Aust. J. Math. Anal. Appl.
Vol. **19** (2022), No. 2, Art. 11, 9 pp.
AJMAA

ADDITIVE MAPPINGS ON SEMIPRIME RINGS FUNCTIONING AS CENTRALIZERS

ABU ZAID ANSARI AND FAIZA SHUJAT

Received 9 March, 2022; accepted 19 September, 2022; published 21 October, 2022.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, ISLAMIC UNIVERSITY OF MADINAH, MADINAH K.S.A.
ansari.abuzaid@gmail.com, ansari.abuzaid@iu.edu.sa

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, TAIBAH UNIVERSITY, MADINAH, K.S.A.
faiza.shujat@gmail.com, fullahkhan@taibahu.edu.sa

ABSTRACT. The objective of this research is to prove that an additive mapping $\mathcal{T} : R \to R$ is a centralizer on R if it satisfies any one of the following identities:
(i) $3\mathcal{T}(x^{3n}) = \mathcal{T}(x^n)x^{2n} + x^n\mathcal{T}(x^n)x^n + x^{2n}\mathcal{T}(x^n)$
(ii) $2\mathcal{T}(x^{2n}) = \mathcal{T}(x^n)x^n + x^n\mathcal{T}(x^n)$
(ii) $\mathcal{T}(x^{3n}) = x^n\mathcal{T}(x^n)x^n$
for all $x \in R$, where $n \geq 1$ is a fixed integer and R is any suitably torsion free semiprime ring. Some results on involution " $*$ " are also presented as consequences of the main theorems. In addition, we will take criticism in account with examples.

Key words and phrases: Semiprime ring; Left (right) centralizer, Involution.

2010 *Mathematics Subject Classification.* Primary 16N60, 16B99. Secondary 16W25, 16R50.

ISSN (electronic): 1449-5910

The authors of the paper extend their sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah, KSA for the support provided to the Post-Publishing Program 1.

1. INTRODUCTION

Our present interpretation is encouraged by the research work of Helgosen [6], who intimated the concept of centralizers (multipliers) on Banach algebras. A potential conception on centralizers of commutative Banach algebra posed by J. K. Wang [18]. Further B. E. Johnson [8] explore previous such ideas on centralizers for topological algebras and continuity of centralizers on Banach algebra. For exhaustive knowledge of related matter, one refer to [9, 10] and references therein. Husain [7] has also investigated centralizers on topological algebras with particular reference to complete metrizable locally convex algebras and topological algebras with orthogonal bases. Later, in [11, 13] authors have studied centralizers and double centralizers on certain topological algebras. Centralizers have also appeared in a variety, among which we mention representation theory of Banach algebras, the study of Banach modules, Hopf algebras (see [4, 5]), the theory of singular integrals, interpolation theory, stochastic processes, the theory of semigroups of operators, partial differential equations and the study of approximation problems (see Larsen [12] for more details).

First of all we need to recollect some basic notions that is useful for our concept. Throughout R will represent an associative ring with identity. A ring R is said to be n-torsion free, where $n > 1$ is an integer, if $nx = 0$ implies $x = 0$ for all $x \in R$. A ring R is called prime if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$, and is known as semiprime if $aRa = \{0\}$ implies $a = 0$. Johnson [8] inaugrated the concept of centralizers in rings as follows: an additive mapping $\mathcal{T} : R \to R$ is called a left (right) centralizer if $\mathcal{T}(xy) = \mathcal{T}(x)y$ $\big(\mathcal{T}(xy) = x\mathcal{T}(y)\big)$ holds for all pairs $x, y \in R$ and is called a Jordan left (Jordan right) centralizer if $\mathcal{T}(x^2) = \mathcal{T}(x)x$ $\big(\mathcal{T}(x^2) = x\mathcal{T}(x)\big)$ holds for all $x \in R$. The concept of centralizers is also known as multipliers (see [19]). We call such map $\mathcal{T}$ a centralizer, in case $\mathcal{T}$ is both left as well as right centralizer. Following Zalar [20], if $k \in R$, then $l_k(x) = kx$ is a left centralizer and $r_k(x) = xk$ is a right centralizer for some fixed element $k \in R$. A map $\mathcal{T} : R \to R$ on a ring R having identity element is a left (right) centralizer if and only if its of the form $l_k(x)$ $\big(r_k(x)\big)$.

A remarkable contribution to the study of centralizers and their properties on prime and semiprime ring has been done like [1, 2, 9, 10, 15, 19]. Molnar [14] in 1995 proved that: if R is a 2-torsion free prime ring and $\mathcal{T} : R \to R$ is an additive map such that $\mathcal{T}(xyx) = \mathcal{T}(x)yx$ for all $x, y \in R$, then $\mathcal{T}$ is a left (right) centralizer. Let us consider a map $\mathcal{T} : R \to R$ of an arbitrary ring is a centralizer if it satisfies the relation $\mathcal{T}(xyx) = x\mathcal{T}(y)x$, for all $x, y \in R$. We can think that what will happen, if we draw our attention for converse situation of the last relation. By considering the converse situation Vukman [16] came across an affirmative answer for semiprime rings. More precisely, author in [16] established that: let R be a 2-torsion semiprime ring and let $\mathcal{T} : R \to R$ be an additive mapping. Suppose that $\mathcal{T}(xyx) = x\mathcal{T}(y)x$, holds for all pairs $x, y \in R$. In this case $\mathcal{T}$ is a centralizer.

Several generalization of previous result has obtained by number of mathematician. An implicit idea found through Theorem 2.3.2 in [3], if R is semiprime ring with extended centroid $\mathcal{C}$ and $\mathcal{T} : R \to R$ is a left and right centralizer, then there exists an element $\lambda \in \mathcal{C}$ such that $\mathcal{T}(x) = \lambda x$ for all $x \in R$. Later, Vukman and Kosi-Ulbl [17] established a result by taking an algebraic equation $3\mathcal{T}(xyx) = \mathcal{T}(x)yx + x\mathcal{T}(y)x + xy\mathcal{T}(x)$. In fact they proved that an additive mapping $\mathcal{T} : R \to R$ is of the form $\mathcal{T}(x) = \lambda x$ if it satisfies the algebraic equation $3\mathcal{T}(xyx) = \mathcal{T}(x)yx + x\mathcal{T}(y)x + xy\mathcal{T}(x)$ for all $x, y \in R$ and $\lambda \in \mathcal{C}$, where R is 2-torsion free semiprime ring with extended centroid $\mathcal{C}$. Inspired by the above literature review, we present our ideas to generalize the notion of centralizer in virtue of some identities mentioned in abstract.

Likewise we also examine such identities in the setting of ring with involution. To develop the proof of our main theorems, we require the following lemma:

Lemma 1.1 ([15, Theorem 1]). *Let R be a 2 torsion free semiprime ring and $\mathcal{T} : R \to R$ be an additive mapping satisfying the condition $2\mathcal{T}(x^2) = \mathcal{T}(x)x + x\mathcal{T}(x)$ for all $x \in R$, then $\mathcal{T}$ is a left and right centralizer on R.*

2. RESULTS ON SEMIPRIME RINGS

We begin our investigation with the following problem:

Theorem 2.1. *Let $n \geq 1$ be a fixed integer and R be any k-torsion free semiprime ring. If $\mathcal{T} : R \to R$ is an additive mapping which satisfies*

$$3\mathcal{T}(x^{3n}) = \mathcal{T}(x^n)x^{2n} + x^n\mathcal{T}(x^n)x^n + x^{2n}\mathcal{T}(x^n) \text{ for all } x \in R, \tag{2.1}$$

then $\mathcal{T}$ is a left and right centralizer on R, where $k \in \{2, 3, n, (3n-1)!\}$.

Proof. We begin with the condition (2.1) and replace x by $x + \mathrm{q}y$ to get

$$3\mathcal{T}(x^{3n}+\binom{3n}{1}(x^{3n-1})\mathrm{q}y+\binom{3n}{2}x^{3n-2}\mathrm{q}^2y^2+...+\mathrm{q}^{3n}y^{3n}) = \mathcal{T}(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+ ...+\mathrm{q}^ny^n)\cdot(x^{2n}+\binom{2n}{1}x^{2n-1}\mathrm{q}y+\binom{2n}{2}x^{2n-2}\mathrm{q}^2y^2+...+\mathrm{q}^{2n}y^{2n})+(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+ ...+\mathrm{q}^ny^n)\cdot\mathcal{T}(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)\cdot(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+ \mathrm{q}^ny^n)+(x^{2n}+\binom{2n}{1}x^{2n-1}\mathrm{q}y+\binom{2n}{2}x^{2n-2}\mathrm{q}^2y^2+...+\mathrm{q}^{2n}y^{2n})\cdot\mathcal{T}(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+ ... + \mathrm{q}^ny^n)$$

Rewrite the above expression by using (2.1) as

$$q\mathcal{P}_1(x, y) + q^2\mathcal{P}_2(x, y) + ... + q^{3n-1}\mathcal{P}_{3n-1}(x, y) = 0,$$

where $\mathcal{P}_i(x, y)$ stand for the coefficients of q^i's for all $i = 1, 2, ..., 3n-1$. If we replace q by $1, 2, ..., 3n-1$, then we find a system of $(3n-1)$ homogeneous equations. It gives us a Vander Monde matrix

$$\begin{bmatrix} 1 & 1 & ... & 1 \\ 2 & 2^2 & ... & 2^{3n-1} \\ ... & & & \\ ... & & & \\ 3n-1 & (3n-1)^2 & ... & (3n-1)^{3n-1} \end{bmatrix}.$$

Which yields that $\mathcal{P}_i(x, y) = 0$ for all $x, y \in R$ and for $i = 1, 2, .., 3n-1$. In particular, We have $\mathcal{P}_1(x, y) = 3\binom{3n}{1}\mathcal{T}(x^{3n-1}y) - \binom{2n}{1}\mathcal{T}(x^n)x^{2n-1}y + \binom{n}{1}\mathcal{T}(x^{n-1}y)x^{2n} - \binom{n}{1}\mathcal{T}(x^n)x^{n-1}y - \binom{n}{1}x^n\mathcal{T}(x^{n-1}y)x^n - \binom{n}{1}x^{2n}\mathcal{T}(x^{n-1}y) - \binom{2n}{1}x^{2n-1}y\mathcal{T}(x^n) = 0$ for all $x, y \in R$. If we put e in place of x in above expression, then we find $9n\mathcal{T}(y) = 2n\mathcal{T}(e)y + n\mathcal{T}(y) + n\mathcal{T}(e)y + n\mathcal{T}(y) + ny\mathcal{T}(e) + n\mathcal{T}(y) + 2ny\mathcal{T}(e)$, for all $y \in R$. This implies that $6n\mathcal{T}(y) = 3n\mathcal{T}(e)y + 3ny\mathcal{T}(e)$, for all $y \in R$. Making use of torsion restrictions on R to obtain

$$2\mathcal{T}(y) = \mathcal{T}(e)y + y\mathcal{T}(e), \text{ for all } y \in R. \tag{2.2}$$

Let us consider the term that $\mathcal{P}_2(x, y) = 0$. This implies that

$$\begin{aligned} 3\binom{3n}{2}\mathcal{T}(x^{3n-2}y^2) \;=\; & \binom{2n}{2}\mathcal{T}(x^n)x^{2n-2}y^2+\binom{n}{1}\binom{2n}{1}\mathcal{T}(x^{n-1}y)x^{2n-1}y \\ & +\binom{n}{2}\mathcal{T}(x^{n-2}y^2)x^{2n}+\binom{n}{2}x^n\mathcal{T}(x^n)x^{n-2}y^2 \\ & +\binom{n}{1}\binom{n}{1}x^{n-1}y\mathcal{T}(x^n)x^{n-1}y+\binom{n}{1}\binom{n}{1}x^n\mathcal{T}(x^{n-1}y)x^{n-1}y \\ & +\binom{n}{2}x^n\mathcal{T}(x^{n-2}y^2)x^n+\binom{n}{1}\binom{n}{1}x^{n-1}y\mathcal{T}(x^{n-1}y)x^n \\ & +\binom{n}{2}x^{n-2}y^2\mathcal{T}(x^n)x^n+\binom{n}{2}x^{2n}\mathcal{T}(x^{n-2}y^2) \\ & +\binom{2n}{1}\binom{n}{1}x^{2n-1}y\mathcal{T}(x^{n-1}y)+\binom{2n}{2}x^{2n-2}y^2\mathcal{T}(x^n). \end{aligned}$$

Reword the above expression by putting e in place of x, we have

$$\begin{aligned} 3\frac{3n(3n-1)}{2}\mathcal{T}(y^2) \;=\; & \frac{2n(2n-1)}{2}\mathcal{T}(e)y^2+2n^2\mathcal{T}(y)y+\frac{n(n-1)}{2}\mathcal{T}(y^2) \\ & +\frac{n(n-1)}{2}\mathcal{T}(e)y^2+n^2\mathcal{T}(y)y+\frac{n(n-1)}{2}\mathcal{T}(y^2) \\ & +n^2y\mathcal{T}(e)y+n^2y\mathcal{T}(y)+\frac{n(n-1)}{2}y^2\mathcal{T}(e) \\ & +\frac{n(n-1)}{2}\mathcal{T}(y^2)+2n^2y\mathcal{T}(y)+\frac{2n(2n-1)}{2}y^2\mathcal{T}(e) \end{aligned}$$

Simplify the above expression to find

$$\begin{aligned} 9n(3n-1)\mathcal{T}(y^2) \;=\; & 2n(2n-1)\mathcal{T}(e)y^2+4n^2\mathcal{T}(y)y+n(n-1)\mathcal{T}(y^2) \\ & +n(n-1)\mathcal{T}(e)y^2+2n^2\mathcal{T}(y)y+n(n-1)\mathcal{T}(y^2) \\ & +2n^2y\mathcal{T}(e)y+2n^2y\mathcal{T}(y)+n(n-1)y^2\mathcal{T}(e) \\ & +n(n-1)\mathcal{T}(y^2)+4n^2y\mathcal{T}(y)+2n(2n-1)y^2\mathcal{T}(e) \end{aligned}$$

Collect the like terms in above expression and make them more comprehensible as $(27n^2-9n-3n^2+3n)\mathcal{T}(y^2)=(4n^2-2n+n^2-n)\mathcal{T}(e)y^2+6n^2\mathcal{T}(y)y+6n^2y\mathcal{T}(y)+(4n^2-2n+n^2-n)y^2\mathcal{T}(e)$, for all $y\in R$. This indicate that $(24n^2-6n)\mathcal{T}(y^2)=(5n^2-3n)2\mathcal{T}(y^2)+6n^2(\mathcal{T}(y)y+y\mathcal{T}(y))+2n^2y\mathcal{T}(e)y$, for all $y\in R$. Again comparing the like terms both side, we come up with $14n^2\mathcal{T}(y^2)=6n^2(\mathcal{T}(y)y+y\mathcal{T}(y))+2n^2y\mathcal{T}(e)y$, for all $y\in R$. Hence, we get by applying the torsion freeness of R

(2.3) $$14\mathcal{T}(y^2)=6(\mathcal{T}(y)y+y\mathcal{T}(y))+2y\mathcal{T}(e)y, \text{ for all } y\in R.$$

Multiplying from right side by x to (2.2), we obtain $2\mathcal{T}(x)x=\mathcal{T}(e)x^2+x\mathcal{T}(e)x$ for all $x\in R$. Multiply from left side by x to (2.2) to get $2x\mathcal{T}(x)=x\mathcal{T}(e)x+x^2\mathcal{T}(e)$ for all $x\in R$. Adding these equations, we find $2(\mathcal{T}(x)x+x\mathcal{T}(x))=2x\mathcal{T}(e)x+2\mathcal{T}(x^2)$, which implies that $2x\mathcal{T}(e)x=2(\mathcal{T}(x)x+x\mathcal{T}(x))-2\mathcal{T}(x^2)$ for all $x\in R$. Using this equation in (2.3), we have $14\mathcal{T}(x^2)=6(\mathcal{T}(x)x+x\mathcal{T}(x))+2(\mathcal{T}(x)x+x\mathcal{T}(x))-2\mathcal{T}(x^2)$ for all $x\in R$. Using Torsion restrictions on R, we get $2\mathcal{T}(x^2)=\mathcal{T}(x)x+x\mathcal{T}(x)$ for all $x\in R$. Using Lemma 1.1, $\mathcal{T}$ is a centralizer on R.

Theorem 2.2. *Let $n\geq 1$ be a fixed integer and R be any k-torsion free semiprime ring. If $\mathcal{T}:R\to R$ is an additive mapping which satisfies*

(2.4) $$2\mathcal{T}(x^{2n})=\mathcal{T}(x^n)x^n+x^n\mathcal{T}(x^n) \text{ for all } x\in R,$$

then $\mathcal{T}$ is a left and right centralizer on R, where $k\in\{2,n,(2n-1)!\}$.

Proof. We proceed with (2.4) and replace x by $x+\mathrm{q}y$ to get

$2\mathcal{T}(x^{2n}+\binom{2n}{1}(x^{2n-1})\mathrm{q}y+\binom{2n}{2}x^{2n-2}\mathrm{q}^2y^2+...+\mathrm{q}^{2n}y^{2n})=\mathcal{T}(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)\cdot(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)+(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)\cdot\mathcal{T}(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)$ for all $x,y\in R$.

Rewrite the above expression by using (2.4) as

$$qf_1(x,y)+q^2f_2(x,y)+...+q^{2n-1}f_{2n-1}(x,y)=0,$$

where $f_i(x,y)$ stand for the coefficients of q^i's for all $i=1,2,...,2n-1$. If we replace q by $1,2,...,2n-1$, then we find a system of $(2n-1)$ homogeneous equations. It gives us a Vander Monde matrix

$$\begin{bmatrix} 1 & 1 & ... & 1 \\ 2 & 2^2 & ... & 2^{2n-1} \\ ... & & & \\ ... & & & \\ 2n-1 & (2n-1)^2 & ... & (2n-1)^{2n-1} \end{bmatrix}.$$

Which yields that $f_i(x,y)=0$ for all $x,y\in R$ and for $i=1,2,..,2n-1$. In particular, We have

$$\begin{aligned} f_1(x,y) &= 2\binom{2n}{1}\mathcal{T}(x^{2n-1}\mathrm{q}y)-\binom{n}{1}\mathcal{T}(x^n)x^{n-1}y-\binom{n}{1}\mathcal{T}(x^{n-1})yx^n \\ &\quad -\binom{n}{1}x^n\mathcal{T}(x^{n-1}y)-\binom{n}{1}x^{n-1}y\mathcal{T}(x^n)=0 \end{aligned} \tag{2.5}$$

Reinstate the above equation by putting e in place of x to have $2.2n\mathcal{T}(y)=n\mathcal{T}(e)y+n\mathcal{T}(y)+n\mathcal{T}(y)+ny\mathcal{T}(e)$. On simplifying the last relation we can obtain $2n\mathcal{T}(y)=n\mathcal{T}(e)y+ny\mathcal{T}(e)$ for all $y\in R$. A torsion restriction given in the hypothesis enable us to write

$$2\mathcal{T}(y)=\mathcal{T}(e)y+y\mathcal{T}(e),\quad \text{for all } y\in R. \tag{2.6}$$

Now consider the following

$$\begin{aligned} f_2(x,y) &= 2\binom{2n}{2}\mathcal{T}(x^{2n-2}y^2)-\binom{n}{2}\mathcal{T}(x^n)x^{n-2}y^2-\binom{n}{1}\binom{n}{1}\mathcal{T}(x^{n-1}y)x^{n-1}y \\ &\quad -\binom{n}{2}\mathcal{T}(x^{n-2}y^2)x^n-\binom{n}{2}x^n\mathcal{T}(x^{n-2}y^2)-\binom{n}{1}\binom{n}{1}x^{n-1}y\mathcal{T}(x^{n-1}y) \\ &\quad -\binom{n}{2}x^{n-2}y^2\mathcal{T}(x^n)=0 \end{aligned}$$

Substitute e for x in above expression to obtain

$$\begin{aligned} 2.\tfrac{2n(2n-1)}{2}\mathcal{T}(y^2) &= \tfrac{n(n-1)}{2}\mathcal{T}(e)y^2+n^2\mathcal{T}(y)y+\tfrac{n(n-1)}{2}\mathcal{T}(y^2) \\ &\quad +\tfrac{n(n-1)}{2}\mathcal{T}(y^2)+n^2y\mathcal{T}(y)+\tfrac{n(n-1)}{2}y^2\mathcal{T}(e) \end{aligned}$$

A simple manipulation yields that $(4n^2-2n-n^2+n-n^2+n)\mathcal{T}(y^2)=n^2(\mathcal{T}(y)y+y\mathcal{T}(y))$. This implies that $2n^2\mathcal{T}(y^2)=n^2(\mathcal{T}(y)y+y\mathcal{T}(y))$. Making use of n^2-torsion freeness of R, we have $2\mathcal{T}(y^2)=\mathcal{T}(y)y+y\mathcal{T}(y)$ for all $y\in R$. Hence $\mathcal{T}$ is carry oneself like centralizer, as desired.

Theorem 2.3. *Let $n\geq 1$ be a fixed integer and R be any k-torsion free semiprime ring. If $\mathcal{T}:R\to R$ is an additive mapping which satisfies*

$$\mathcal{T}(x^{3n})=x^n\mathcal{T}(x^n)x^n \text{ for all } x\in R, \tag{2.7}$$

then $\mathcal{T}$ is a left and right centralizer on R, where $k\in\{2,3,n,(3n-1)!\}$.

Proof. Replacing x by $x+\mathrm{q}y$ in (2.7), we get

$\mathcal{T}(x^{3n}+\binom{3n}{1}(x^{3n-1})\mathrm{q}y+\binom{3n}{2}x^{3n-2}\mathrm{q}^2y^2+...+\mathrm{q}^{3n}y^{3n})=(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)\cdot\mathcal{T}(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)\cdot(x^n+\binom{n}{1}x^{n-1}\mathrm{q}y+\binom{n}{2}x^{n-2}\mathrm{q}^2y^2+...+\mathrm{q}^ny^n)$

Rewrite the above expression by using (2.1) as

$$q\mathcal{R}_1(x,y)+q^2\mathcal{R}_2(x,y)+...+q^{3n-1}\mathcal{R}_{3n-1}(x,y)=0,$$

where $\mathcal{R}_i(x,y)$ stand for the coefficients of q^i's for all $i=1,2,...,3n-1$. If we replace q by $1,2,...,3n-1$, then we find a system of $(3n-1)$ homogeneous equations. It gives us a Vander Monde matrix

$$\begin{bmatrix} 1 & 1 & ... & 1 \\ 2 & 2^2 & ... & 2^{3n-1} \\ ... & & & \\ ... & & & \\ 3n-1 & (3n-1)^2 & ... & (3n-1)^{3n-1} \end{bmatrix}.$$

Which yields that $\mathcal{R}_i(x,y)=0$ for all $x,y\in R$ and for $i=1,2,..,3n-1$. In particular, We have $\mathcal{R}_1(x,y)=\binom{3n}{1}\mathcal{T}(x^{3n-1}y)-\binom{n}{1}x^n\mathcal{T}(x^{n-1}y)x^n-\binom{n}{1}x^n\mathcal{T}(x^n)x^{n-1}y-\binom{n}{1}x^{n-1}y\mathcal{T}(x^n)x^n=0$ for all $x,y\in R$. Replacing x by e and making use of $2n$-torsion restriction to get

$$2\mathcal{T}(y)=\mathcal{T}(e)y+y\mathcal{T}(e),\ \text{for all } y\in R. \tag{2.8}$$

Let us consider the term that $\mathcal{P}_2(x,y)=0$. This implies that

$$\begin{aligned} 3\binom{3n}{2}\mathcal{T}(x^{3n-2}y^2) &= \binom{n}{2}x^n\mathcal{T}(x^n)x^{n-2}y^2+\binom{n}{1}\binom{n}{1}x^{n-1}y\mathcal{T}(x^n)x^{n-1}y \\ &+\binom{n}{1}\binom{n}{1}x^n\mathcal{T}(x^{n-1}y)x^{n-1}y+\binom{n}{2}x^n\mathcal{T}(x^{n-2}y^2)x^n \\ &+\binom{n}{1}\binom{n}{1}x^{n-1}y\mathcal{T}(x^{n-1}y)x^n+\binom{n}{2}x^{n-2}y^2\mathcal{T}(x^n)x^n. \end{aligned}$$

Reword the above expression by putting e in place of x, we have

$$\begin{aligned} 3\frac{3n(3n-1)}{2}\mathcal{T}(y^2) &= \binom{n}{2}\mathcal{T}(e)y^2+\binom{n}{1}\binom{n}{1}y\mathcal{T}(e)y \\ &+\binom{n}{1}\binom{n}{1}\mathcal{T}(y)y+\binom{n}{2}\mathcal{T}(y^2)x^n \\ &+\binom{n}{1}\binom{n}{1}y\mathcal{T}(y)+\binom{n}{2}y^2\mathcal{T}(e)x^n. \end{aligned}$$

Simplify the above expression using the same steps as we did in last theorems to find $3n^2\mathcal{T}(y^2)=n^2(\mathcal{T}(y)y+y\mathcal{T}(y))+n^2y\mathcal{T}(e)y$, for all $y\in R$. Hence, we get by applying the torsion freeness of R

$$3\mathcal{T}(y^2)=(\mathcal{T}(y)y+y\mathcal{T}(y))+y\mathcal{T}(e)y,\ \text{for all } y\in R. \tag{2.9}$$

Multiplying from right side by x to (2.8), we obtain $2\mathcal{T}(x)x=\mathcal{T}(e)x^2+x\mathcal{T}(e)x$ for all $x\in R$. Multiply from left side by x to (2.8) to get $2x\mathcal{T}(x)=x\mathcal{T}(e)x+x^2\mathcal{T}(e)$ for all $x\in R$. Adding these equations, we find $2(\mathcal{T}(x)x+x\mathcal{T}(x))=2x\mathcal{T}(e)x+2\mathcal{T}(x^2)$, which implies that $x\mathcal{T}(e)x=\mathcal{T}(x)x+x\mathcal{T}(x)-\mathcal{T}(x^2)$ for all $x\in R$. Using this equation in (2.9), we have $4\mathcal{T}(x^2)=2(\mathcal{T}(x)x+x\mathcal{T}(x))$ for all $x\in R$. Use torsion restrictions on R to get $2\mathcal{T}(x^2)=\mathcal{T}(x)x+x\mathcal{T}(x)$ for all $x\in R$. Using Lemma 1.1, $\mathcal{T}$ is a centralizer on R.

The following example is in favour of our theorems:

Example 2.1. *Let* $R=\left\{\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} \mid a,b,c\in 2\mathbb{Z}_8\right\}$ *is a ring under matrix addition and matrix multiplication, where* $\mathbb{Z}_8$ *denotes the ring of integers addition and multiplication modulo 8. Define a mapping* $\mathcal{T}:R\to R$ *by* $\mathcal{T}\left(\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}\right)=\begin{pmatrix} 0 & b \\ 0 & 0\end{pmatrix}$ *for all* $a,b\in 2\mathbb{Z}_8$*. It is clear that* R *is neither a semiprime ring nor a 2-torsion free and* $\mathcal{T}$ *satisfy the identity of Theorem 2.1, Theorem 2.2 and Theorem 2.3(*$n>1$*) but* $\mathcal{T}$ *is not a centralizer on* R*, hence semiprimeness hypothesis is crucial for the above theorems.*

3. RESULTS ON SEMIPRIME $*$-RINGS

Next, an additive mapping $* : R \to R$ is called involution if it satisfies $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a $*$-ring or ring with involution. An additive mapping $\mathcal{T} : R \to R$ is called a left (resp. right)$*$-centralizer if $\mathcal{T}(xy) = \mathcal{T}(x)y^*$ (resp. $\mathcal{T}(xy) = x^*\mathcal{T}(y)$) holds for all $x, y \in R$ and $\mathcal{T}$ is called a left (resp. right) Jordan $*$-centralizer if for all $x \in R$, $\mathcal{T}(x^2) = \mathcal{T}(x)x^*$ (resp. $\mathcal{T}(x^2) = x^*\mathcal{T}(x)$). If $\mathcal{T}$ is both left and right Jordan $*$-centralizer of R then it is called Jordan $*$-centralizer of R. If $\mathcal{T} : R \to R$ is both left and right Jordan $*$-centralizers, then obviously $\mathcal{T}$ satisfies $2\mathcal{T}(x^{2n}) = \mathcal{T}(x^n)(x^*)^n + (x^*)^n\mathcal{T}(x^n)$, $3\mathcal{T}(x^{3n}) = \mathcal{T}(x^n)(x^*)^{2n} + (x^*)^n\mathcal{T}(x^n)(x^*)^n + (x^*)^{2n}\mathcal{T}(x^n)$ and $\mathcal{T}(x^{3n}) = (x^*)^n\mathcal{T}(x^n)(x^*)^n$ for all $x \in R$ but the converse is not true in general. The present paper deals with the study of this problem. In fact, it is shown that an additive mappings $\mathcal{T}$ on a suitable torsion free restricted semiprime ring R satisfying $3\mathcal{T}(x^{3n}) = \mathcal{T}(x^n)(x^*)^{2n} + (x^*)^n\mathcal{T}(x^n)(x^*)^n + (x^*)^{2n}\mathcal{T}(x^n)$, $2\mathcal{T}(x^{2n}) = \mathcal{T}(x^n)(x^*)^n + (x^*)^n\mathcal{T}(x^n)$ and $\mathcal{T}(x^{3n}) = (x^*)^n\mathcal{T}(x^n)(x^*)^n$ for all $x \in R$, is a $*$-centralizer of R. To prove our main results, we need the following lemma.

Lemma 3.1 ([2, Corollary 2.1])**.** *Let R be a 2 torsion free semiprime ring with involution $*$ and $\mathcal{T} : R \to R$ be an additive mapping satisfying the condition $2\mathcal{T}(x^2) = \mathcal{T}(x)x^* + x^*\mathcal{T}(x)$ for all $x \in R$, then $\mathcal{T}$ is a $*$-centralizer on R.*

Next, start main result of this part.

Theorem 3.2. *Let $n \geq 1$ be a fixed integer and R be any k-torsion free semiprime ring with involution $*$. If $\mathcal{T} : R \to R$ is an additive mapping which satisfies any one of the following identities:*

(i) $3\mathcal{T}(x^{3n}) = \mathcal{T}(x^n)(x^*)^{2n} + (x^*)^n\mathcal{T}(x^n)(x^*)^n + (x^*)^{2n}\mathcal{T}(x^n)$
(ii) $2\mathcal{T}(x^{2n}) = \mathcal{T}(x^n)(x^*)^n + (x^*)^n\mathcal{T}(x^n)$
(iii) $\mathcal{T}(x^{3n}) = (x^*)^n\mathcal{T}(x^n)(x^*)^n$ *for all* $x \in R$.

Then, $\mathcal{T}$ is a $*$-centralizer on R, where $k \in \{2, 3, n, (2n-1)!, (3n-1)!\}$.

Proof. Define a mapping $S : R \to R$ such that $S(x) = \mathcal{T}(x^*)$ for all $x \in R$. It is clear that S is an additive mapping on R. Now, consider

$$\begin{aligned} 3S(x^{3n}) &= 3\mathcal{T}((x^{3n})^*) \\ &= \mathcal{T}(x^*)(x)^{2n} + (x)^n\mathcal{T}(x^*)(x)^n + (x)^{2n}\mathcal{T}(x^*) \\ &= S(x^n)x^{2n} + x^nS(x^n)x^n + x^{2n}S(x^n) \text{ for all } x \in R. \end{aligned}$$

Using Theorem 2.1, we obtain that S is a centralizer on R. Hence, $S(xy) = S(x)y = xS(y)$ for all $x, y \in R$. This implies that $\mathcal{T}(x^*)^2 = \mathcal{T}(x^*)x = x\mathcal{T}(x^*)$ for all $x \in R$. Now, replacing x by x^* and using Lemma 3.1, we get required result. Next, consider

$$\begin{aligned} 2S(x^{2n}) &= 2\mathcal{T}((x^{2n})^*) \\ &= \mathcal{T}(x^*)^n(x^n) + (x^n)\mathcal{T}(x^*)^n \\ &= S(x^n)x^n + x^nS(x)^n \text{ for all } x \in R. \end{aligned}$$

Using Theorem 2.2, we obtain that S is a centralizer on R. Hence, using the same arguments, we get required result. Further, take

$$\begin{aligned} S(x^{3n}) &= \mathcal{T}((x^{3n})^*) \\ &= (x)^n\mathcal{T}(x^*)(x)^n \\ &= x^nS(x^n)x^n \text{ for all } x \in R. \end{aligned}$$

Using Theorem 2.3, we get required result.

Example 3.1. *Let* $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in 2\mathbb{Z}_8 \right\}$ *is a ring with involution* $* : R \to R$ *by* $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$ *for all* $a, b, c \in 2\mathbb{Z}_8$ *under matrix addition and matrix multiplication, where* $\mathbb{Z}_8$ *denotes the ring of integers addition and multiplication modulo 8. Define a mapping* $\mathcal{T} : R \to R$ *by* $\mathcal{T}\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ *for all* $a, b, c \in 2\mathbb{Z}_8$. *It is clear that* $\mathcal{T}$ *satisfy the identities in Theorem 3.2 ((ii)* $n > 1$*) and* R *is neither a 2-torsion free semiprime ring nor* $\mathcal{T}$ *is a centralizer on* R*, hence semiprimeness hypothesis is crucial for* **Theorem 3.2**.

4. CONJECTURE

In view of the above literature review, we impose two conjectures here as follows:

Theorem 4.1. *An additive mapping* $\mathcal{T} : R \to R$ *is a centralizer on* R *if it satisfies any one of the following identities:*

(i) $3\mathcal{T}(x^{3n}) = \mathcal{T}(x^n)x^{2n} + x^n\mathcal{T}(x^n)x^n + x^{2n}\mathcal{T}(x^n)$
(ii) $2\mathcal{T}(x^{2n}) = \mathcal{T}(x^n)x^n + x^n\mathcal{T}(x^n)$
(ii) $\mathcal{T}(x^{3n}) = x^n\mathcal{T}(x^n)x^n$

for all $x \in R$*, where* $n \geq 1$ *is a fixed integer and* R *is any suitably torsion free semiprime ring not necessarily with identity.*

Theorem 4.2. *Let* $n \geq 1$ *be a fixed integer and* R *be any suitably torsion free semiprime* $*$*-ring not necessarily with identity. If* $\mathcal{T} : R \to R$ *is an additive mapping which satisfies any one of the following identities:*

(i) $3\mathcal{T}(x^{3n}) = \mathcal{T}(x^n)(x^*)^{2n} + (x^*)^n\mathcal{T}(x^n)(x^*)^n + (x^*)^{2n}\mathcal{T}(x^n)$
(ii) $2\mathcal{T}(x^{2n}) = \mathcal{T}(x^n)(x^*)^n + (x^*)^n\mathcal{T}(x^n)$
(iii) $\mathcal{T}(x^{3n}) = (x^*)^n\mathcal{T}(x^n)(x^*)^n$ *for all* $x \in R$.

Then, $\mathcal{T}$ *is a* $*$*-centralizer on* R.

REFERENCES

[1] A. Z. ANSARI AND F. SHUJAT, Semiprime rings with involution and centralizers, *J. Appl. Math. & Informatics*, (3-4) **40** (2022), pp. 709–717.

[2] M. ASHRAF, AND M. R. MOZUMDER, On Jordan α-$*$-centralizers in semiprime rings with involution, *Int. J. Contemp. Math. Sciences*, (23) **7** (2012), pp. 1103–1112.

[3] K.I. BEIDAR,, W. S. MATTINDALE, AND A. V. MIKHALEV, *Ring with Generalized Identities*, Markel Dekker Inc. New York, 1996.

[4] A. V. DAELE, Multiplier Hopf algebras, *Trans. Amer. Math. Soc.*, **342** (1994), pp. 917–932.

[5] A. V. DAELE AND Y. ZHANG, A Survey on multiplier Hopf algebras, *Hopf algebras and quantum groups* (Brussels, 1998), pp. 269–306.

[6] S. HELGOSEN, Multipliers of Banach algebras, *Ann. of Math.*, **64** (1956) pp. 240–254.

[7] T. HUSAIN, Multipliers of Topological algebras, *Dessertation Math.* (Rozprawy Mat.) **285** (1989) pp. 44.

[8] B. E. JOHNSON, An introduction to the theory of centralizers, *Proc. London Math. Soc.*, **14** (1964), pp. 299–320 .

[9] B.E. JOHNSON, Centralizers on certain topological algebras, *J. London Math. Soc.*, **39** (1964), pp. 603–614.

[10] B. E. JOHNSON, Continuity of centralizers on Banach algebras, *J. London Math. Soc.*, **41** (1966), pp. 639–640.

[11] L. A. KHAN,N. MOHAMMAD AND A. B. THAHEEM, Double multipliers on topological algebras, *Internat. J. Math. Math. Sci.*, **22** (1999), pp. 629–636.

[12] R. LARSEN, *An Introduction to the Theory of Multipliers*, Springer-Verlag, Berlin, 1971.

[13] N. MOHAMMAD, L. A. KHAN, AND A. B. THAHEEM, On closed range multipliers on topological algebras, *Scientiae Math. Japonica*, **53** (2001), pp. 89–96.

[14] L. MOLNAR, On centralizers of an H^***-** algebra, *Publ. Math. Debrecen*,(1-2) **46** (1995), pp. 89–95.

[15] J. VUKMAN, An identity related to centralizers in semiprime rings, *Comment. Math. Univ. Carol.*, (3) **40** (1999), pp. 447–456.

[16] J. VUKMAN, Centralizers of semiprime ring, *Comment. Math. Univ. Caroline*, (2) **42**(2001), 237-245.

[17] J. VUKMAN AND I. KOSI-ULBL, An equation related to centralizers in semiprime rings, *Glasnik Matematicki*, (58) **38** (2003), pp. 253–261

[18] J. K. WANG, Multipliers of commutative Banach algebras, *Pacific. J. Math.*, **11** (1961), pp. 1131–1149.

[19] J. G. WENDEL, Left centralizers and isomorphisms of group algebras, *Pacific J. Math.*, **2** (1952), pp. 251–266.

[20] B. ZALAR, On centralizers of semiprime rings, *Comment. Math. Univ. Carol.*, **32** (1991), pp. 609–614.