Aust. J. Math. Anal. Appl.
Vol. **19** (2022), No. 1, Art. 8, 5 pp.
AJMAA

THE BOUNDEDNESS OF GRADIENT SOLUTIONS OF P-LAPLACIAN TYPE

CORINA KARIM, KHOIRUNISA, RATNO BAGUS EDY WIBOWO

Received 18 January, 2022; accepted 27 April, 2022; published 27 May, 2022.

DEPARTMENT OF MATHEMATICS, UNIVERSITAS BRAWIJAYA, VETERAN STREET, MALANG, INDONESIA.
co_mathub@ub.ac.id
khrnisa@student.ub.ac.id
rbagus@ub.ac.id

ABSTRACT. In this paper, we give the boundedness of gradient solutions result for p-Laplacian systems only in the singular case. The Lebesgue space for initial data belong to guarantee the local boundedness of gradient solutions.

Key words and phrases: Boundedness of gradient; p-Laplacian systems; Singular case.

2010 *Mathematics Subject Classification.* Primary 35K67, 35K55. Secondary 35D30.

ISSN (electronic): 1449-5910

This paper partially supported by the doctoral grant No. 44/UN10.F09/PN/2020 at Mathematics and Natural Sciences Faculty, Universitas Brawijaya.

1. INTRODUCTION

In this paper, we give the p-Laplacian type equation

$$(1.1)\qquad \begin{cases} \partial_t u - \text{div}(|Du|^{p-2}Du) = 0 & \text{di } (0,T)\times\Omega \\ u(0,x) = u_0(x) & \text{on } \partial_p(0,T)\times\Omega \end{cases},$$

where Ω is a bounded domain in $\mathbb{R}^m, m \geq 2$, with smooth boundary $\partial\Omega$, $\frac{2m}{m+2} < p < 2, u : (0,T)\times\Omega \to \mathbb{R}^n$, and $u = (u_i), i = 1, 2, ..., n$ be a vectorial function on parabolic cylinders Q.

In 2017, a weak solution of (1.1) is constructed by using a varitional (like) method as in [8] and [11] was construct a weak solution of (1.1) for singular case by using Galerkin method. The Hölder regularity for parabolic systems (1.1) , where u is real valued and scalar case, goes back to the fundamental work of DiBenedetto and Friedman, see [1, 2, 3, 4].

Moreover, [13] was studied the local boundedness for u is a vectorial case, the technique used resembles the classical approach going back to Campanato, to prove local Schauder estimates. Then the kind of regularity enjoyed by the solutions of the comparison problems is inherited by the original solution by mean of a delicate comparison technique. This is indeed the crucial technical point. While in the normal elliptic case this is done on a sequence of standard shrinking balls, in the parabolic case, this must be done in a very careful way. Namely, the so-called intrinsic geometry must be used. This notion, introduced and deeply studied by DiBenendetto in the 80s, prescribes that the cylinders used are stretched in the time direction by a factor that depends on the solution itself. This allows to rebalance the inohomogeneity of the equation considered, see [12, 6, 7].

However, the local Hölder regularity of weak solutions of (1.1) was studied by [10] for Degenerate parabolic type only. The emphasis here is the boundedness of gradient of weak solution of (1.1) only for singular case, as continuity from our pevious paper (see [9]), in which the intrinsic geometry ([5, 2, 3]) changes its type.

We consider the definition of weak solution as below.

Definition 1.1. A function u is a weak solution of (1.1), if and only if $u \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^n))$ $\cap L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^n))$ and satisfies

$$(1.2)\qquad \int_{(0,T)X\Omega} \partial_t u\,\varphi + |Du|^{p-2}Du\cdot D\varphi\,dz = 0,$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^n))$ with $\partial_t\varphi \in L^2(Q,\mathbb{R}^n)$ and $T > 0$.

Our main theorem is the following:

Theorem 1.1. *If u is a weak solution of (1.1) and $\frac{2m}{m+2} < p < 2$, then there exist $C_1 > 0$ and $C_2 > 0$ such that for all $Q(\rho^2, \lambda^{\frac{p-2}{2}}\rho)(z_o) \subset Q$, $\sigma = \frac{2\tau}{1+\tau}$ and $0 < \tau < \frac{1}{2}$*

$$(1.3)\qquad \sup_{Q((\sigma\rho)^2,\lambda^{\frac{p-2}{2}}\sigma\rho))} |Du| \leq C_1\,\lambda^{\frac{m(p-2)}{p(m+2)-2m}} \left(⨍_{Q(\rho^2,\lambda^{\frac{p-2}{2}}\rho)} |Du|^p\,dz \right)^{\frac{2}{p(m+2)-2m}} + C_2\lambda,$$

holds, where C_1 depends on m, p, σ and C_2 is arbitary constant.

2. RESULTS

Let u is a weak solution of (1.1) in $Q(\rho^2, \lambda^{\frac{p-2}{2}}\rho)(z_o) \subset Q$, and set the intrinsic geometry for $\frac{2m}{m+2} < p < 2$,

$$t = t_0 + \rho^2 s, \quad x = x_0 + \lambda^{\frac{p-2}{2}} \rho y, \quad \tilde{z} = (s, y), \tag{2.1}$$

$$v(s,y) = \frac{u(t_0 + \rho^2 s, x_0 + \lambda^{\frac{p-2}{2}} \rho y)}{\lambda^{\frac{p}{2}} \rho}, \quad 0 \leq \rho < 1, \tag{2.2}$$

hence the equation (1.1) in $Q(\rho^2, \lambda^{\frac{p-2}{2}} \rho)(z_o)$, can be written to the equation in $Q(1,1)(0,0)$:

$$\partial_t v - \text{div}(|Dv|^{p-2} Dv) = 0. \tag{2.3}$$

Next, we will proof of our **Theorem 1.1**

Proof. Set our intrinsic geometry as in (2.1) and (2.2) then the equation (1.1) in $Q(\rho^2, \lambda^{\frac{p-2}{2}} \rho)(z_o)$, (1.1) can be reduced to equation (2.3) in $Q(1,1)(0,0)$ such that

$$\sup_{Q(\sigma^2,\sigma)} |Dv| \leq C \left(\fint_{Q(1,1)} |Dv|^p \, dz \right)^{\frac{2}{p(m+2)-2m}} + C. \tag{2.4}$$

Let take the testing function $\varphi = v\eta^p \xi$, where η is a linear cutoff function, such that $\eta = 1$ in $B(r)$, supp$(D\eta) \subset B(1)$ and $0 \leq \partial_t \xi \leq \frac{C}{\rho^2 - r^2}$. By using the reverse Poincaré's inequality Lemma and the Sobolev inequality, we have the following reverse Hölder inequality,

$$\begin{aligned} \int_{Q(r)} |Dv|^{\frac{\alpha+p}{2} + \frac{m+2}{2m}(\alpha+2)} \, dz &\leq \int_{-r^2}^{0} \left(\int_{B(r)} |Dv|^{\frac{m+2}{2m}(\alpha+2)\frac{2m}{m+2}} \, dx \right)^{\frac{m+2}{2m}} \times \\ &\quad \left(\int_{B(r)} (|Dv|^{\frac{\alpha+p}{2}})^{\frac{2m}{m-2}} \, dx \right)^{\frac{m-2}{2m}} dt \\ &\leq \sup_{-r^< t<0} \left(\int_{B(r)} |Dv|^{(\alpha+2)} \, dx \right)^{\frac{m+2}{2m}} \times \\ &\quad \int_{-r^2}^{0} \left(\int_{B(r)} (|Dv|^{\frac{\alpha+p}{2}})^{\frac{2m}{m-2}} \, dx \right)^{\frac{m-2}{2m}} \end{aligned} \tag{2.5}$$

We let

$$\begin{aligned} &\alpha_k = \frac{p(m+2) - 2m}{2}(1 + \frac{1}{m})^k - \frac{p(m+2) - 2m}{2} + p - 2; \;\; \theta = 1 + \frac{1}{m}; \\ &R_k = \sigma + \frac{1-\sigma}{2^k}; \;\; R_0 = 1. \end{aligned}$$

Choose $r = \frac{\rho}{2}$ and make iteration on $k = 0, 1, 2, ...,$ we have

$$\left(\frac{1}{|Q(R_{k+1})|}\int_{Q(R_{k+1})}|Dv|^{\alpha_{k+1}+2}\,dz+1\right)^{\frac{1}{\theta^{k+1}}} \le C\frac{|Q(R_k)|^{\frac{1}{\theta^k}}}{|Q(R_{k+1})|^{\frac{1}{\theta^{k+1}}}}\frac{(\alpha_k+2)^{\frac{3}{\theta^k}}}{(R_k-R_{k+1})^{\frac{2}{\theta^k}}} \times\left(\frac{1}{|Q(R_k)|}\int_{Q(R_k)}|Dv|^{\alpha_k+2}\,dz+1\right)^{\frac{1}{\theta^k}}$$

$$\vdots$$

$$\le\prod_{i=0}^{\infty}C\frac{|Q(R_i)|^{\frac{1}{\theta^i}}}{|Q(R_{i+1})|^{\frac{1}{\theta^{i+1}}}}\frac{(\alpha_i+2)^{\frac{3}{\theta^i}}}{(R_i-R_{i+1})^{\frac{2}{\theta^i}}} \times\left(\frac{1}{|Q(R_0)|}\int_{Q(R_0)}|Dv|^{\alpha_0+2}\,dz+1\right)^{\frac{1}{\theta^0}}. \tag{2.6}$$

In fact, we have

$$\alpha_i=\frac{p(m+2)-2m}{2}(1+\frac{1}{m})^i-\frac{p(m+2)-2m}{2}+p-2;\;\; \theta=1+\frac{1}{m};$$
$$R_i=\sigma+\frac{1-\sigma}{2^i};\;\; R_0=1,$$

The constant in (2.6) can be evaluated as

$$\begin{aligned}\overline{C}(m,p,\sigma)=&\prod_{i=0}^{\infty}C\frac{|Q(R_i)|^{\frac{1}{\theta^i}}}{|Q(R_{i+1})|^{\frac{1}{\theta^{i+1}}}}\frac{(\alpha_i+2)^{\frac{3}{\theta^i}}}{(R_i-R_{i+1})^{\frac{2}{\theta^i}}}\\ \le&\prod_{i=0}^{\infty}(C_1)^{\sum_{i=1}^{\infty}\frac{1}{\theta^i}}\left(1+\frac{1}{m}\right)^{\sum_{i=1}^{\infty}\frac{3i}{\theta^i}}(2)^{\sum_{i=1}^{\infty}\frac{2i}{\theta^i}}\left(\sigma+\frac{1-\sigma}{2^i}\right)^{\frac{m+2}{\theta^i}}\left(\sigma+\frac{1-\sigma}{2^{i+1}}\right)^{-\frac{m+2}{\theta^{i+1}}}\\ =&(C_1)^{m+2}\left(1+\frac{1}{m}\right)^{3\tilde{c}}(2)^{2\tilde{c}}(1)^{m+2},\end{aligned}\tag{2.7}$$

where $C_1=C2^8(1-\sigma)^{-2}$ and

$$\lim_{i\to\infty}\frac{(i+1)}{\theta^{i+1}}\Big/\frac{i}{\theta^i}=\frac{1}{\theta}<1.$$

Thus for all i it holds that

$$\left(\fint_{Q(R_i)}|Dv|^{\alpha_i+2}dz\right)^{\frac{1}{\theta^i}}\le\overline{C}\fint_{Q(R_0)}|Dv|^p\,dz+C. \tag{2.8}$$

In fact, we use

$$\lim_{i\to\infty}\frac{\alpha_i+2}{\theta^i}=\frac{p(m+2)-2m}{2}.$$

From this estimate, we conclude that for any $\frac{2m}{m+2}<p<2$

$$\sup_{Q(\sigma^2,\sigma)}|Dv|\le\overline{C}\left(\fint_{Q(1)}|Dv|^p dz\right)^{\frac{2}{p(m+2)-2}}+C. \tag{2.9}$$

If we use scaling in (2.2) then (2.9) becomes to (1.3).

REFERENCES

[1] H. CHOE, Hölder regularity for the gradient of solutions of certain singular parabolic systems, Commun. *Partial Differential Equations*, **16** (11) (1991), pp. 1709-1732.

[2] E. DIBENEDETTO and A. Friedman, Regularity of solutions of nonlinear degenerate parabolic systems, *J. reine angew. Math.*, **349** (1984), pp. 83-128.

[3] E. DIBENEDETTO and A. Friedman, Hölder estimates for nonlinear degenerate parabolic systems, *J. Reine Angew. Math.*, **357** (1985), pp. 1-22.

[4] E. DIBENEDETTO and A. Friedman, Adendum to Hölder estimates for nonlinear degenerate parabolic systems", *J. reine angew. Math.*, **363** (1985), pp. 217-220.

[5] E. DIBENEDETTO, *Degenerate Parabolic Equations, Universitext*, Springer-Verlag, New York, NY, pp. xv+387. 1993.

[6] C. KARIM and M. MISAWA, Hölder regularity for singular parabolic systems of *p*-Laplacian type, *Advances in Differential Equations*, **vol.20** (7-8) (2015), pp. 741-772.

[7] C. KARIM and M. MASAHI, Gradient Hölder regularity for nonlinear parabolic systems of *p*-Laplacian type, *Differential and Integral Equations*, **vol.29** (3-4) (2016), pp. 201-228.

[8] C. KARIM and M. MASAHI, Existence of global weak solutions for Cauchy-Dirichlet problem for evolutional *p*-Laplacian systems, *AIP Conference Proceedings*, **1913** (2017), pp. 1-4.

[9] C. KARIM, Local Boundedness of weak solutions for singular parabolic systems of *p*-Laplacian type, *The Australian Journal of Mathematical Analysis and Applications*, **vol. 15**, Issue 2, Article 8 (2018), pp.1-5.

[10] C. KARIM, Local Hölder Regularity of weak solutions for Degenerate parabolic type, *AIP Conference Proceedings*, **2192** (2019), pp. 1-4.

[11] P.R. AKBAR, C. KARIM, R. B. E. WIBOWO, Existence of global weak solutions for Singular Parabolic Systems of *p*-Laplacian Type, *Journal of Physics: Conference Series*, **1562** (2020), pp. 1-8.

[12] M. MISAWA, A Hölder estimate for nonlinear parabolic systems of *p*-Laplacian type, *J. Differential Equations*, **254** (2013), pp. 847-878.

[13] M. MISAWA, Local Hölder regularity of gradients for evolutional p-Laplacian systems, *Annali di Matematica*, **181** (2002), pp. 389-405.