Aust. J. Math. Anal. Appl.
Vol. **19** (2022), No. 2, Art. 10, 19 pp.
AJMAA

ESTIMATION FOR BOUNDED SOLUTIONS OF SOME NONLINEAR INTEGRAL INEQUALITIES WITH DELAY IN SEVERAL VARIABLES

SMAKDJI MOHAMED ELHADI, DENCHE MOUHAMED AND KHELLAF HASSANE

Received 21 April, 2022; accepted 23 September, 2022; published 21 October, 2022.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FRÈRES MENTOURI, PO BOX 25000, AIN ELBAY, CONSTANTINE, ALGERIA.
khellafhassane@umc.edu.dz

ABSTRACT. In this paper, some new nonlinear retarded integral inequalities of Gronwall-Bellman type for functions of two and n-independents variables are investigated. The derived results can be applied in the study of differential-integral equations with time delay. An example is given to illustrate the application of our results.

Key words and phrases: Retarded integral inequality, Gronwall-Bellman type inequality, n-independent variables, Infinite integral, Differential equations.

2000 *Mathematics Subject Classification.* Primary 26D10, 26D15. Secondary 26D07, 45D05.

ISSN (electronic): 1449-5910

I would like to thank the referees very much for their valuable suggestions on improving this paper.

1. INTRODUCTION

The integral inequalities which provide explicit bounds on unknown functions have played a fundamental role in the development of the theory of differential and integral [13, 14], and can be used as handy tools in the study of existence, boundedness, uniqueness, stability and other qualitative properties of solutions of differential and integral equations. Over the years, many nonlinear retarded Gronwall-Bellman inequalities were discussed by several authors, who either reproved and generalized them in many different ways (see [3, 7, 11, 12]).
In his study of boundedness of solutions to linear second order differential equations, Pachpatte [13] established the following nonlinear integral inequality.

$$u(t) \leq a + \int_{t_0}^{t} f(s)w(u(s))ds, \tag{1.1}$$

where $a > 0$ is a constant. Replacing t by a function $b(t)$ in (1.1), Lipovan [10] investigates the retarded Gronwall-Bellman inequality :

$$u(t) \leq a + \int_{t_0}^{t} f(s)w(u(s))ds + \int_{b(t_0)}^{b(t)} g(s)w(u(s))ds. \tag{1.2}$$

These inequalities have been generalized to more than one variable see for example [2, 4, 8, 9].
In 2005, Zhao and Meng [15] studied the following new nonlinear retarded integral inequality:

Lemma 1.1. *Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$. Let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function and let c be a nonnegative constant. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on $\mathbb{R}_+$. If $u, f \in (\mathbb{R}_+, \mathbb{R}_+)$ and*

$$\varphi(u(t)) \leq c + \int_{\alpha(x)}^{\infty} f(s)\psi(u(s))ds, \quad t \in \mathbb{R}_+, \tag{1.3}$$

then for $0 \leq T \leq t < \infty$,

$$u(t) \leq \varphi^{-1}\left(G^{-1}\left[G(c) + \int_{\alpha(t)}^{\infty} f(s)ds\right]\right), \tag{1.4}$$

Where $G(z) = \int_{z_0}^{z} \frac{ds}{\psi[\varphi^{-1}(s)]}, \ \ z \geq z_0 > 0.$

He also established the following main result

$$\varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} \left[f(s)u(s)\psi(u(s)) + g(s)u(s)\right] ds. \tag{1.5}$$

Similar to (1.5), in 2016, Hunng and Wang [6] established some new retarded integral inequalities in two independent variables (Lemma 2.1 and 2.2). However, in some situations, it is desirable to investigate some inequalities of the above type where a constant c is replaced by a function $c(x)$ and the linear $g(s)u(t)$ in integral functions in (1.5) is replaced by the nonlinear case $\Phi\left(u(t)\right) w(u(x))$.
In this paper, our results concern with integral inequalities involving infinite integral for functions with such a function $f_j(x)$ term outside the integrals, which gives us another generalizations in different form in the case of n-independents variables as we will see in Lemma 3.1, Theorem 3.3 and Theorem 3.4 in Section 3.
Motivated by the inequalities (1.3) and (1.5) of Zhao and Meng in [15] and by the works of Hunng -Wang and Lipovan presented in [6, 12], our main aim here is to establish some nonlinear retarded integral inequalities involving infinite integrals in several variables. So in this

paper we discuss more general forms of the following integral inequality:

$$\begin{aligned}\varphi(u(x)) &\leq c(x)+\sum_{j=1}^{n_1} f_j(x)\int_{\widetilde{\alpha_j}(x)}^{\widetilde{\infty}} a_j(x,t)\Phi\left(u(t)\right)dt \\ &+\sum_{k=1}^{n_2} g_k(x)\int_{\widetilde{\beta_k}(x)}^{\widetilde{\infty}} b_k(x,t)\Phi\left(u(t)\right)w(u(t))dt, \quad x,t\in\mathbb{R}_+^n, \end{aligned} \tag{1.6}$$

where $c(x)\geq 0$ is a function in $C(\mathbb{R}_+^n,\mathbb{R}_+)$, $f_j(x)$ and $g_k(x)$ be nondecreasing continuous functions for all $x\in\mathbb{R}_+^n$ and $\widetilde{\alpha}_j(x)=(\alpha_{j1}(x_1),\alpha_{j2}(x_2),\ldots,\alpha_{jn}(x_n))\in\mathbb{R}_+^n$, $\widetilde{\beta}_k(x)=(\beta_{k1}(x_1),\beta_{k2}(x_2),\ldots,\beta_{kn}(x_n))\in\mathbb{R}_+^n$ for $j=1,2,\ldots,n_1$ and $k=1,2,\ldots,n_2$, where $\varphi,\Phi\in C(\mathbb{R}_+,\mathbb{R}_+)$.
Furthermore, we show that some results of [6, 12, 15] can be deduced from our results in some special cases. As applications and motivated by the works in [6, 12], we give the boundedness of the solutions of Volterra-Fredholm integral equation with delay.

2. INTEGRAL INEQUALITIES IN TWO VARIABLES

In this section, we state and prove some new nonlinear retarded integral inequalities of Gronwall-Bellman type, which are further generalizations for some known results in the case of two independents variables.
Throughout the present section, all the functions which appear in the inequalities are assumed to be real valued of two independents variables which are nonnegative and continuous.

Lemma 2.1. *Let $c\in C(\mathbb{R}_+^n,\mathbb{R}_+)$, $w\in C(\mathbb{R}_+,\mathbb{R}_+)$ be a nondecreasing function with $w(u)>0$ on $(0,\infty)$ and let $a_j(x,y,s,t)\in C(\mathbb{R}_+^2\times\mathbb{R}_+^2,\mathbb{R}_+)$be nondecreasing functions in (x,y) for every (s,t) fixed for any $j=1,\ldots,n_1$. Let $\alpha_j,\beta_j\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be nondecreasing functions with $\alpha_j(x)\geq x$,$\beta_j(y)\geq y$ on $\mathbb{R}_+$ for $j=1,2,\ldots,n_1$. Let $\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$ be a strictly increasing function with $\lim_{x\to+\infty}\varphi(x)=+\infty$. If $u\in C(\mathbb{R}_+^n,\mathbb{R}_+)$ and*

$$\varphi(u(x,y))\leq c(x,y)+\sum_{j=1}^{n_1}\left(\int_{\alpha_j(x)}^{\infty}\int_{\beta_j(y)}^{\infty} a_j(x,,y,s,t)w(u(s,t))dsdt\right), \tag{2.1}$$

then for any $(x,y)\in\mathbb{R}_+^2$ with $0\leq x^*\leq x<\infty$ and $0\leq y^*\leq y<\infty$, we have

$$u(x)\leq\varphi^{-1}\left(G^{-1}\left[G(c(x,y))+\sum_{j=1}^{n_1}\int_{\alpha_j(x)}^{\infty}\int_{\beta_j(y)}^{\infty} a_j(x,,y,s,t)dsdt\right]\right). \tag{2.2}$$

Where

$$G(z)=\int_c^z\frac{ds}{w(\varphi^{-1}\left(z(s)\right))},\qquad c>0,\ \ z\in(0,+\infty), \tag{2.3}$$

*and φ^{-1},G^{-1} are respectively the inverse of φ and G, on condition that $G(+\infty)=+\infty$, and the real numbers $(x^*,y^*)\in\mathbb{R}_+^2$, are chosen so that $G(c(x,y))+\sum_{j=1}^{n_1}\int_{\alpha_j(x)}^{\infty}\int_{\alpha_j(x)}^{\infty} a_j(x,,y,s,t)dsdt\in Dom(G^{-1})$
and $G^{-1}\left[G(c(x,y))+\sum_{j=1}^{n_1}\int_{\alpha_j(x)}^{\infty}\int_{\alpha_j(y)}^{\infty} a_j(x,,y,s,t)dsdt\right]\in Dom(\varphi^{-1})$ for all $(x,y)\in[x^*,\infty)\times[y^*,\infty)$.*

Remark 2.1. We can regard Lemma 2.1 as a generalized form of a Gronwall–Bellman inequality (1.1) with advanced argument in two independent variables and infinite integration.

Remark 2.2. It is interesting to note that in the special case when $c(x) = c$ (positive constant), $n_1 = j = 1$ and $a_1(x, y, s, t) = f(x)$ for all $x \in \mathbb{R}_+$then the inequality (2.1) reduces to the Zhao and Meng result in [15, Lemma 2.1].

Remark 2.3. Since the proofs resemble each other, we give the details for Lemma 3.1, Theorem 3.2 and Theorem 3.4 only, the proofs of the remaining inequalities can be completed by following the proofs of the inequalities in n-independents variables (see section 3)

Theorem 2.2. *Let $c \in C(\mathbb{R}^2_+, \mathbb{R}_+)$, $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_1(u)$,*
$w_1(u) > 0$ on $(0, \infty)$ and let $a_j(x, y, s, t)$ and $b_k(x, y, s, t) \in C(\mathbb{R}^2_+ \times \mathbb{R}^2_+, \mathbb{R}_+)$ be nondecreasing functions in (x, y) for every (s, t) fixed and $j = 1, 2, \ldots, n_1,\ k = 1, 2, \ldots, n_2$. Let $\alpha_{ji}, \beta_{ki} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\alpha_{ji}(x) \geq x$ and $\beta_{ki}(y) \geq y$ on $\mathbb{R}_+$ for $i = 1, 2,\ j = 1, 2, \ldots, n_1,\ k = 1, 2, \ldots, n_2\, \forall x, y \in \mathbb{R}_+$. Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a strictly increasing function with $lim_{x\to\infty}\varphi(x) = \infty$. If $u \in C(\mathbb{R}^n_+, \mathbb{R}_+)$ and

$$\begin{aligned} \varphi(u(x,y)) \;&\leq\; c(x,y) + \sum_{j=1}^{n_1} \int_{\alpha_{j1}(x)}^{\infty} \int_{\alpha_{j2}(y)}^{\infty} a_j(x,y,s,t) w_1(u(s,t)) dsdt \\ &+ \sum_{k=1}^{n_2} \int_{\beta_{k1}(x)}^{\infty} \int_{\beta_{k2}(y)}^{\infty} b_k(x,y,s,t) w_2(u(s,t)) dsdt, \end{aligned} \tag{2.4}$$

then for any $(x, y) \in \mathbb{R}^2_+$ with $0 \leq \zeta_1 \leq x < \infty$ and $0 \leq \zeta_2 \leq y < \infty$.
***(a)** In the case $w_2(u) \leq w_1(u)$, for any $(x, y) \in \mathbb{R}^n_+$, there exists $(\zeta_1, \zeta_2) \in \mathbb{R}^2_+$, so that for all $0 \leq \zeta_1 \leq x < \infty$ and $0 \leq \zeta_2 \leq y < \infty$, we have*

$$\begin{aligned} u(x) \;&\leq\; \varphi^{-1}\left(G_1^{-1}\left[G_1(c(x)) + \sum_{j=1}^{n_1} \int_{\alpha_{j1}(x)}^{\infty} \int_{\alpha_{j2}(y)}^{\infty} a_j(x,y,s,t) dsdt \right.\right. \\ &\left.\left. + \sum_{k=1}^{n_2} \int_{\beta_{k1}(x)}^{\infty} \int_{\beta_{k2}(y)}^{\infty} b_k(x,y,s,t) dsdt\right]\right). \end{aligned} \tag{2.5}$$

***(b)** In the case $w_2(u) \geq w_1(u)$, for any $(x, y) \in \mathbb{R}^n_+$, there exists $(\widetilde{\zeta}_1, \zeta_2) \in \mathbb{R}^2_+$, so that for all $0 \leq \widetilde{\zeta}_1 \leq x < \infty$ and $0 \leq \widetilde{\zeta}_2 \leq y < \infty$, we have*

$$\begin{aligned} u(x) \;&\leq\; \varphi^{-1}\left(G_2^{-1}\left[G_2(c(x)) + \sum_{j=1}^{n_1} \int_{\alpha_{j1}(x)}^{\infty} \int_{\alpha_{j2}(y)}^{\infty} a_j(x,y,s,t) dsdt \right.\right. \\ &\left.\left. + \sum_{k=1}^{n_2} \int_{\beta_{k1}(x)}^{\infty} \int_{\beta_{k2}(y)}^{\infty} b_k(x,y,s,t) dsdt\right]\right). \end{aligned} \tag{2.6}$$

Where

$$G_\tau(z) = \int_{z_0}^{z} \frac{ds}{w_\tau(\varphi^{-1}(s))}, \qquad z \geq z_0 > 0, \quad (\tau = 1, 2), \tag{2.7}$$

and $\varphi^{-1}, G_\tau^{-1}$ are respectively the inverse of φ and G_τ, on condition that $G_\tau(+\infty) = +\infty$, and the real numbers $\zeta_\tau, \widetilde{\zeta}_\tau \in \mathbb{R}_+$ are chosen so that
$G_\tau(c(x))+\sum_{j=1}^{n_1} \int_{\alpha_{j1}(x)}^{\infty} \int_{\alpha_{j2}(y)}^{\infty} a_j(x,y,s,t)dsdt+\sum_{k=1}^{n_2} \int_{\beta_{k1}(x)}^{\infty} \int_{\beta_{k2}(y)}^{\infty} b_k(x,y,s,t)dsdt \in Dom(G_\tau^{-1})$
for all $x \in [\zeta_\tau, \infty)$ and $y \in [\widetilde{\zeta}_\tau, \infty)$ for $(\tau = 1, 2)$ respectively.

Many interesting corollaries can also be obtained from the above results

Corollary 2.3. *(Gronwall–Bellman inequality in tow variables) Let* $c(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, *and let* $a(x,y,s,t)$*and* $b(x,y,s,t) \in C(\mathbb{R}_+^2 \times \mathbb{R}_+^2, \mathbb{R}_+)$ *be nondecreasing functions in* x, y *for every* s,t *fixed. Where* $\alpha_{ji}, \beta_{ki} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $\alpha_{ji}(t_i) \geq t_i$ *and* $\beta_{ki}(t_i) \geq t_i$ *on* $\mathbb{R}_+$ *for* $i = 1, 2,\ j = 1, 2, \dots, n_1$ *and* $k = 1, 2, \dots, n_2$*. Let* p, q *be nonnegative constants* $(p > q \geq 0)$*. If* $u \in C(\mathbb{R}_+^2, \mathbb{R}_+)$

$$u(x,y)^p \leq c(x,y) + \sum_{j=1}^{n_1} \int_{\alpha j_1(x)}^{\infty} \int_{\alpha_{j2}(y)}^{\infty} a_j(x,y,s,t) u^q(s,t) ds dt$$

$$+ \sum_{k=1}^{n_2} \int_{\beta_{k1}(x)}^{\infty} \int_{\beta_{k2}(y)}^{\infty} b_k(x,,y,s,t) u^q(s,t) ds dt, \tag{2.8}$$

then for any $(x,y) \in \mathbb{R}_+^2$*, there exists* $(x^*, y^*) \in \mathbb{R}_+^2$*, so that for all* $0 \leq x^* \leq x < \infty$ *and* $0 \leq y^* \leq y < \infty$*, we have*

$$u(x,y) \leq \left(G^{-1} \left[G(c(x,y)) + \sum_{j=1}^{n_1} \int_{\alpha_{j1}(x)}^{\infty} \int_{\alpha_{j2}(y)}^{\infty} a_j(x,y,s,t)) dt \right.\right.$$

$$\left.\left. + \sum_{k=1}^{n_2} \int_{\beta_{k1}(x)}^{\infty} \int_{\beta_{k2}(y)}^{\infty} b_k(x,y,s,t) ds dt \right] \right)^{1/p}, \tag{2.9}$$

with

$$G(z) = \int_{z_0}^{z} \frac{ds}{s^{q/p}}, \qquad z \geq z_0 > 0. \tag{2.10}$$

Where G^{-1} *is the inverse of* G*,on condition that* $G(+\infty) = +\infty$*, and the real numbers* $(x^*, y^*) \in \mathbb{R}_+^2$ *are chosen so that* $G(c(x,y)) + \sum_{j=1}^{n_1} \int_{\alpha_{j1}(x)}^{\infty} \int_{\alpha_{j2}(y)}^{\infty} a_j(x,y,s,t))dt$ $+ \sum_{k=1}^{n_2} \int_{\beta_{k1}(x)}^{\infty} \int_{\beta_{k2}(y)}^{\infty} b_k(x,y,s,t) ds dt \in Dom(G^{-1})$ *for all* $x \in [x^*, \infty)$ *and* $y \in [y^*, \infty)$*.*

Remark 2.4. It is interesting to note that in the special case when $n_1 = j = 1$ and $\beta_{j1}(x) = x\ (x \in \mathbb{R}_+)$ then the inequality given in Corollary 2.3 reduces to the El-Owaidy result [5] in the case of infinite integration.

Remark 2.5. II the special case $(j = 1)$when $c(x,y) = u_0$ (positive constant), $a_1(x,y,s,t) = g(s,t),\ b_1(x,y,s,t) = h(s,t)$, for all $x, y \in \mathbb{R}_+$ then the inequality (2.1) reduces to Theorem 2.2 in [1] in the case of infinite integration.

3. FURTHER GENERALIZATIONS IN N-INDEPENDENTS VARIABLES

In this section, we state and prove some new retarded nonlinear integral inequalities of Gronwall-Bellman type, which are further generalizations for some known results in the case of n independents variables, these inequalities can be used in the analysis of various problems in the theory of retarded nonlinear differential equations.

Throughout the present section, all the functions which appear in the inequalities are assumed to be real valued of $n-$variables which are nonnegative and continuous. All integrals are assumed to exist on their domains of definitions. For $x = (x_1, x_2, \dots, x_n),\ t = (t_1, t_2, \dots, t_n) \in \mathbb{R}_+^n$ and $\widetilde{\infty} = (+\infty, +\infty, \dots, +\infty)$, we denote

$$\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} dt = \int_{\alpha_{j1}(x_1)}^{+\infty} \int_{\alpha_{j2}(x_2)}^{+\infty} \cdots \int_{\alpha j_n(x_2)}^{+\infty} dt_n \cdots dt_1, \quad j = 1, \dots, n_1,$$

$$\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} dt = \int_{\beta_{k1}(x_1)}^{\infty} \int_{\beta_{k2}(x_2)}^{\infty} \cdots \int_{\beta_{kn}(x_2)}^{\infty} dt_n \cdots dt_1, \qquad k = 1, \dots, n_2,$$

with $n_1, n_2 \in \mathbb{N}^*$. For $x, t \in \mathbb{R}^n_+$, we shall write $x \le t < \widetilde{\infty}$ whenever $x_i \le t_i < +\infty$, with $i = 1, 2, \dots, n.$

We denote $D = D_1 D_2 \cdots D_n$, where $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, \dots, n$. We use the usual convention of writing $\sum_{s \in \Phi} u(s) = 0$ if Φ is empty set.
$\widetilde{\alpha_j}(x) = (\alpha_{j1}(x_1), \alpha_{j2}(x_2), \dots, \alpha j_n(x_n)) \in \mathbb{R}^n_+$ for $j = 1, 2, \dots, n_1,$
$\widetilde{\beta}_k(x) = (\beta_{k1}(x_1), \beta_{k2}(x_2), \dots, \beta_{kn}(x_n)) \in \mathbb{R}^n_+$. for $k = 1, 2, \dots, n_2.$
Our main results read as the follows.

Lemma 3.1. *Let $c \in C(\mathbb{R}^n_+, \mathbb{R}_+)$, $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $w(u) > 0$ on $(0, \infty)$ and let $a_j(x, t) \in C(\mathbb{R}^n_+ \times \mathbb{R}^n_+, \mathbb{R}_+)$be nondecreasing functions in x for every t fixed for any $j = 1, 2, \dots, n_1$. Let $\alpha_{ji} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\alpha_{ji}(t_i) \ge t_i$ on $\mathbb{R}_+$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n_1$. Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a strictly increasing function with $lim_{x \to +\infty} \varphi(x) = +\infty$.If $u \in C(\mathbb{R}^n_+, \mathbb{R}_+)$*

$$\varphi(u(x)) \le c(x) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x, t) w(u(t)) dt, \qquad t \in \mathbb{R}^n, \tag{3.1}$$

then for any $x \in \mathbb{R}^n_+$, then there exists $x^* \in \mathbb{R}^n_+$ with $0 \le x^* \le x < \infty$, we have

$$u(x) \le \varphi^{-1} \left(G^{-1} \left[G(c(x)) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x, t) dt \right] \right). \tag{3.2}$$

Where

$$G(z) = \int_c^z \frac{ds}{w(\varphi^{-1}(s))}, \qquad c > 0, \ z \in (0, +\infty), \tag{3.3}$$

and φ^{-1}, G^{-1} are respectively the inverse of φ and G, on condition that $G(+\infty) = +\infty$, and the real numbers $x^* \in \mathbb{R}^n_+$, are chosen so that $G(c(x)) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x, t) dt \in Dom(G^{-1})$ and $G^{-1} \left[G(c(x)) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x, t) dt \right] \in Dom(\varphi^{-1})$ for all $x \in [x^*, \infty)$.

Proof. **1)** If $c(x) > 0$ for all $x \in \mathbb{R}^n_+$, fixing arbitrary $y = (y_1, \dots, y_n) \in \mathbb{R}^n_+$ with $0 \le x^* \le y \le x < \widetilde{\infty}$, we define on $[y, \widetilde{\infty})$ a function $z(x)$ by

$$z(x) = c(y) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y, t) w(u(t)) dt, \tag{3.4}$$

then $z(x)$ is positive and nonincreasing function in each variables $x_i \in [y_i, +\infty)$, and

$$u(x) \le \varphi^{-1}(z(x)), \qquad x \in [y; \widetilde{\infty}). \tag{3.5}$$

we know that differentiating (3.4) and using (3.5) and the monotonicity of φ and w, we deduce that

$$D_1 D_2 \cdots D_n z(x) = (-1)^n \sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}(x)) w(u(\widetilde{\alpha}_j(x))) \alpha'_{j1} \alpha'_{j2} \cdots \alpha'_{jn}, \tag{3.6}$$

for all $j = 1, 2, \dots, n_1$.

i) If n is even and since $\alpha_{ji}(t_i) \geq t_i$ on $\mathbb{R}_+$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_1$, we have $z(\widetilde{\alpha}_j(x)) \leq z(x)$, then

$$\begin{aligned} D_1 D_2 \cdots D_n z(x) &= -\sum_{j=1}^{n_1} a(y, \widetilde{\alpha}_j(x)) w(u(\widetilde{\alpha}_j(x))) \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn}, \\ &\geq -\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) w(\varphi^{-1}(z(\widetilde{\alpha}_j(x)))) \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn}, \\ &\geq -\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) w(\varphi^{-1}(z(x))) \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn}, \end{aligned} \tag{3.7}$$

for all $x \in [y, \widetilde{\infty})$ and $j = 1, 2, \ldots, n_1$. Since

$$w(\varphi^{-1}(z(x))) \geq w(\varphi^{-1}(z(\widetilde{\infty}))) = w(\varphi^{-1}(c(y))) > 0. \tag{3.8}$$

Using (3.7) and (3.8), we have

$$\frac{D_1 D_2 \cdots D_n z(x)}{w(\varphi^{-1}(z(x)))} \geq -\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn}, \tag{3.9}$$

using $[D_1 D_2 \cdots D_{n-1} z(x)] \times D_n z(x) < 0$ and $(\varphi^{-1})' \geq 0, w' \geq 0$ and (3.9), we have

$$\begin{aligned} D_n\left(\frac{D_1 D_2 \cdots D_{n-1} z(x)}{w(\varphi^{-1}(z(x)))}\right) &\geq \frac{D_1 D_2 \cdots D_n z(x)}{w(\varphi^{-1}(z(x)))} \\ &\geq -\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn}, \end{aligned}$$

Fixing $x_1, x_{2,}, \ldots, x_{n-1}$, setting $x_n = t_n$ and integrating (3.10) from x_n to ∞, we obtain

$$\begin{aligned} -\frac{D_1 D_2 \cdots D_{n-1} z(x)}{w(\varphi^{-1}(z(x)))} \geq\ & -\sum_{j=1}^{n_1} \left[\int_{\alpha_{jn}(x_n)}^{\infty} a(y, , \alpha_{j1}(x_1), \alpha_{j1}(x_2), \ldots, \alpha_{jn-1}(x_{n-1}), t_n) \right. \\ & \left. \times \alpha'_{j1}(x_1)\alpha'_{j2}(x_2)\cdots\alpha'_{jn-1}(x_{n-1}) dt_n \right] \end{aligned}$$

Using the same method above, we obtain

$$\begin{aligned} & (-1)^{n-1}\frac{D_1 z(x)}{w(\varphi^{-1}(z(x)))} \\ \geq\ & -\sum_{j=1}^{n_1}\left(\int_{\alpha_{j2}(x_2)}^{\infty}\cdots\int_{\alpha_{jn}(x_n)}^{\infty} a(y, \alpha_{j1}(x_1), t_2, \ldots, t_{n-1}, t_n)\alpha'_{j1}(x_1) dt_n \cdots dt_2\right). \end{aligned}$$

Since n is even, we have

$$\begin{aligned} & \frac{D_1 z(x)}{w(\varphi^{-1}(z(x)))} \\ \geq\ & -\sum_{j=1}^{n_1}\left(\int_{\alpha_{j2}(x_2)}^{\infty}\cdots\int_{\alpha_{jn}(x_n)}^{\infty} a(y, \alpha_{j1}(x_1), t_2, \ldots, t_{n-1}, t_n)\alpha'_{j1}(x_1) dt_n \cdots dt_2\right), \end{aligned} \tag{3.10}$$

for all $x \in [y, \widetilde{\infty})$. Integrating (3.10) from x_1 to $+\infty$, we obtain

$$G(z(+\infty, x_2, \dots, x_n)) - G(z(x)) \geq -\sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t)dt,$$

for all $x \in [y, \widetilde{\infty})$, which implies that

$$G(z(x)) \leq G(c(y)) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t)dt,$$

we have

$$z(y) \leq G^{-1}\left[G(c(y)) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(y)}^{\widetilde{\infty}} a_j(y,t)dt\right], \tag{3.11}$$

for any arbitrary numbers $y \in \mathbb{R}_+^n$, with $x^* \leq y$ and G is defined by (3.3). From (3.11) and (3.5) we obtain the following inequality

$$u(y) \leq \varphi^{-1}\left(G^{-1}\left[G(c(y)) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(y)}^{\widetilde{\infty}} a_j(y,t)dt\right]\right).$$

Since y are arbitrary numbers with $x^* \leq y$, we obtain the result (3.2).
ii) In the case of n is odd, using the same method in (i) with n is odd, we obtain the result in Lemma 3.1, we omit the details here.
2) If $c(x) \geq 0$, we carry out the above procedure in (i) and (ii) with $c(x) + \epsilon$ instead of $c(x)$, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (3.2). This completes the proof. ∎

Remark 3.1. Under the same hypothesis as in the previous Lemma, and If

$$u(x) \leq c(x) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)w(\varphi^{-1}u(t))dt, \qquad t \in \mathbb{R}^n, \tag{3.12}$$

where $c(x) \in C(\mathbb{R}_+^n, \mathbb{R}_+)$, then the inequality (3.2) also holds. In fact, we can observe that in the special case $(n = 2$ and $j = 1)$ the inequality (3.12) reduced to the main results in [6, Lemma 2.1] without the monotonicity condition on $H(x, y)$.

Remark 3.2. We can also regard Lemma 3.1 as a generalized form of a Gronwall inequality (2.1) with advanced argument in n-independent variables.

Remark 3.3. It is interesting to note that in the special case when $c(x) = c$ (positive constant), $n = 1, j = 1$ and $a_1(x,t) = f(x)$ for all $x \in \mathbb{R}_+$ then the inequality (3.1) reduces to the Zhao and Meng result in [15, Lemma 2.1].

Theorem 3.2. *Let* $c \in C(\mathbb{R}_+^n, \mathbb{R}_+)$, w_1,
$w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $w_1(u), w_1(u) > 0$ *on* $(0, \infty)$ *and let* $a_j(x,t)$*and* $b_k(x,t) \in C(\mathbb{R}_+^n \times \mathbb{R}_+^n, \mathbb{R}_+)$*be nondecreasing functions in* x *for every* t *fixed for* $j = 1, 2, \dots, n_1$ *and* $k = 1, 2, \dots, n_2$. *Let* $\alpha_{ji}, \beta_{ki} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $\alpha_i(t_i) \geq t_i$ *and* $\beta_i(t_i) \geq t_i$*on* $\mathbb{R}_+$ *for* $i = 1, 2, \dots, n,\ j = 1, 2, \dots, n_1$ *and* $k = 1, 2, \dots, n_2$. *Let* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be a strictly increasing function with* $lim_{x\to\infty}\varphi(x) = \infty$. *If* $u \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *and*

$$\varphi(u(x)) \leq c(x) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha} j(x)}^{\widetilde{\infty}} a_j(x,t)w_1(u(t))dt + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)w_2(u(t))dt, \tag{3.13}$$

then for any $x \in \mathbb{R}_+^n$ with $0 \le \zeta_1 \le x < \infty$,
***(a)** In the case $w_2(u) \le w_1(u)$, for any $x \in \mathbb{R}_+^n$, there exists $\zeta_1 \in \mathbb{R}_+^n$, so that for all $0 \le \zeta_1 \le x < \infty$, we have*

$$u(x) \le \varphi^{-1}\left(G_1^{-1}\left[G_1(c(x)) + \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_k(x,t)dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)dt\right]\right). \tag{3.14}$$

***(b)** In the case $w_2(u) \ge w_1(u)$, for any $x \in \mathbb{R}_+^n$, there exists $\zeta_2 \in \mathbb{R}_+^n$, so that for all $0 \le \zeta_2 \le x < \infty$, we have*

$$u(x) \le \varphi^{-1}\left(G_2^{-1}\left[G_2(c(x)) + \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)dt\right]\right). \tag{3.15}$$

Where

$$G_\tau(z) = \int_{z_0}^{z} \frac{ds}{w_\tau(\varphi^{-1}(s))}, \qquad z \ge z_0 > 0, \quad (\tau = 1,2), \tag{3.16}$$

and $\varphi^{-1}, G_\tau^{-1}$ are respectively the inverse of φ and G_τ, on condition that $G_i(+\infty) = +\infty$, and the real numbers $\zeta_\tau \in \mathbb{R}_+^n$ are chosen so that $G_\tau(c(x)) + \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)dt \in Dom(G_\tau^{-1})$ and
$G_\tau^{-1}\left[G_\tau(c(x)) + \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)dt\right] \in Dom(\varphi^{-1})$ for all $x \in [\zeta_\tau, \infty)$ for $(\tau = 1,2)$ respectively.

Proof. If $c(x) > 0$ for all $x \in \mathbb{R}_+^n$, fixing arbitrary $y = (y_1, \dots, y_n) \in \mathbb{R}_+^n$ with $0 \le \zeta_\tau \le y \le x < \infty$ $(\tau = 1,2)$, we define on $[y, \widetilde{\infty})$ a function $z(x)$ by

$$z(x) = c(y) + \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t)w_1(u(t))dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t)w_2(u(t))dt, \tag{3.17}$$

then $z(x)$ is positive and nonincreasing function in each variables $x_i \in [y_i, \infty)$, and

$$u(x) \le \varphi^{-1}(z(x)), \qquad x \in [y; \widetilde{\infty}). \tag{3.18}$$

We know that differentiating (3.17) and using (3.18) and the monotonicity of φ and w, we deduce that

$$\begin{aligned} D_1 D_2 \cdots D_n z(x) &= (-1)^n \sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) w_1(u(\widetilde{\alpha}_j(x)))\alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\ &\quad +(-1)^n \sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_2(u(\widetilde{\beta}_k(x)))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}. \end{aligned}$$

in the case of n is even, then

$$\begin{aligned} D_1 D_2 \cdots D_n z(x) &\ge -\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) w_1(\varphi^{-1}(z(x)))\alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\ &\quad -\sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_2(\varphi^{-1}(z(x)))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}. \end{aligned}$$

(a) When $w_2(u) \le w_1(u)$, we deduce that

$$\begin{aligned} D_1D_2\cdots D_n z(x) \;&\ge\; -\sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))w_1(\varphi^{-1}(z(x)))\alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\ &\quad -\sum_{k=1}^{n_2} b_k(y,\widetilde{\beta}_k(x))w_1(\varphi^{-1}(z(x)))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}, \end{aligned} \tag{3.19}$$

for all $x \in [y,\widetilde{\infty})$,

$$w_1(\varphi^{-1}(z(x))) \ge w_1(\varphi^{-1}(z(\widetilde{\infty}))) = w_1(\varphi^{-1}(c(y))) > 0. \tag{3.20}$$

Using (3.19) and (3.20), we have

$$\frac{D_1D_2\cdots D_n z(x)}{w_1(\varphi^{-1}(z(x)))} \ge -\sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))\alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} - \sum_{k=1}^{n_2} b_k(y,\widetilde{\beta}_k(x))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}.$$

Using similar procedures as from (3.5) to (3.11) in the proof of Lemma 3.1 (i) and(ii), we can get the desired bound of $u(x)$ in (3.14). By continuity, (3.14) also holds for the case $c(x) \ge 0$. ∎

Remark 3.4. In the special case $(n = 1, j = 1$ and $k = 1)$when $c(x) = u_0$ (positive constant), $a(x,t) = g(x)$, $b(x,t) = h(x)$, $\widetilde{\beta}(x) = x$, $\widetilde{\alpha}(x) = \alpha(x)$ for all $x \in \mathbb{R}_+$ then the inequality (3.1) reduces to Theorem 2.2 in [1] in the case of infinite integration.

Theorem 3.3. *Let the functions $u, c, a, b, w_1, w_1, \alpha_{ji}$ and β_{ki} be defined as in Theorem 3.2. Moreover, let p, q be nonnegative constants $(p > q \ge 0)$.*

(A$_1$***)*** *If $u \in C(\mathbb{R}^n_+, \mathbb{R}_+)$ and*

$$u(x)^p \le c(x) + \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)u^q(t)dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)u^q(t)w_1(u(t))dt, \tag{3.21}$$

then for any $x, t \in \mathbb{R}^n_+$, there exists $x^* \in \mathbb{R}^n_+$, so that for all $0 \le x^* \le x < \infty$,with we have

$$u(x) \le \left(\Psi_1^{-1}\left[\Psi_1(p(x)) + \frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)dt\right]\right)^{p/(p-q)}. \tag{3.22}$$

Where

$$p(x) = c^{(p-q)/p}(x) + \frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)dt, \tag{3.23}$$

$$\Psi_1(\delta) = \int_{\delta_0}^{\delta}\frac{ds}{w_1(s^{1/(p-q)})}, \qquad \delta > \delta_0 > 0. \tag{3.24}$$

Here Ψ_1^{-1} is the inverse of Ψ_1, on condition that $\Psi_1(+\infty) = +\infty$, and the real numbers $x^* \in \mathbb{R}^n_+$ are chosen so that $\Psi_1(p(x)) + \frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)dt \in Dom(\Psi_1^{-1})$ and for all $x \in [x^*, \infty)$.

(A$_2$) *If* $u \in C(\mathbb{R}^n_+, \mathbb{R}_+)$ *and*

$$\begin{aligned} u^p(x) &\leq c(x) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t) u^q(t) w_1(u(t)) dt \\ &\quad + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t) u^q(t) w_2(u(t)) dt, \end{aligned} \tag{3.25}$$

(i) *In the case* $w_2(u) \leq w_1(u)$*, for any* $x \in \mathbb{R}^n_+$*, there exists* $\zeta_1 \in \mathbb{R}^n_+$*, so that for all* $0 \leq \zeta_1 \leq x < \infty$*, with we have*

$$u(x) \leq \left(\Psi_2^{-1}\left[\Psi_2\left(c^{(p-q)/p}(x)\right) + e(x)\right]\right)^{1/(p-q)}. \tag{3.26}$$

(ii) *In the case* $w_2(u) \geq w_1(u)$ *, for any* $x \in \mathbb{R}^n_+$ *, there exists* $\zeta_2 \in \mathbb{R}^n_+$ *, so that for all* $0 \leq \zeta_2 \leq x < \infty$*, with we have*

$$u(x) \leq \left(\Psi_1^{-1}\left[\Psi_1\left(c^{(p-q)/p}(x)\right) + e(x)\right]\right)^{1/(p-q)}. \tag{3.27}$$

With

$$e(x) = \frac{p-q}{p}\left[\sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t) dt + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t) dt\right], \tag{3.28}$$

$$\Psi_\tau(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_i\left(s^{1/(p-q)}\right)}, \quad \tau = 1, 2. \tag{3.29}$$

Where Ψ_τ^{-1} *are the inverse* Ψ_τ $(\tau = 1, 2)$, *on condition that* $\Psi_\tau(+\infty) = +\infty$*, and the real numbers* $\zeta_\tau \in \mathbb{R}^n_+$ *are chosen so that* $\left(\Psi_i\left(c^{(p-q)/p}(x)\right) + e(x)\right) \in Dom(\Psi_i^{-1})$ *for all* $x \in [\zeta_\tau, \infty)$ *for* $(\tau = 1, 2)$ *respectively.*

Proof. **(A1)** If $c(x) > 0$ for all $x \in \mathbb{R}^n_+$, fixing arbitrary $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ with $0 \leq x^* \leq y \leq x < \infty$, we define on $[y, \widetilde{\infty})$ a function $z(x)$ by

$$\begin{aligned} z(x) &= c(y) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t) u^q(t) dt \\ &\quad + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t) u^q(t) w_1(u(t)) dt, \end{aligned} \tag{3.30}$$

then $z(x)$ is positive and nonincreasing function in each variables $x_i \in [y_i, \infty)$, and

$$u(x) \leq z(x)^{1/p}, \qquad x \in [y; \widetilde{\infty}). \tag{3.31}$$

we know that differentiating (3.30) and using (3.31) and the monotonicity of φ and w and in the case of n is even, then we deduce that

$$\begin{aligned} D_1D_2\cdots D_nz(x) &= -\sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))u^q(\widetilde{\alpha}_j(x))\alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\ &\quad -\sum_{k=1}^{n_2} b_k(y,\widetilde{\beta}_k(x))u^q(\widetilde{\beta}_k(x))w_1(u(\widetilde{\beta}_k(x)))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}, \\ &\geq z^{q/p}(x)\left[-\sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))\alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn}\right. \\ &\quad \left.\sum_{k=1}^{n_2} -b_k(y,\widetilde{\beta}_k(x))w_1(z^{1/p}(x))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}\right], \end{aligned} \tag{3.32}$$

for all $x\in[y,\widetilde{\infty})$

$$z^{q/p}(x)\geq z^{q/p}(\widetilde{\infty})=z^{q/p}(c(y))>0. \tag{3.33}$$

Using (3.32) and (3.33), we have

$$\begin{aligned} \frac{D_1D_2\cdots D_nz(x)}{z^{q/p}(x)} &\geq -\sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))\alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\ &\quad -\sum_{k=1}^{n_2} -b_k(y,\widetilde{\beta}_k(x))w_1(z^{1/p}(x))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}. \end{aligned}$$

Using similar procedures as in the proof of Lemma 3.1 (i), we obtain

$$\begin{aligned} &(-1)^{n-1}\ \frac{D_1z(x)}{z^{p/q}(x)} \\ \geq\ & -\sum_{j=1}^{n_1}\int_{\alpha_{j2}(x_2)}^{\infty}\cdots\int_{\alpha_{jn-1}(x_{n-1})}^{\infty}\int_{\alpha_{jn}(x_n)}^{\infty} a_j(y,\alpha_{j1}(x_1),t_2,\ldots,t_{n-1},t_n)\alpha'_{j1}(x_1)dt_n\cdots dt_2 \\ & -\sum_{k=1}^{n_2}\int_{\beta_{k2}(x_2)}^{\infty}\cdots\int_{\beta_{kn-1}(x_{n-1})}^{\infty}\int_{\beta_{kn}(x_n)}^{\infty} b_k(y,\beta_{k1}(x_1),t_2,\ldots,t_{n-1},t_n) \\ & \times w_1(z^{1/p}(\beta_{k1}(x_1),t_2,\ldots,t_{n-1},t_n))\beta'_{k1}(x_1)dt_n\cdots dt_2, \end{aligned}$$

since n is even, we obtain

$$\begin{aligned} &(-1)^{n-1}\ \frac{D_1z(x)}{z^{p/q}(x)} \\ \geq\ & -\sum_{j=1}^{n_1}\int_{\alpha_{j2}(x_2)}^{\infty}\cdots\int_{\alpha_{jn-1}(x_{n-1})}^{\infty}\int_{\alpha_{jn}(x_n)}^{\infty} a_j(y,\alpha_{j1}(x_1),t_2,\ldots,t_{n-1},t_n)\alpha'_{j1}(x_1)dt_n\cdots dt_2 \\ & -\sum_{k=1}^{n_2}\int_{\beta_{k2}(x_2)}^{\infty}\cdots\int_{\beta_{kn-1}(x_{n-1})}^{\infty}\int_{\beta_{kn}(x_n)}^{\infty} b_k(y,\beta_{k1}(x_1),t_2,\ldots,t_{n-1},t_n) \\ & \times w_1(z^{1/p}(\beta_{k1}(x_1),t_2,\ldots,t_{n-1},t_n))\beta'_{k1}(x_1)dt_n\cdots dt_2, \end{aligned} \tag{3.34}$$

for all $x\in[y,\widetilde{\infty})$.

Integrating (3.34) from x_1 to $+\infty$, we obtain

$$\frac{p}{p-q}z^{(p-q)/p}(y)-\frac{p}{p-q}z^{(p-q)/p}(x) \geq -\sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t)dt - \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t)w_1(z^{1/p}(t))dt,$$

for all $x \in [y,\widetilde{\infty})$, we have

$$z^{(p-q)/p}(x) \leq c^{(p-q)/p}(y)+\frac{p-q}{p}\sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t)dt + \frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t)w_1(z^{1/p}(t))dt. \tag{3.35}$$

Defining $r_1(x)$ as $r_1(x)=z^{(p-q)/p}(x)$,(3.35) can be rewritten as

$$r_1(x) \leq p(y)+\frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t)w_1(r_1^{1/(p-q)}(t))dt, \tag{3.36}$$

where $p(y)$ is defined by (3.23). Now applying Lemma 3.1 to (3.36), we get

$$r_1(x) \leq \Psi_1^{-1}\left[\Psi_1(p(y))+\frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t)dt\right].$$

From (3.35) and for any arbitrary y, we obtain

$$z(y) \leq \Psi_1^{-1}\left[\Psi_1(p(y))+\frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(y)}^{\widetilde{\infty}} b_k(y,t)dt\right]^{p/(p-q)}.$$

Since y are arbitrary numbers $x^* \leq y,$ the desired bound for $u(x)$ appeared in (3.22) directly. By continuity, (3.22) also holds for the case $c(x) \geq 0$ and n is odd number. **(A2)** If $c(x) > 0$ for all $x \in \mathbb{R}_+^n$, fixing arbitrary $y=(y_1,\ldots,y_n) \in \mathbb{R}_+^n$ with $0 \leq \zeta_1 \leq y \leq x < \infty$, we define on $[y,\widetilde{\infty})$ a function $z(x)$ by

$$z(x) = c(y)+\sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t)u^q(t)w_1(u(t))dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(y)}^{\widetilde{\infty}} b_k(y,t)u^q(t)w_2(u(t))dt, \tag{3.37}$$

then $z(x)$ is positive and nonincreasing function in each variables $x_i \in [y_i,\infty)$, and

$$u(x) \leq z(x)^{1/p}, \qquad x \in [y;\widetilde{\infty}). \tag{3.38}$$

we know that differentiating (3.37) and using (3.38) and the monotonicity of $z^{1/p}$ and w_τ, we deduce that

$$D_1D_2\cdots D_nz(x) = (-1)^n a(y,\widetilde{\alpha}(x))u^q(\widetilde{\alpha}(x))w_1(u(\widetilde{\alpha}(x)))\alpha_{j1}^{'}\alpha_{j2}^{'}\cdots\alpha_{jn}^{'} + (-1)^n\sum_{k=1}^{n_2} b_k(y,\widetilde{\beta}_k(x))u^q(\widetilde{\beta}_k(x))w_2(u(\widetilde{\beta}_k(x)))\beta_{k1}^{'}\beta_{k2}^{'}\cdots\beta_{kn}^{'},$$

(i) When $w_2(u) \leq w_1(u)$ and if n is even, then

$$\begin{aligned} D_1 D_2 \cdots D_n z(x) &= -\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) u^q(\widetilde{\alpha}_j(x)) w_1(u(\widetilde{\alpha}_j(x))) \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\ &\quad -\sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) u^q(\widetilde{\beta}_k(x)) w_2(u(\widetilde{\beta}_k(x))) \beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}, \\ &\geq z^{q/p}(x)\left[-\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) w_2(z^{1/p}(x))'_{j1}\alpha'_{j2}\cdots\alpha'_{jn}\right. \\ &\quad \left. -\sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_2(z^{1/p}(x)) \beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}\right], \end{aligned} \tag{3.39}$$

for all $x \in [y, \widetilde{\infty})$

$$z^{q/p}(x) \geq z^{q/p}(\widetilde{\infty}) = z^{q/p}(c(y)) > 0. \tag{3.40}$$

Using (3.39) and (3.40), we have

$$\begin{aligned} \frac{D_1 D_2 \cdots D_n z(x)}{z^{q/p}(x)} &\geq -\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) w_2(z^{1/p}(x))'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\ &\quad -\sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_2(z^{1/p}(x)) \beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}, \end{aligned}$$

using similar procedures as is Lemma 3.1, for all $x \in [y, \widetilde{\infty})$ we obtain

$$\begin{aligned} z^{(p-q)/p}(x) &\leq c^{(p-q)/p}(y) + \frac{p-q}{p}\sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t) w_2(z^{1/p}(t))dt \\ &\quad + \frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t) w_2(z^{1/p}(t))dt. \end{aligned} \tag{3.41}$$

Setting $r_1(x) = z^{(p-q)/p}(x)$,(3.41) can be rewritten as

$$\begin{aligned} r_1(x) &\leq c^{(p-q)/p}(y) + \frac{p-q}{p}\sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t) w_2(r_1^{1/(p-q)}(t))dt \\ &\quad + \frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(y,t) w_2(r_1^{1/(p-q)}(t))dt, \end{aligned} \tag{3.42}$$

Now applying Theorem 3.2 to (3.42), and since y are arbitrary numbers $\zeta_1 \leq y,$ the desired bound for $u(x)$ appeared in (3.26) directly. By continuity, (3.26) also holds for the case $c(x) \geq 0$. ∎

Remark 3.5. Theorem 3.3 reduced to [11, Theorem 2.2] in the case of one variable (with an infinite integration limits), when $b_k(x,t) = 0$, $w_1(t) = 1$, $j = 1$ and $n = 1$.

Theorem 3.4. *Let the functions $u, c, a, b, w, \widetilde{\alpha}_j$ and $\widetilde{\beta}_k$ $(j = 1, 2, \ldots, n_1,\ k = 1, 2, \ldots, n_2)$ be defined as in Theorem 3.3. Moreover, Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a strictly increasing function with $\lim_{x\to\infty}\varphi(x) = \infty$, and $\Phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing function with $\Phi(x) > 0$ for* *all* $x \in \mathbb{R}_+^n$. *Let* $d_j(x)$ *and* $l_k(x)$ *be nondecreasing continuous functions for all* $x \in \mathbb{R}_+^n$. *If* $u \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *and*

$$
\begin{aligned}
\varphi(u(x)) \;\leq\; & c(x) + \sum_{j=1}^{n_1} f_j(x) \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)\Phi(u(t))\,dt \\
& + \sum_{k=1}^{n_2} g_k(x) \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)\Phi(u(t))\,w(u(t))dt,
\end{aligned}
\tag{3.43}
$$

then for any $x, t \in \mathbb{R}_+^n$ *, there exists* $x^* \in \mathbb{R}_+^n$ *, shut as for all* $0 \leq x^* \leq x < \infty$,*we have*

$$
u(x) \leq \varphi^{-1}\left(G^{-1}\left[\Omega_1^{-1}\left(\Omega_1(\eta(x)) + \sum_{k=1}^{n_2} g_k(x) \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)dt \right)\right]\right). \tag{3.44}
$$

Where

$$
\eta(x) = G(c(x)) + \sum_{j=1}^{n_1} f_j(x) \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)dt, \tag{3.45}
$$

$$
G(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{\Phi(\varphi^{-1}(s))}, \qquad \delta > \delta_0 > 0, \tag{3.46}
$$

$$
\Omega_1(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w\left(\varphi^{-1}(G^{-1}(s))\right)}, \qquad \delta > \delta_0 > 0. \tag{3.47}
$$

Here Ω_1^{-1} *is the inverse of* Ω_1, *on condition that* $\Omega_1(+\infty) = +\infty$, *and the real numbers* $x^* \in \mathbb{R}_+^n$ *are chosen so that* $\Omega_1(\eta(x)) + \int_{\widetilde{\beta}(x)}^{\widetilde{\infty}} b(x,t)dt \in Dom(\Omega_1^{-1})$, $\Omega_1^{-1}\left(\Omega_1(\eta(x)) + \int_{\widetilde{\beta}(x)}^{\widetilde{\infty}} b(x,t)dt\right) \in Dom(G^{-1})$ *and* $G^{-1}\left[\Omega_1^{-1}\left(\Omega_1(\eta(x)) + \int_{\widetilde{\beta}(x)}^{\widetilde{\infty}} b(x,t)dt\right)\right] \in Dom(\varphi^{-1})$ *for all* $x \in [x^*, \infty)$.

Proof. If $c(x) > 0$ for all $x \in \mathbb{R}_+^n$, fixing arbitrary $y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$ with $0 \leq x^* \leq y \leq x < \infty$, we define on $[y, \widetilde{\infty})$ a function $z(x)$ by

$$
z(x) = c(y) + \sum_{j=1}^{n_1} f_j(y) \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y,t)\Phi(u(t))\,dt + \sum_{k=1}^{n_2} g_k(y) \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)\Phi(u(t))\,w(u(t))dt, \tag{3.48}
$$

then $z(x)$ is positive and nonincreasing function in each variables $x_i \in [y_i, +\infty)$, and

$$
u(x) \leq \varphi^{-1}(z(x)), \qquad x \in [y; \widetilde{\infty}). \tag{3.49}
$$

We know that differentiating (3.48) and using (3.49) and the monotonicity of φ , Φ and w, we deduce that

$$
\begin{aligned}
D_1 D_2 \cdots D_n z(x) \;=\; & (-1)^n \sum_{j=1}^{n_1} f_j(y) a_j(y, \widetilde{\alpha}_j(x)) \Phi(u(\widetilde{\alpha}_j(x))) \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn} \\
& + (-1)^n \sum_{k=1}^{n_2} g_k(y) b_k(y, \widetilde{\beta}_k(x)) \Phi(u(\widetilde{\beta}_k(x))) w(u(\widetilde{\beta}_k(x))) \beta'_{k1}\beta'_{k2}\cdots\beta'_{kn},
\end{aligned}
$$

in the case of n is even, and since $\widetilde{\alpha}_j(x), \widetilde{\beta}_k(x) \geq x$ for all $j = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$, we have $z(\widetilde{\alpha}_j(x)) \leq z(x)$ and $z(\widetilde{\beta}_k(x)) \leq z(x)$, then

$$\begin{aligned} D_1 D_2 \cdots D_n z(x) &= -\sum_{j=1}^{n_1} f_j(y) a_j(y, \widetilde{\alpha}_j(x)) \Phi(u(\widetilde{\alpha}_j(x))) \alpha'_{j1} \alpha'_{j2} \cdots \alpha'_{jn} \\ &\quad - \sum_{k=1}^{n_2} g_k(y) b_k(y, \widetilde{\beta}_k(x)) \Phi(u(\widetilde{\beta}_k(x))) w(u(\widetilde{\beta}_k(x))) \beta'_{k1} \beta'_{k2} \cdots \beta'_{kn}, \\ &\geq \Phi\left(\varphi^{-1}(z(x)\right) \left[-\sum_{j=1}^{n_1} f_j(y) a_j(y, \widetilde{\alpha}_j(x)) \alpha'_{j1} \alpha'_{j2} \cdots \alpha'_{jn} \right. \\ &\quad \left. - \sum_{k=1}^{n_2} g_k(y) b_k(y, \widetilde{\beta}_k(x)) w(\varphi^{-1}(z(x)) \beta'_{k1} \beta'_{k2} \cdots \beta'_{kn} \right], \end{aligned} \tag{3.50}$$

for all $x \in [y, \widetilde{\infty})$ and $j = 1, 2, \ldots, n_1$ and $k = 1, 2, \ldots, n_2$

$$\Phi\left(\varphi^{-1}(z(x)\right) \geq \Phi\left(\varphi^{-1}(z(\widetilde{\infty})\right) = \Phi(c(y)) > 0. \tag{3.51}$$

Using (3.50) and (3.51), we obtain

$$\begin{aligned} \frac{D_1 D_2 \cdots D_n z(x)}{\Phi(\varphi^{-1}(x))} &\geq -\sum_{j=1}^{n_1} f_j(y) a_j(y, \widetilde{\alpha}_j(x)) \alpha'_{j1} \alpha'_{j2} \cdots \alpha'_{jn} \\ &\quad - \sum_{k=1}^{n_2} g_k(y) b_k(y, \widetilde{\beta}_k(x)) w(\varphi^{-1}(z(x))) \beta'_{k1} \beta'_{k2} \cdots \beta'_{kn}, \end{aligned}$$

using similar procedures as in the proof of Lemma 3.1, we obtain

$$\begin{aligned} G(c(y)) - G(z(x)) &\geq -\sum_{j=1}^{n_1} f_j(y) \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(y, t) dt \\ &\quad - \sum_{k=1}^{n_2} g_k(y) \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b(y, \widetilde{\beta}_k(x)) w(\varphi^{-1}(z(t)) dt, \end{aligned}$$

where G is defined in (3.46), for all $x \in [y, \widetilde{\infty})$, we have

$$\begin{aligned} G(z(x)) &\leq G(c(y)) + \sum_{j=1}^{n_1} f_j(y) \int_{\widetilde{\alpha}_j(y)}^{\widetilde{\infty}} a_j(y, t) dt \\ &\quad + \sum_{k=1}^{n_2} g_k(y) \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b(y, \widetilde{\beta}_k(x)) w(\varphi^{-1}(z(t)) dt. \end{aligned}$$

Setting $r_1(x) = G(z(x))$,(3.52) can be rewritten as

$$r_1(x) \leq \eta(y) + \sum_{k=1}^{n_2} g_k(y) \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b(y, \widetilde{\beta}_k(x)) w(\varphi^{-1}(z(t)) dt. \tag{3.52}$$

Where η is defined in (3.45). Now applying Theorem 3.2 to (3.52), we get

$$z(y) \leq G^{-1}\left(\Omega_1^{-1}\left[\Omega_1(\eta(y)) + \int_{\widetilde{\beta}(y)}^{\widetilde{\infty}} b(y, t) dt \right] \right).$$

Since y are arbitrary numbers $x^* \leq y$, the desired bound for $u(x)$ appeared in (3.44) directly. By continuity, (3.44) also holds for the case $c(x) \geq 0$. ∎

Remark 3.6. Under the same hypothesis as in the previous Theorem, we can obtain easily the estimation of the following inequality

$$\begin{aligned} \varphi\left(u(x)\right) &\leq c(x) + \sum_{j=1}^{n_1} f_k(x) \int_{\widetilde{\alpha}_j(x)}^{\widetilde{\infty}} a_j(x,t)\Phi\left(u(t)\right) w_1(u(t))dt \\ &\quad + \sum_{k=1}^{n_2} g_k(x) \int_{\widetilde{\beta}_k(x)}^{\widetilde{\infty}} b_k(x,t)\Phi\left(u(t)\right) w_2(u(t))dt, \end{aligned} \tag{3.53}$$

for any $x, t \in \mathbb{R}^n_+$. Details are omitted here.

Remark 3.7. As one can see, the established results above mainly deal with Gronwall-Bellman type integral inequalities involving infinite integral for functions with n independent variables. And they are different from the main results presented in [15]. It is interesting to note that in the special case $n = 1(\mathbb{R}_+)$ and $j = k = 1$ when $c(x_1, \ldots, x_n) = c$ (positive constant), and $a_1(x_1, \ldots, x_n, t_1, t_2, \ldots, t_n) = f(x)$, $b_1(x_1, \ldots, x_n, t_1, t_2, \ldots, t_n) = g(x)$ and $(\alpha_{j1}(x_1), \ldots, \alpha_{jn}(x_n)) = (\beta_{k1}(x_1), \ldots, \beta_{k1}(x_n)) = \alpha(x)$, for all $x \in \mathbb{R}_+$ and $\Phi(u) = u$ then the inequality (3.43) reduces to the Zhao and Meng main result in [15, Theorem 3.1] (Inequality (1.5) in the case of one variable).

Remark 3.8. As in Corollaries 2.3, other new Gronwall-Bellman type integral inequalities of one and two variables can be obtained from Theorems 3.3 and 3.4 by choosing suitable functions for φ,w and Φ. Details are omitted here.

4. AN APPLICATION

In this section, motivated by the works in [9, 6], we give the boundedness of the solutions of Volterra-Fredholm integral equation with delay and infinity upper limit

$$\begin{aligned} u^p(x,y) &= c(x,y) + \int_{\alpha_1(x)}^{\infty} \int_{\alpha_2(y)}^{\infty} K_1(x,y,s,t,u(s,t))dsdt \\ &\quad + \int_{\beta_1(x)}^{\infty} \int_{\beta_2(x)}^{\infty} K_2(x,y,s,t,u(s,t)dsdt. \end{aligned} \tag{4.1}$$

Where $u(x,y), c(x,y) \in C(\mathbb{R}^2_+, \mathbb{R})$, with $\alpha_i, \beta_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions such that $\alpha_i(x) \geq x$, $\alpha_i(+\infty) = +\infty$ $(\beta_i(x) \geq x$, $\beta_i(+\infty) = +\infty)$ and $K_i \in C(\mathbb{R}^2_+ \times \mathbb{R}^2_+ \times \mathbb{R}, \mathbb{R})$ for $i = 1, 2$.

Proposition 4.1. *Suppose that* $u(x,y)$ *is a solution of (4.1) and the functions* $K_i \in C(\mathbb{R}^2_+ \times \mathbb{R}^2_+ \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$ *satisfy the following conditions*

$$|K_1(x,y,s,t,u(s,t))| \leq f_1(x,y,s,t)\,|u(s,t)|^{\frac{p}{2}}, \tag{4.2}$$

$$|K_2(x,y,s,t,u(s,t))| \leq f_2(x,y,s,t)\,|u(s,t)|^{\frac{p}{2}}. \tag{4.3}$$

Where $f_i \in C(\mathbb{R}^2_+ \times \mathbb{R}^2_+, \mathbb{R}_+)$ $i = 1, 2$, *are nondecreasing function in* x, y *for every* s, t *fixed, then we have*

$$\begin{aligned} |u(x,y)| &\leq \left\{ \sqrt{|c(x,y)|} + \frac{1}{2} \int_{\alpha_1(x)}^{\infty} \int_{\alpha_2(y)}^{\infty} f_1(x,y,s,t)dsdt \right. \\ &\quad \left. + \frac{1}{2} \int_{\beta_1(x)}^{\infty} \int_{\beta_2(y)}^{\infty} f_2(x,y,s,t)dsdt \right\}^{\frac{2}{p}}, \end{aligned} \tag{4.4}$$

Proof. By the conditions (4.2) and (4.3) , from (4.1) we have

$$
\begin{aligned}
|u(x,y)|^p &= |c(x,y)| + \int_{\alpha_1(x)}^{\infty}\int_{\alpha_2(y)}^{\infty} f_1(x,y,s,t)\ |u(s,t)|^{p/2}\,dsdt \\
&\quad + \int_{\beta_1(x)}^{\infty}\int_{\beta_2(x)}^{\infty} f_2(x,y,s,t)\,|u(s,t)|^{p/2}\,dsdt, \qquad (4.5)
\end{aligned}
$$

a suitable application of Corollary 2.3 (with $q = \frac{p}{2}$) to (4.5) yields

$$
\begin{aligned}
|u(x,y)| &\leq \left(G^{-1}\left[G(c(x,y)) + \int_{\alpha_1(x)}^{\infty}\int_{\alpha_2(y)}^{\infty} f_1(x,y,s,t)dsdt\right.\right. \\
&\quad \left.\left. + \int_{\beta_1(x)}^{\infty}\int_{\beta_2(y)}^{\infty} f_2(x,y,s,t)dsdt\right]\right)^{1/p}, \qquad (4.6)
\end{aligned}
$$

with

$$
G(z) = \int_{z_0}^{z}\frac{ds}{s^{1/2}} = 2\sqrt{z} - 2\sqrt{z_0}, \qquad z \geq z_0 > 0, \qquad (4.7)
$$

$$
G^{-1}(z) = \left(\frac{z + 2\sqrt{z_0}}{2}\right)^2, \; z \geq z_0 > 0. \qquad (4.8)
$$

From (4.6), (4.7) and (4.8) , we obtain the following estimation

$$
\begin{aligned}
|u(x,y)| &\leq \left\{\frac{G(c(x,y)) + \int_{\alpha_1(x)}^{\infty}\int_{\alpha_2(y)}^{\infty} f_1(x,y,s,t)dsdt}{2}\right. \\
&\quad \left. + \frac{\int_{\beta_1(x)}^{\infty}\int_{\beta_2(y)}^{\infty} f_2(x,y,s,t)dsdt + 2\sqrt{z_0}}{2}\right\}^{\frac{2}{p}} \\
&\leq \left\{\frac{2\sqrt{c(x,y)} - 2\sqrt{z_0} + \int_{\alpha_1(x)}^{\infty}\int_{\alpha_2(y)}^{\infty} f_1(x,y,s,t)dsdt}{2}\right. \\
&\quad \left. + \frac{\int_{\beta_1(x)}^{\infty}\int_{\beta_2(y)}^{\infty} f_2(x,y,s,t)dsdt + 2\sqrt{z_0}}{2}\right\}^{\frac{2}{p}},
\end{aligned}
$$

That is the desired estimation (4.4). ∎

Remark 4.1. By using Theorem 3.3, we can obtain estimate of solutions of Volterra-Fredholm integral equation (4.1) in $\mathbb{R}_+^n$.

5. CONCLUSION :

In this paper, we established several new retarded nonlinear Gronwall-Bellman type integral inequalities containing integration on infinite intervals in two independent variables in Theorem 2.2, Theorem 3.2 and Theorem 3.4, and gave their specific cases in Corollary 2.3, which can be used in the analysis of the qualitative properties to solutions of integral equations in n independent variables. In the last section, we also presented the application to research the boundedness of solutions of the initial boundary value problem for hyperbolic partial delay differential equations with delay.

Using our method one can further study the integral inequality containing integration on infinite intervals in two independent variables and with more dimensions.

REFERENCES

[1] A. ABDELDAIM, Nonlinear retarded integral inequalities of Gronwall-Bellman type and applications, *J. Maths. Inequalities*, **10** (2016), 285-299.

[2] A. BOUDELIOU and H. KHELLAF, On some delay nonlinear integral inequalities in two independent variables, *J. Inequal Appl.*, **14** (2015), pp. 22-313.

[3] M. DENCHE and H. KHELLAF, Integral inequalities similar to Gronwall inequality, *Electron. J. Diff. Eqns.* **176** (2007),pp. 1-14.

[4] S. S. DRAGOMIR and Y. H. KIM, Some integral inequalities for functions of two variables, *Electron. J. Diff. Eqns.*, **10** (2003), PP. 1–13.

[5] H. El-OWAIDY, A. ABDELDAIM and A. A. El-DEEB, On some new nonlinear retarded integral inequalities with iterated integrals and their applications in differential-integral equations, *Mathematical Sciences Letters journal*, **3** (2014), pp. 157-164.

[6] J. HUANG and W. S. WANG, Some Volterra-Fredholm type nonlinear inequalities involving four iterated infenite integral and application, *J. Maths. Inequalities*, **10** (2016), pp. 1105-1118.

[7] H. KHELLAF and M. SMAKDJI, Nonlinear delay integral inequalities for multi-variable functions, *Electron. J. Diff. Eqns*, **169** (2011), pp. 1-14.

[8] H. KHELLAF, M. SMAKDJI and M. DENCHE, Integral inequalities with time delay in two independent variables, *Electron. J. Diff. Eqns*, **117** (2014), pp. 1-16.

[9] Y.H. KIM, On some new integral inequalities for functions in one and two variables, *Acta Math. Sin. (Engl. Ser.)*,, **21** (2005),pp. 423-434.

[10] O. LIPOVAN, A retarded Gronwall-like inequality and its applications, *J. Math. Anal. Appl.*, **251** (2000), pp. 389-401.

[11] O. LIPOVAN, Integral inequalities for retarded Volterra equations, *J. Math. Anal. Appl.* , **322** (2006), pp. 349-358.

[12] O. LIPOVAN, A retarded integral inequality and its applications, *J. Math. Anal. Appl.*, **285** (2003), pp. 436-443.

[13] B.G. PACHPATTE, Explicit bounds on certain integral inequalities, *J. Math. Anal. Appl.*, **267** (2002), pp. 48-61.

[14] B.G. PACHPATTE, Inequalities for differential and integral equations, Academic Press, New York, 1998.

[15] X. ZHAO and F. MENG, On some advanced integral inequalities and their applications, *J. Inequal. Pure and Appl. Math.*, **6** (2005), pp. 1-8.