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UNIQUENESS PROBLEMS FOR DIFFERENCE POLYNOMIALS SHARING A NON-ZERO POLYNOMIAL OF CERTAIN DEGREE WITH FINITE WEIGHT

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ABSTRACT. In this paper, we prove a result on the value distribution of difference polynomials sharing higher order derivatives of meromorphic functions which improves some earlier results. At the same time, we also prove possible uniqueness relation of entire functions when the difference polynomial generated by them sharing a non zero polynomial of certain degree. The result obtained in the paper will improve and generalize a number of recent results in a compact and convenient way.

Key words and phrases: Meromorphic function; Entire function; Uniqueness; Weighted Sharing; Difference operators.

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1. INTRODUCTION AND PRELIMINARIES

In this Literature Survey, let f be non-constant meromorphic in the complex plane, we assumed that the reader is familiar with the notations of Nevanlinna theory [6]. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. Let us say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities and $f(z), g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities. Here we adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f)=O(T(r,f))$ as $r\to\infty$, possibly outside a set of finite linear measure, denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=O(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite logarithmic measure

$$\sigma(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}$$

In this we say that a finite value z_0 is called a fixed point of f if $f(z_0)=z_0$ or z_0 is a zero of $f(z)-z$. For the sake of simplicity we also use the notation

$$m^*:=\begin{cases}0, & \text{if } m=0\\ m, & \text{if } m\in\mathbb{N}\end{cases}$$

Let $f(z)$ be a transcendental meromorphic function, n be a positive integer. During the last few decades many authors investigated the value distribution of f^nf'. Specially in 1959, W.K. Hayman (see [5]) proved the following Theorem.
We now explain following definitions and notations which are used in the paper.

Definition 1.1. [9] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\le k$ and k times if $m>k$. Then

$$N_k(r,a;f)=\bar{N}(r,a;f)+\bar{N}(r,a;f\mid\ge 2)+\ldots+\bar{N}(r,a;f\mid\ge k).$$

Clearly $N_1(r,a;f)=\bar{N}(r,a;f)$.

Definition 1.2. [8][9] Let k be a non-negative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$, we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\le k$ and $\mathrm{k}+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.
The definition implies that if f,g share a value a with weight k, then z_0 is an a point of f with multiplicity $m(\le k)$ if and only if it is an a-point of g with multiplicity $m(\le k)$ and z_0 is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $n(>k)$, where m is not necessarily equal to n.
We write f,g share (a,k) to mean that f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for any integer $p, 0\le p<k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively.

Theorem 1.1. [5] *Let $f(z)$ be a transcendental meromorphic function and $n(\ge 3)$ is an integer. Then $f^nf'=1$ has infinitely many solutions.*

The case $n = 2$ was settled by Mues [14] *in 1979. Bergweiler and Eremenko* [1] *showed that $ff' - 1$ has infinitely many zeros.*
For an analog of the above results Laine and Yang investigated the value distribution of difference products of entire functions in the following manner.

Theorem 1.2. [10] *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2, f^n(z)f(z + c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*
Afterwards, Liu and Yang [13] *improved Theorem 1.2 and obtained the next result.*

Theorem 1.3. [13] *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2, f^n(z)f(z + c) - p(z)$ has infinitely many zeros, where $p(z)$ is a non-zero polynomial.*

Next we recall the uniqueness result corresponding to Theorem 1.1, obtained by Yang and Hua [17] which may be considered a gateway to a new research in the direction of sharing values of differential polynomials.

Theorem 1.4. [13] *Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n f'$ and $g^n g'$ share 1CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where $c_1, c_2, c \in \mathbb{C}$ satisfying $4\left(c_1 c_2\right)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2001, Fang and Hong[4] studied the uniqueness of differential polynomials of the form $f^n(f - 1)f'$ and proved the following uniqueness result.

Theorem 1.5. [4] *Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let $n \geq 11$ be a positive integer. If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share the value 1CM, then $f = g$.*

In 2004, Lin and Yi[12] extended the above result in view of the fixed point and they proved the following.

Theorem 1.6. [12] *Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let $n \geq 7$ be a positive integer. If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share $z\text{CM}$, then $f = g$.*

In 2010, Zhang[19] got a analogue result in difference.

Theorem 1.7. [19] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer.If $f(z)^n(f(z) - 1)f(z + c)$ and $g(z)^n(g(z) - 1)g(z + c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

In 2010, Qi, Yang and Liu[15] obtained the difference counterpart of Theorem 1.4 by proving the following theorem.

Theorem 1.8. [15] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and c be a nonzero complex constant; let $n \geq 6$ be an integer. If $f^n f(z + c)$ and $g^n g(z + c)$ share z CM, then $f \equiv t_1 g$ for a constant t_1 that satisfies $t_1^{n+1} = 1$.*

Theorem 1.9. [15] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and c be a nonzero complex constant; let $n \geq 6$ be an integer. If $f^n f(z + c)$ and $g^n g(z + c)$ share 1CM, then $fg \equiv t_2$ or $f \equiv t_3 g$ for some constants t_2 and t_3 that satisfy $t_3^{n+1} = 1$.*

X.M. Li et. al. [11] [Theorem 1.9] replaced the fixed point sharing in the above two theorems to sharing a polynomial with $\deg < \frac{n+1}{2}$.
So we see that there are many generalization in terms of difference operator. The purpose of this paper is to study the uniqueness problem for more general difference polynomials namely

$f^nP(f)f(z+c)$ and $g^nP(g)g(z+c)$ sharing a non-zero polynomial so that improved version of all the above results can be unified under a single result. We also relax the nature of sharing with the notion of weighted sharing introduced in [8]-[9].
Keeping the above question in mind, in 2020, A.Banerjee and S.Majumder [20] proved the following results.

Theorem 1.10. [20] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, c be a non-zero complex constant and let $p(z)$ be a nonzero polynomial with $\deg(p) \leq n-1$, $n(\geq 1), m^*(\geq 0)$ be two integers such that $n > m^*+5$. Let $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ be a nonzero polynomial. If $f^nP(f)f(z+c)-p$ and $g^nP(g)g(z+c)-p$ share $(0,2)$, then*
(I) when $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ is a nonzero polynomial, one of the following three cases holds:
(I1) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n), a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,
(I2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^n\left(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0\right) - \omega_2^n\left(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0\right)$,
(I3) $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i\omega^i \not\equiv 0$, for $i \in \{0, 1, \ldots, m\}$, if $p(z)$ is a nonzero constant b, then $f(z) = e^{\alpha(z)}, g = e^{\beta(z)}$, where $\alpha(z), \beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n+i+1)d} = b^2$;
(II) when $P(\omega) = \omega^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;
(III) when $P(\omega) = (\omega-1)^m (m \geq 2)$, one of the following two cases holds:
(III1) $f(z) \equiv g(z)$,
(III2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^n(\omega_1 - 1)^m\omega_1(z+c) - \omega_2^n(\omega_2-1)^m\omega_2(z+c)$;
(IV) when $P(\omega) \equiv c_0$, one of the following two cases holds:
(IV1) $f \equiv tg$ for some constant t such that $t^{n+1} = 1$,
(IV2) $f(z) = e^{\alpha(z)}, g = e^{\beta(z)}$, where $\alpha(z), \beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $c_0^2 e^{(n+1)d} = b^2$.

Question: Could we further reduce Theorem 1.10 under relax sharing hypothesis?
In this direction, We prove the following main result

2. MAIN RESULTS

Theorem 2.1. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, c be a non-zero complex constant and let $p(z)$ be a nonzero polynomial with $\deg(p) \leq n-1$, $n(\geq 1), m^*(\geq 0)$ be two integers such that $n > m^*+5$. Let $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ be a nonzero polynomial. If $f^nP(f)f(z+c)-p$ and $g^nP(g)g(z+c)-p$ share $(0,l)$, then*
(i) $l \geq 2, m = 0$, and $n \geq 2m+6$
(ii) $l \geq 2, m = \infty$, and $n \geq m+5$
(iii) $l = 1, m = 0$, and $n \geq 5m+17$
(iv) $l = 0, m = 0$, and $n \geq 7m+23$
(v)$f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$,

3. SOME LEMMAS

In this section, we present some lemmas which will be needed in the result. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We also denote by H, the following function

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right) \tag{3.1}$$

Lemma 3.1. [16] *Let $f(z)$ be a non-constant meromorphic function and $a_n(z)(\not\equiv 0)$,$a_{n-1}(z)$, $\cdots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r,f)$ for $i = 0,1,2,\ldots,n$. Then*

$$T\left(r, a_n f^n + a_{n-1}f^{n-1} + \ldots + a_1 f + a_0\right) = nT(r,f) + S(r,f).$$

Lemma 3.2. [2] *Let $f(z)$ be a meromorphic function of finite order σ, and let c be a fixed nonzero complex constant. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 3.3. [2] *Let $f(z)$ be a meromorphic function of finite order $\sigma, c \neq 0$ be fixed. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r,f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).$$

Lemma 3.4. [4] *Let $f(z)$ be an entire function of finite order σ, c be a fixed nonzero complex constant and let $n \in \mathbb{N}$ and $P(\omega)$ be defined as in Theorem 1.1 Then for each $\varepsilon > 0$, we have*

$$T\left(r, f^n P(f) f(z+c)\right) = T\left(r, f^{n+1}P(f)\right) + O\left(r^{\sigma-1+\varepsilon}\right)$$

Proof. By Lemma 3.2 we have

$$\begin{aligned} T\left(r, f^n P(f) f(z+c)\right) &= m\left(r, f^n P(f) f(z+c)\right) \\ &\leq m\left(r, f^n P(f) f(z+c)\right) + m\left(r, \frac{f(z+c)}{f(z)}\right) \\ &\leq m\left(r, f^{n+1}P(f)\right) + O\left(r^{\sigma-1+\varepsilon}\right) \\ &= T\left(r, f^{n+1}P(f)\right) + O\left(r^{\sigma-1+\varepsilon}\right). \end{aligned}$$

Also we have

$$\begin{aligned} T\left(r, f^{n+1}P(f)\right) &= m\left(r, f^n P(f) f(z+c)\right) \\ &\leq m\left(r, f^n P(f) f(z+c)\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) \\ &\leq m\left(r, f^n P(f) f(z+c)\right) + O\left(r^{\sigma-1+\varepsilon}\right) \\ &\leq T\left(r, f^n P(f) f(z+c)\right) + O\left(r^{\sigma-1+\varepsilon}\right). \end{aligned}$$

Therefore $T\left(r, f^n P(f) f(z+c)\right) = T\left(r, f^{n+1}P(f)\right) + O\left(r^{\sigma-1+\varepsilon}\right)$. ∎

Note. Under the condition of Lemma 3.4, by Lemma 3.1 we have $S\left(r, f^n P(f) f(z+c)\right) = S(r,f)$.

Lemma 3.5. [3] *Let $f(z)$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$N(r,0;f(z+c)) \leq N(r,0;f(z)) + S(r,f)$, $\quad N(r,\infty;f(z+c)) \leq N(r,\infty;f) + S(r,f)$,
$\bar{N}(r,0;f(z+c)) \leq \bar{N}(r,0;f(z)) + S(r,f)$, $\quad \bar{N}(r,\infty;f(z+c)) \leq \bar{N}(r,\infty;f) + S(r,f)$

Lemma 3.6. *Let $f(z)$ be a transcendental entire function of finite order σ, c be a fixed nonzero complex constant, $n(\geq 1), m^*(\geq 0)$ be two integers and let $a(z)(\not\equiv 0,\infty)$ be a small function of f. If $n > 1$, then $f^n P(f) f(z+c) - a(z)$ has infinitely many zeros.*

Proof. Let $\Phi = f^nP(f)f(z+c)$. Now in view of Lemma 3.5 and the second theorem for small functions (see [18]) we get

$$\begin{aligned} T(r,\Phi) &\leq \bar{N}(r,0;\Phi) + \bar{N}(r,\infty;\Phi) + \bar{N}(r,a(z);\Phi) + (\varepsilon + o(1))T(r,f) \\ &\leq \bar{N}\left(r,0;f^nP(f)\right) + \bar{N}(r,0;f(z+c)) + \bar{N}(r,a(z);\Phi) + (\varepsilon + o(1))T(r,f) \\ &\leq 2\bar{N}(r,0;f) + \bar{N}(r,0;P(f)) + \bar{N}(r,a(z);\Phi) + (\varepsilon + o(1))T(r,f) \\ &\leq (2+m^*)\,T(r,f) + \bar{N}(r,a(z);\Phi) + (\varepsilon + o(1))T(r,f), \end{aligned}$$

for all $\varepsilon > 0$.
From Lemmas 3.1 and 3.4 we get

$$(n+m^*+1)\,T(r,f) \leq (2+m^*)\,T(r,f) + \bar{N}(r,a(z);\Phi) + (\varepsilon + o(1))T(r,f).$$

Take $\varepsilon < 1$. Since $n > 1$ from above one can easily say that $\Phi - a(z)$ has infinitely many zeros. ∎

This completes the lemma.

Lemma 3.7. [9] *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following holds: (i) $T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r,f) + S(r,g)$, (ii) $fg \equiv 1$, (iii) $f \equiv g$.*

Lemma 3.8. *[Hadamard Factorization Theorem] Let $f(z)$ be an entire function of finite order ρ with zeros $a_1, a_2, \ldots$, each zeros is counted as often as its multiplicity. Then f can be expressed in the form*

$$f(z) = Q(z)e^{\alpha(z)},$$

where $\alpha(z)$ is a polynomial of degree not exceeding $[\rho]$ and $Q(z)$ is the canonical product formed with the zeros of f.

Lemma 3.9. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, $c \in \mathbb{C}\backslash\{0\}$ and $p(z)$ be a nonzero polynomial such that $\deg(p) \leq n-1$, where $n \in \mathbb{N}$. Let $P(\omega)$ be a nonzero polynomial defined as in Theorem 1.1 Suppose*

$$f^nP(f)f(z+c)g^nP(g)g(z+c) \equiv p^2.$$

Then $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i\omega^i \not\equiv 0$, for $i \in \{0,1,\ldots,m\}$. If $p(z) = b \in \mathbb{C}\backslash\{0\}$, then $f(z) = e^{\alpha(z)}, g = e^{\beta(z)}$, where $\alpha(z), \beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n+i+1)d} = b^2$.

Proof. Suppose

$$f^nP(f)f(z+c)g^nP(g)g(z+c) \equiv p^2. \tag{3.2}$$

We consider the following cases:

Case 1: Let $\deg(p(z)) = l(\geq 1)$.

From the assumption that f and g are two transcendental entire functions, we deduce by 3.2 that $N\left(r,0;f^nP(f)\right) = O(\log r)$ and $N\left(r,0;g^nP(g)\right) = O(\log r)$. First we suppose that $P(\omega)$ is not a nonzero monomial. For the sake of simplicity let $P(\omega) = \omega - a$ where $a \in \mathbb{C}\backslash\{0\}$. Clearly $\Theta(0;f)+\Theta(a;f) = 2$, which is impossible for an entire function. Thus $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i\omega^i \not\equiv 0$ for some $i \in \{0,1,\ldots,m\}$ and so 3.2 reduces to

$$a_i^2 f^{n+i}f(z+c)g^{n+i}g(z+c) \equiv p^2. \tag{3.3}$$

From 3.3 it follows that $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$. Now by Lemma 3.8 we obtain that $f = h_1e^{\alpha_1}$ and $f = h_2e^{\beta_1}$, where h_1, h_2 are two nonzero polynomials and α_1 and β_1 are two non-constant polynomials.

By virtue of the polynomial $p(z)$, from 3.3 we arrive at a contradiction.
Case 2: Let $p(z) = b \in \mathbb{C}\backslash\{0\}$.
Then from 3.2 we have

$$f^n P(f) f(z+c) g^n P(f) g(z+c) \equiv b^2. \tag{3.4}$$

Now from the assumption that f and g are two non-constant entire functions, we deduce by 3.4 that $f^n P(f) \neq 0$ and $g^n P(g) \neq 0$. By Picard's theorem, we claim that $P(\omega) = a_i \omega^i \not\equiv 0$ for $i \in \{0, 1, \ldots, m\}$, otherwise the Picard's exception values are atleast three, which is a contradiction. Then 3.4 reduces to

$$a_i^2 f^{n+i} f(z+c) g^{n+i} g(z+c) \equiv b^2. \tag{3.5}$$

Hence by Lemma 3.8 we obtain that

$$f = e^{\alpha}, \quad g = e^{\beta}, \tag{3.6}$$

where $\alpha(z), \beta(z)$ are two non-constant polynomials.
Now from 3.5 and 3.6 we obtain

$$(n+i)(\alpha(z) + \beta(z)) + \alpha(z+c) + \beta(z+c) \equiv d_1,$$

where $d_1 \in \mathbb{C}$, i.e.,

$$(n+i)\left(\alpha'(z) + \beta'(z)\right) + \alpha'(z+c) + \beta'(z+c) \equiv 0. \tag{3.7}$$

Let $\gamma(z) = \alpha'(z) + \beta'(z)$. Then from 3.7 we have

$$(n+i)\gamma(z) + \gamma(z+c) \equiv 0. \tag{3.8}$$

We assert that $\gamma(z) \equiv 0$. It not suppose $\gamma(z) \not\equiv 0$. Note that if $\gamma(z) \equiv d_2 \in \mathbb{C}$, from 3.8 we must have $d_2 = 0$. Suppose that $\deg(\gamma) \geq 1$. Let $\gamma(z) = \sum_{i=1}^{m} b_i z^i$, where $b_m \neq 0$. Therefore the co-efficient of z^m in $(n+i)\gamma(z) + \gamma(z+c)$ is $(n+1+i)b_m \neq 0$. Thus we arrive at a contradiction from 3.8. Hence $\gamma(z) \equiv 0$, i.e., $\alpha + \beta \equiv d \in \mathbb{C}$. Also from 3.5 we have $a_i^2 e^{(n+i+1)d} = b^2$.
This completes the proof. ∎

Lemma 3.10. *Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C}\backslash\{0\}$ and $p(z)$ be a nonzero polynomial such that $\deg(p) \leq n-1$, where $n \in \mathbb{N}$. Let $P(\omega)$ be defined as in Theorem 1.1 with at least two of $a_i, i = 0, 1, \cdots m$ are nonzero. Then*

$$f^n P(f) f(z+c) g^n P(g) g(z+c) \not\equiv p^2.$$

Proof. Proof of the lemma follows from Lemma 3.9 ∎

Lemma 3.11. [6] *Let f, g be two non-constant meromorphic functions such that they share (1,1) and $H \not\equiv 0$. Then*

$$N(r, 1; F| = 1) = N(r, 1; G| = 1) \leq N(r, H) + S(r, F) + S(r, G)$$

Lemma 3.12. *Let f be a transcendental meromorphic function of finite order and let $F = f^n P(f) f(z+c)$, where n is positive integer. Then*

$$(n-2)T(r, f) \leq T(r, F) + S(r, f)$$

Proof. from Lemmas 3.1 and 3.2 and first fundamental theorem, we obtain

$$\begin{aligned}(n+1)T(r,f) &\leq T(r,f^{n+1}) + S(r,f)\\ &\leq T(r,\frac{fF}{P(f)f(z+c)}) + S(r,f)\\ &\leq T(r,F) + T(r,\frac{f}{P(f)f(z+c)}) + S(r,f)\\ &\leq T(r,F) + m(r,\frac{P(f)f(z+c)}{f}) + N(r,\frac{P(f)f(z+c)}{f}) + S(r,f)\\ &\leq T(r,F) + 3T(r,f) + S(r,f)\\ (n-2)T(r,f) &\leq T(r,F) + S(r,f)\end{aligned}$$

∎

This completes the lemma.

4. Proof of Main Results

Proof. Let $F = f^n P(f) f(z+c)$ and $G = g^n P(g) g(z+c)$.
Case 1: Suppose $H \not\equiv 0$
Keeping in view of Lemma 3.1, we get by applying Second Fundamental theorem of Nevanlinna of F and G that

(4.1)
$$\begin{aligned}(n+m+1)[T(r,f)+T(r,g)] \leq\ & \overline{N}(r,0;F) + \overline{N}(r,1;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G)\\ &+ \overline{N}(r,1;G) + \overline{N}(r,\infty;G) - \overline{N}(r,0;F') - \overline{N}(r,0;G')\\ &+ S(r,f) + S(r,g).\end{aligned}$$

(i) $l \geq 2$ **and** $m = 0$**,** then using Lemmas 3.1, 3.3, 3.11 and 4.1 we obtain

$$\begin{aligned}(n+m+1)[T(r,f)+T(r,g)] \leq\ & N_2(r,0;F) + \overline{N}(r,\infty;F) + N_2(r,0;G)\\ &+ \overline{N}(r,\infty;G) - \overline{N}(r,0;F') - \overline{N}(r,0;G')\\ &+ S(r,f) + S(r,g),\end{aligned}$$

now by applying Lemma 3.7

$$\begin{aligned}(n+m+1)[T(r,f)+T(r,g)] \leq\ & N_2(r,0;F) + \overline{N}(r,\infty;F) + N_2(r,0;G) + \overline{N}(r,\infty;G)\\ &- \overline{N}_*(r,\infty;F,G) - (l-\frac{3}{2})\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\ (n+m+1)[T(r,f)+T(r,g)] \leq\ & (3+m+3+2m)T(r,f) + T(r,g) + S(r,f) + S(r,g)\end{aligned}$$

This implies that
$(n-2m-5)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g),$
which contradict to the fact that $n \geq 2m+6$.
(ii) $l \geq 2$ **and** $m = \infty$**,** then using Lemmas 3.1,3.3,3.11 and 4.1 we obtain

$$\begin{aligned}(n+m+1)[T(r,f)+T(r,g)] \leq\ & N_2(r,0;F) + \overline{N}(r,\infty;F) + N_2(r,0;G)\\ &+ \overline{N}(r,\infty;G) - \overline{N}(r,0;F') - \overline{N}(r,0;G')\\ &+ S(r,f) + S(r,g),\end{aligned}$$

now by applying Lemma 3.7

$$(n+m+1)[T(r,f)+T(r,g)] \leq N_2(r,0;F)+\overline{N}(r,\infty;F)+N_2(r,0;G)+\overline{N}(r,\infty;G) - \overline{N}_*(r,\infty;F,G)-(l-\frac{3}{2})\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)$$

$$(n+m+1)[T(r,f)+T(r,g)] \leq (3+m+2+m)T(r,f)+T(r,g)+S(r,f)+S(r,g).$$

This implies that
$(n-m-4)[T(r,f)+T(r,g)] \leq S(r,f)+S(r,g)$,
which contradict to the fact that $n \geq m+5$.
(iii) $l=1$ **and** $m=0$**,** then using Lemmas 3.1,3.3,3.11 and 4.1 we obtain

$$(n+m+1)[T(r,f)+T(r,g)] \leq N_2(r,0;F)+\overline{N}(r,\infty;F)+N_2(r,0;G) + \overline{N}(r,\infty;G)-\overline{N}(r,0;F')-\overline{N}(r,0;G') + S(r,f)+S(r,g),$$

now by applying Lemma 3.7

$$(n+m+1)[T(r,f)+T(r,g)] \leq N_2(r,0;F)+\overline{N}(r,\infty;F)+N_2(r,0;G)+\overline{N}(r,\infty;G) - \overline{N}_*(r,\infty;F,G)-(l-\frac{3}{2})\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)$$

$$(n+m+1)[T(r,f)+T(r,g)] \leq (6m+17)T(r,f)+T(r,g)+S(r,f)+S(r,g).$$

This implies that
$(n-5m-16)[T(r,f)+T(r,g)] \leq S(r,f)+S(r,g)$,
which contradict to the fact that $n \geq 5m+17$.
(iv) $l=0$ **and** $m=0$**,** then using Lemmas 3.1, 3.3, 3.11 and 4.1 we obtain

$$(n+m+1)[T(r,f)+T(r,g)] \leq N_2(r,0;F)+\overline{N}(r,\infty;F)+N_2(r,0;G) + \overline{N}(r,\infty;G)-\overline{N}(r,0;F')-\overline{N}(r,0;G') + S(r,f)+S(r,g),$$

now by applying Lemma 3.7

$$(n+m+1)[T(r,f)+T(r,g)] \leq N_2(r,0;F)+\overline{N}(r,\infty;F)+N_2(r,0;G)+\overline{N}(r,\infty;G) - \overline{N}_*(r,\infty;F,G)-(l-\frac{3}{2})\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)$$

$$(n+m+1)[T(r,f)+T(r,g)] \leq (8m+23)T(r,f)+T(r,g)+S(r,f)+S(r,g).$$

This implies that
$(n-7m-22)[T(r,f)+T(r,g)] \leq S(r,f)+S(r,g)$,
which contradict to the fact that $n \geq 7m+23$.
Case 2: Suppose $H \equiv 0$ by integration 3.1 we get

$$\frac{1}{F-1} = \frac{BG+A-B}{G-1}, \tag{4.2}$$

where A, B are constants and A$\neq$ 0 from 4.2 it is clear that F and G share $(1,\infty)$. We now consider following cases.
(i) Let $B \neq 0$ **and** $A \neq B$. If $B=-1$ then from 4.2 we have,

$$F = \frac{-A}{G-A-1}$$

Therefore

$$\overline{N}(r, A+1; G) = N(r, 0; p) = S(r, g),$$

by using Lemma 3.12 and Nevanlinna second fundamental theorem we get

$$\begin{aligned}(n-2)T(r,g) &\leq T(r, g^n P(g)g(z+c)) + S(r,g) \\ &\leq T(r,G) + S(r,g) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,A+1;G) + S(r,g) \\ &\leq \overline{N}(r,\infty; g^n P(g)g(z+c)) + \overline{N}(r,0; g^n P(g)g(z+c)) + S(r,g) + S(r,g) \\ &\leq (5+2m)T(r,g) + S(r,g),\end{aligned}$$

which is a contradiction since $n \geq 7 + 2m.$
If $B \neq -1$ from 4.2

$$F - (1 + \frac{1}{B}) = \frac{-A}{B^2(G + \frac{A-B}{B})},$$

so,

$$\overline{N}(r, \frac{B-A}{B}; G) = S(r, g),$$

by Lemma 3.12 and the same argument as used in the case when $B = -1$, we can get a contradiction.
(ii) Let $B \neq 0$ **and** $A = B$. If $B = -1$ then from 4.2 we have,

$$F(z)G(z) \equiv 1,$$
$$i.e, f^n P(f) f(z+c) g^n P(g) g(z+c) \equiv p^2(z),$$

where $f^n P(f) f(z+c) - p$ and $g^n P(g) g(z+c) - p$ share 0 CM.
If $B \neq -1$ from 4.2

$$\frac{1}{F} = \frac{BG}{(1+B)(G-1)},$$

so,

$$\overline{N}(r, \frac{B-A}{B}; G) = S(r, g)$$

therefore

$$\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F) = S(r, g),$$

so Lemma 3.12 and second fundamental theorem we get

$$\begin{aligned}(n-2)T(r,g) &\leq T(r, g^n P(g)g(z+c)) + S(r,g) \\ &\leq T(r,G) + S(r,g) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{1}{1+B};G) + S(r,g) \\ &\leq \overline{N}(r,\infty; g^n P(g)g(z+c)) + \overline{N}(r,0; g^n P(g)g(z+c)) \\ &+ \overline{N}(r,0; f^n P(f)f(z+c)) + S(r,g) + S(r,g) \\ &\leq (7+3m)T(r,f) + T(r,g) + S(r,g) + S(r,f),\end{aligned}$$

which is a contradiction since $n \geq 9 + 3m.$
(iii) If $B = 0$ from 4.2

$$F = \frac{G-1}{A} + 1. \tag{4.3}$$

If $A \neq 1$ then from 4.3 we obtain

$$\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in case 2.
therefore $A = 1$ and from 4.3 we obtain

$$F(z) \equiv G(z),$$

i.e,

$$f^n P(f) f(z+c) \equiv g^n P(g) g(z+c). \tag{4.4}$$

Let $h = \frac{f}{g}$ and then substituting $f = gh$ in 4.4

$$g^n h^n P(f) f(z+c) = g^n P(g) g(z+c) h^{n+1} = \frac{f P(g) g(z+c)}{g P(f) f(z+c)},$$

If h is not a constant, then we have

$$\begin{aligned}
(n+1)T(r,h) &\leq T(r, \frac{f}{P(f)f(z+c)}) + T(r, \frac{P(g)g(z+c)}{g}) + S(r,f) + S(r,g) \\
&\leq T(r, \frac{P(f)f(z+c)}{g}) + T(r, \frac{P(g)g(z+c)}{g}) + S(r,f) + S(r,g) \\
&\leq N(r, \frac{P(f)f(z+c)}{g}) + N(r, \frac{P(g)g(z+c)}{g}) + S(r,f) + S(r,g) \\
&\leq (7+3m)[T(r,f) + T(r,g)] + S(r,f) + S(r,g).
\end{aligned}$$

Combining above inequality with

$$T(r,h) = T(r, \frac{f}{g}) = T(r,f) + T(r,g) + S(r,f) + S(r,g).$$

We obtain

$$(n - 6 - 3m)[T(r,f) + T(r,g)] \leq S(r,f) + S(r,g),$$

$(n-6-3m)[T(r,f)+T(r,g)]$ which is impossible.
therefore h is a constant, then substitute $f = gh$ in 4.4 we have $h^{(n+1)} \equiv 1$.
therefore $f = tg$ where t is a constant,$t^{(n+1)} \equiv 1$.

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