorem, we characterize the convergence of a sequence by using its subsequences.

Theorem 3.6. *A sequence (x_n) is statistically ϕ-convergent to x if and only if any statistically ϕ-dense subsequence (x_{n_k}) of (x_n) is statistically ϕ-convergent to x.*

Proof. Assume that a sequence (x_n) is statistically ϕ-convergent to x and let (x_{n_k}) be any statistically ϕ-dense subsequence in (x_n). We are going to prove that (x_{n_k}) is statistically ϕ-convergent to x. Suppose the contrary is true, then there exists an $\epsilon > 0$ such that

$$\lim \frac{1}{n}|B_{n,\epsilon}| > 0,$$

where $B_{n,\epsilon} = \{k \leq n : \ \phi(x_{n_k} - x) \geq \epsilon\}$. Since

$$A_{n,\epsilon} = \{k \leq n : \ \phi(x_n - x) \geq \epsilon\} \supset B_{n,\epsilon},$$

then $\lim \frac{1}{n}|A_{n,\epsilon}| > 0$. This contradicts to the hypothesis.

For the converse, since (x_n) is a statistically ϕ-dense subsequence of itself, then the assertion follows. ∎

4. STATISTICAL ϕ-LIMIT SUPERIOR AND STATISTICAL ϕ-LIMIT INFERIOR

Throughout this section, the statement $\delta(K) \neq 0$ means $\delta(K) > 0$ or $K = \emptyset$.

Definition 4.1. For any sequence $x = (x_n)$, let

$$B_x = \{b \in \mathbb{R} : \ \delta(\{n \in \mathbb{N} : \ \phi(x_n) > b\}) \neq 0\}$$

and

$$A_x = \{b \in \mathbb{R} : \ \delta(\{n \in \mathbb{N} : \ \phi(x_n) < b\}) \neq 0\}$$

We define

(i) statistical ϕ-limit superior of (x_n) by

$$stat - \phi - \limsup x_n = \sup B_x$$

provided $B_x \neq \emptyset$. If $B_x = \emptyset$, we set

$$stat - \phi - \limsup x_n = -\infty$$

(ii) statistical ϕ-limit inferior of (x_n) by

$$stat - \phi - \liminf x_n = \inf A_x$$

provided $A_x \neq \emptyset$. If $A_x = \emptyset$, we set

$$stat - \phi - \limsup x_n = \infty$$

Example 4.1. *Let ϕ be an Orlicz function defined by $\phi(x) = x^2$ and (x_n) a sequence given by*

$$x_n = \begin{cases} \sqrt{n} & \text{, n is an odd square} \\ \sqrt{2} & \text{, n is an even square} \\ 1 & \text{, n is an odd nonsquare} \\ 0 & \text{, n is an even nonsquare} \end{cases}$$

Then

$$B_x = \{b \in \mathbb{R} : \ \delta(\{n \in \mathbb{N} : \ \phi(x_n) > b\}) \neq 0\} = (-\infty, \ 1)$$

and

$$A_x = \{b \in \mathbb{R} : \ \delta(\{n \in \mathbb{N} : \ \phi(x_n) < b\}) \neq 0\} = (0, `\infty)$$

So,

$$stat - \phi - \limsup x_n = 1 \qquad \text{and} \qquad stat - \phi - \liminf x_n = 0$$

Theorem 4.1. *For any sequence (x_n),*

(4.1) $$stat - \phi - \liminf x_n \leq stat - \phi - \limsup x_n.$$

Proof. It is obvious whenever $stat - \phi - \liminf x_n = -\infty$ or $stat - \phi - \limsup x_n = \infty$.

Now we assume that $stat - \phi - \liminf x_n = \alpha \neq -\infty$ and $stat - \phi - \limsup x_n = \beta \neq \infty$. For any $\epsilon > 0$, we have

$$\delta(\{n : \ \phi(x_n) > \frac{\beta}{2}\}) = 0,$$

so

$$\delta(\{n : \ \phi(x_n) \leq \frac{\beta}{2}\}) = 1$$

This implies that

$$\delta\{n : \ \phi(x_n) < \beta\}) = 1$$

Hence, $\beta + \epsilon \in A_x$. Since $\alpha = \inf A_x$, then $\alpha \leq \beta + \epsilon$. ∎

The equality in (4.1) is achieved whenever the sequence $(\phi(x_n))$ is statistically convergent.

Theorem 4.2. *Let ϕ be an Orlicz function. For any sequence (x_n), $stat - \phi - \liminf x_n = stat - \phi - \limsup x_n$ if and only if $(\phi(x_n))$ is statistically convergent.*

Proof. Suppose that $(\phi(x_n))$ is statistically convergent, say to x. Then for any $\epsilon > 0$,

$$\delta(\{n \in \mathbb{N} : \ |\phi(x_n) - x| \geq \epsilon\}) = 0$$

So,

$$\delta(\{n \in \mathbb{N} : \ \phi(x_n) \geq x + \epsilon\}) = 0,$$

which means that $stat - \phi - \liminf x_n \leq x + \epsilon$. We have also

$$\delta(\{n \in \mathbb{N} : \ \phi(x_n) \leq x - \epsilon\}) = 0,$$

i.e. $stat - \phi - \limsup x_n \geq x - \epsilon$.

For the converse, assume that $stat - \phi - \liminf x_n = stat - \phi - \limsup x_n = x$. For any $\epsilon > 0$, we have

$$\delta(\{n \in \mathbb{N} : \ \phi(x_n) > x + \epsilon\}) = 0$$

and

$$\delta(\{n \in \mathbb{N} : \ \phi(x_n) < x - \epsilon\}) = 0$$

These imply that

$$\begin{aligned} \delta(\{n \in \mathbb{N} : \ |\phi(x_n) - x| > \epsilon\}) &= \delta(\{n \in \mathbb{N} : \ \phi(x_n) < x - \epsilon\}) + \\ &\quad \delta(\{n \in \mathbb{N} : \ \phi(x_n) > x + \epsilon\}) = 0 \end{aligned}$$

This means $(\phi(x_n))$ is statistically convergent to x. ∎

5. CONCLUDING REMARKS

In this paper, we successfully defined a concept of statistical ϕ-convergence, as a generalization of statistical convergence. We have also successfully observed and formulated some basic properties as well as some topological properties of statistical ϕ-convergent sequences.

In Section 2, we can prove the uniqueness of a st-ϕ-limit of a statistical ϕ-convergent sequence and a calculus of statistically ϕ-convergence as well. We can also show that the statistically ϕ-monotonicity and ϕ-boundedness of a sequence (x_n) imply the statistically convergence of the sequence $(\phi(x_n))$.

In Section 3, we can formulate a characterization of statistical ϕ-convergent sequences. Meanwhile, the equality of

$$stat - \phi - \liminf x_n = stat - \phi - \limsup x_n$$

is characterized by the statistically convergence of $(\phi(x_n))$. It is formulated in Section 4.

For future works, we will observe the applications of statistical convergence in fixed point theory and others.

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