Aust. J. Math. Anal. Appl.
Vol. **19** (2022), No. 1, Art. 3, 14 pp.
AJMAA

ON TRIGONOMETRIC APPROXIMATION OF CONTINUOUS FUNCTIONS BY DEFERRED MATRIX MEANS

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Received 14 April, 2021; accepted 13 January, 2022; published 10 February, 2022.

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ABSTRACT. In this paper, for the first time, we introduce the deferred matrix means which contain the well-known generalized deferred Nörlund, deferred Nörlund, deferred Riesz, deferred Cesàro means introduced earlier by others, and a new class of sequences (predominantly a wider class than the class of Head Bounded Variation Sequences). In addition, using the deferred matrix means of Fourier series of a continuous function, we determine the degree of approximation of such function via its modulus of continuity and a positive mediate function.

Key words and phrases: Deferred matrix transformation; Degree of approximation; Fourier series; Modulus of continuity.

2010 *Mathematics Subject Classification.* Primary 42A10. Secondary 41A25.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND MOTIVATION

The degree of approximation of 2π-periodic and continuous functions, using various means of partial sums of its Fourier series, has been studied by Sahney and Goel in [21], by Holland and Sahney in [6], and by second author of this paper in [2]-[4]. In 2004, Leindler [15] replaced the monotonicity conditions in results of Chandra presented in [4]. In his paper, Leindler summed up all results in one theorem, but their proofs were given separately. Namely, he generalized Chandra's results using a broader class of matrices $(a_{n,k})$. These results are extended further in [7], they are generalized in [8], and are treated in [23] as well, where some assumptions are removed, but the obtained results have a weaker degree of approximation. Very recently, see [9] and [10], we have studied the same topic, proving similar but different results, using the so-called the generalized deferred Voronoi-Nörlund means of partial sums of their Fourier series (see also [12], [13]). Also, we have to mention the results of Németh [20] who, in the sense a slightly wider class of functions, complementing Chandra and Leindler's results for $0 < \alpha \leq 1$. In fact, he considered the limit missing case $\alpha = 0$ using a specific modulus of continuity. Once more, was Leindler [17] who showed, as an example, that integral-type conditions can be replaced by sequence-type assumptions to Chandra's theorems and its generalizations.

We will not write all results obtained by others, however for our purpose we are going to write only those that are connected directly to those lately presented in this research paper.

Let $f(x)$ be a 2π- periodic continuous function. Let $s_n(f) := s_n(f;x)$ denote the n-th partial sum of its Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

at x, i.e.

$$s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx),$$

and let $\omega(\delta) = \omega(\delta, f)$ denote the modulus of continuity of f.

Let $A := (a_{n,k})$ $(k, n = 0, 1, \dots)$ be a lower triangular infinite matrix of real numbers and let the A-transform of $\{s_n(f;x)\}$ be given by

$$T_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} s_k(f;x) \quad (n = 0, 1, \dots).$$

The deviation

$$\|T_{n,A}(f) - f\| = \sup_{0 \leq x \leq 2\pi} |T_{n,A}(f;x) - f(x)|$$

was estimated in [3] and [4] for monotonic sequences $\{a_{n,k}\}_{k=0}^{n}$ in next four theorems (we will not recall all results mentioned at the beginning of this section).

Theorem 1.1. *Let $\{a_{n,k}\}$ satisfy the following conditions:*

$$a_{n,k} \geq 0 \qquad \text{and} \qquad \sum_{k=0}^{n} a_{n,k} = 1, \tag{1.1}$$

$$a_{n,k} \leq a_{n,k+1} \quad (k = 0, 1, \dots, n-1; n = 0, 1, \dots). \tag{1.2}$$

Suppose $\omega(t)$ is such that

$$\int_u^{\pi} t^{-2}\omega(t)dt = O\left(H(u)\right) \quad (u \to +0), \tag{1.3}$$

where $H(u) \geq 0$ *and*

$$\int_0^t H(u)du = O\left(tH(t)\right) \quad (t \to +0). \tag{1.4}$$

Then

$$\|T_{n,A}(f) - f\| = O\left(a_{n,n} H(a_{n,n})\right).$$

Theorem 1.2. *Let (1.1), (1.2) and (1.3) hold. Then*

$$\|T_{n,A}(f) - f\| = O\left(\omega(\pi/n)\right) + O\left(a_{n,n} H(\pi/n)\right).$$

If, in addition, $\omega(t)$ *satisfies (1.4) then*

$$\|T_{n,A}(f) - f\| = O\left(a_{n,n} H(\pi/n)\right).$$

Theorem 1.3. *Let us assume that (1.1) and*

$$a_{n,k} \geq a_{n,k+1} \quad (k = 0, 1, \ldots, n-1;\ n = 0, 1, \ldots) \tag{1.5}$$

hold. Then

$$\|T_{n,A}(f) - f\| = O\left(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1} \omega(\pi/k) \sum_{r=0}^{k+1} a_{n,r}\right).$$

Theorem 1.4. *Let (1.1), (1.3), (1.4) and (1.5) hold. Then*

$$\|T_{n,A}(f) - f\| = O\left(a_{n,0} H(a_{n,0})\right).$$

To reveal our intention, we need firstly to write some notations, notions and conditions, to be used later on this paper.

Let $c = \{c_n\}$ and $b = \{b_n\}$ be sequences of non-negative integers with conditions

$$c_n < b_n, \quad n = 1, 2, \ldots$$

and

$$\lim_{n \to \infty} b_n = +\infty.$$

We consider $A_0 := (a_{b_n,k})$ to be a lower triangular infinite matrix of real numbers such that

$$a_{b_n,k} \geq 0 \qquad \text{and} \qquad \sum_{k=c_n+1}^{b_n} a_{b_n,k} = 1, \tag{1.6}$$

and the A_0-transform of $\{s_k(f;x)\}$ given by

$$T^{c,b}_{n,A_0}(f;x) := \sum_{k=c_n+1}^{b_n} a_{b_n,k} s_k(f;x).$$

The transformation (mean) $T^{c,b}_{n,A_0}(f;x)$ will be called *deferred matrix* transformation (mean) and the reason of this label will be clarified in the sequel. For this, taking

$$a_{b_n,k} = \begin{cases} \frac{p_{b_n-k}}{P_0^{b_n-c_n-1}} & \text{if } c_n + 1 \leq k \leq b_n \\ 0 & \text{elsewhere}, \end{cases}$$

where $P_0^{b_n-c_n-1} := \sum_{k=0}^{b_n-c_n-1} p_k \neq 0$, in $T^{c,b}_{n,A_0}(f;x)$, we obtain the deferred Nörlund transformation

$$DN^{c,b}_n(f;x) = \frac{1}{P_0^{b_n-c_n-1}} \sum_{k=c_n+1}^{b_n} p_{b_n-k} s_k(f;x)$$

(see [5]).

Also, taking

$$a_{b_n,k} = \begin{cases} \frac{p_k}{P^{b_n}_{c_n+1}} & \text{if } c_n+1 \le k \le b_n \\ 0 & \text{elsewhere}, \end{cases}$$

where $P^{b_n}_{c_n+1} := \sum_{m=c_n+1}^{b_n} p_m \neq 0$, in $T^{c,b}_{n,A_0}(f;x)$, we obtain the deferred Riesz transformation (see [5])

$$DR^{c,b}_n(f;x) = \frac{1}{P^{b_n}_{c_n+1}} \sum_{k=c_n+1}^{b_n} p_k s_k(f;x).$$

Another deferred transformation has been introduced recently, which indeed serves as an example (particular case) of deferred matrix means $T^{c,b}_{n,A_0}(f;x)$. Namely, the convolution of two sequences $\{p_n\}$ and $\{q_n\}$ of non-negative real numbers is defined by

$$R^{c,b;p,q}_n := \sum_{k=c_n+1}^{b_n} p_k q_{b_n-k} \neq 0.$$

Whence, taking

$$a_{b_n,k} = \begin{cases} \frac{p_{b_n-k} q_k}{R^{c,b;p,q}_n} & \text{if } c_n+1 \le k \le b_n \\ 0 & \text{elsewhere}, \end{cases}$$

in $T^{c,b}_{n,A_0}(f;x)$, we obtain the generalized deferred Nörlund transformation

$$DR^{c,b;p,q}_n(f;x) = \frac{1}{R^{c,b;p,q}_n} \sum_{k=c_n+1}^{b_n} p_{b_n-k} q_k s_k(f;x)$$

see [22].

Furthermore, taking

$$a_{b_n,k} = \begin{cases} \frac{1}{b_n-c_n} & \text{if } c_n+1 \le k \le b_n \\ 0 & \text{elsewhere}, \end{cases}$$

in $T^{c,b}_{n,A_0}(f;x)$, we obtain the deferred Cesàro transformation

$$DC^{c,b}_n(f;x) = \frac{1}{b_n-c_n} \sum_{k=c_n+1}^{b_n} s_k(f;x)$$

(see [1], page 414), which obviously justifies the label *deferred matrix means* for $T^{c,b}_{n,A_0}(f;x)$ which clearly its introduce is motivated based on all above examples.

Next two classes of numerical sequences are introduced in [14].

A sequence $\mathbf{w} := \{w_n\}$ of non-negative numbers tending to zero is called of *Rest Bounded Variation*, or briefly $\mathbf{w} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |w_n - w_{n+1}| \le K(\mathbf{w}) w_m$$

for all natural numbers m, where $K(\mathbf{w})$ is a positive constant depending only on $\mathbf{w}$.

A sequence $\mathbf{w} := \{w_n\}$ of non-negative numbers will be called of *Head Bounded Variation*, or briefly $\mathbf{w} \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |w_n - w_{n+1}| \le K(\mathbf{w}) w_m$$

for all natural numbers m, or only for all $m \leq N$ if the sequence $\mathbf{w}$ has only finite nonzero terms, and the last nonzero term is w_N.

Now we recall the following class of sequences (see [15]).

A sequence $\mathbf{w} := \{w_k\}$ of non-negative numbers tending to zero belongs to $RBVS_+^{r,\delta}$ class, if it has the property

$$\sum_{k=m}^{\infty} |w_k - w_{k+1}| \leq \frac{K(\mathbf{w})}{m^{r+1+\delta}} \sum_{n=1}^{m} n^{r+1} w_n$$

for all natural numbers m, where $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and $K(\mathbf{w})$ is a positive constant depending only on sequence $\mathbf{w}$.

The $RBVS_+^{r,\delta}$ class has been introduced by Leindler [16], who showed that it is a wider class than the R_0^+BVS class. In fact, if $0 < \delta \leq 1$ and $w \in R_0^+BVS$, then $w \in RBVS_+^{r,\delta}$ also holds true. Indeed,

$$w_m \leq m^{1-\delta} w_m \leq K(w) m^{-r-1-\delta} \sum_{n=1}^{m} n^{r+1} w_n.$$

Subsequently, the embedding relations

$$R_0^+BVS \subset RBVS_+^{r,\delta} \tag{1.7}$$

and

$$HBVS \subset HBVS_+^{r,\delta} \tag{1.8}$$

hold true (see the class $HBVS_+^{r,\delta}$ in sequel).

Moreover, we introduce another class of numerical sequences (motivated from the definition of $HBVS$ class). A sequence $\mathbf{w} := \{w_n\}$ of non-negative numbers belongs to $HBVS_+^{r,\delta}$, if it has the property

$$\sum_{n=0}^{m-1} |w_n - w_{n+1}| \leq \frac{K(\mathbf{w})}{m^{r+1+\delta}} \sum_{n=1}^{m} n^{r+1} w_n$$

for all natural numbers m, or only for all $m \leq N$ if the sequence $\mathbf{w}$ has only finite nonzero terms, and the last nonzero term is w_N, where $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$.

It is clear that condition $0 < K(\mathbf{w}) \leq K < \infty$ needs to be assumed, where K is a positive constant. Conditions to be assumed on the entries $a_{b_n,k}$ of the matrix A_0 are that for all b_n and $1 \leq m \leq b_n$,

$$\sum_{k=m}^{\infty} |a_{b_n,k} - a_{b_n,k+1}| \leq \frac{K}{m^{r+1+\delta}} \sum_{j=1}^{m} j^{r+1} a_{b_n,j} \tag{1.9}$$

and

$$\sum_{k=1}^{m-1} |a_{b_n,k} - a_{b_n,k+1}| \leq \frac{K}{m^{r+1+\delta}} \sum_{j=1}^{m} j^{r+1} a_{b_n,j} \tag{1.10}$$

hold, where $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$.

The objective of this paper is to prove the analogues of Theorems 1.1 – 1.4, using deferred matrix means $T_{n,A_0}^{c,b}(f;x) := \sum_{k=c_n+1}^{b_n} a_{b_n,k} s_k(f;x)$ instead of $T_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} s_k(f;x)$, and applying conditions (1.9) and (1.10) instead of (1.5) and (1.2), respectively. To do this, we need some helpful lemmas given in next section.

2. AUXILIARY LEMMAS

Lemma 2.1 ([4]). *If (1.3) and (1.4) hold then*

$$\int_0^v t^{-1}\omega(t)dt = \mathcal{O}\left(vH(v)\right) \quad (v \to +0).$$

Lemma 2.2 ([10]). *If (1.3) and (1.4) hold then*

$$\int_0^{\pi/b_m} \omega(t)dt = \mathcal{O}\left(b_m^{-2}H(\pi/b_m)\right).$$

Lemma 2.3. *Let $r \in \mathbb{N}\cup\{0\}$, $0<\delta\le 1$. If $\{a_{b_n,k}\}$ satisfies (1.9), then uniformly in $t\in(0,\pi]$,*

(2.1)
$$|\mathbb{K}_{c_n,0}^{b_n}(t)| := \left|\sum_{k=0}^{b_n-c_n-1} a_{b_n,c_n+1+k}\sin\left(c_n+k+\frac{3}{2}\right)t\right| = A_{c_n,b_n;\ell} + \mathcal{O}\left(\frac{1}{\ell^{r+1+\delta}}\sum_{j=1}^{\ell} j^{r+1}a_{b_n,j}\right),$$

where $\ell = \left[\frac{\pi}{t}\right]$ and $A_{c_n,b_n;\ell} := \sum_{k=0}^{\ell} a_{b_n,c_n+1+k}$.

If $\{a_{b_n,k}\}$ satisfies (1.10), then

$$|\mathbb{K}_{c_n,0}^{b_n}(t)| = \mathcal{O}\left(\frac{1}{tb_n^{r+1+\delta}}\sum_{j=1}^{b_n} j^{r+1}a_{b_n,j} + \frac{a_{b_n,b_n}}{t}\right). \tag{2.2}$$

Proof. Let us denote

$$\mathbb{K}_{c_n,s}^{b_n}(t) := \sum_{k=s}^{b_n-c_n-1} a_{b_n,c_n+1+k}\sin\left(c_n+k+\frac{3}{2}\right)t.$$

The above quantity (using summation by parts) can be transformed as

$$\begin{aligned}
\mathbb{K}_{c_n,s}^{b_n}(t) &= -\sum_{k=s}^{b_n-c_n-2}\Delta a_{b_n,c_n+1+k}\sum_{j=1}^{k}\sin\left(c_n+j+\frac{3}{2}\right)t \\
&\quad -a_{b_n,c_n+1+s}\sum_{j=1}^{s-1}\sin\left(c_n+j+\frac{3}{2}\right)t + a_{b_n,b_n}\sum_{j=1}^{b_n-c_n-1}\sin\left(c_n+j+\frac{3}{2}\right)t \\
&= \frac{1}{2\sin\frac{t}{2}}\left[-\sum_{k=s}^{b_n-c_n-2}\Delta a_{b_n,c_n+1+k}\left(\cos(c_n+2)t-\cos(c_n+k+2)t\right)\right. \\
&\quad -a_{b_n,c_n+1+s}\left(\cos(c_n+2)t-\cos(c_n+s+1)t\right) \\
&\quad \left.+a_{b_n,b_n}\left(\cos(c_n+2)t-\cos(b_n+1)t\right)\right]
\end{aligned} \tag{2.3}$$

and since $a_{b_n,k}\ge 0$, supposing that $b_n\ge\ell$, we have

$$\begin{aligned}
|\mathbb{K}_{c_n,0}^{b_n}(t)| &\le \sum_{k=0}^{\ell} a_{b_n,c_n+1+k} + \left|\mathbb{K}_{c_n,\ell}^{b_n}(t)\right| \\
&\le A_{b_n,\ell} + \frac{\pi}{t}\left[\sum_{k=\ell}^{b_n-c_n-2}|\Delta a_{b_n,c_n+1+k}| + a_{b_n,c_n+1+\ell} + a_{b_n,b_n}\right].
\end{aligned} \tag{2.4}$$

Because of $b_n\ge\ell$ and $\{a_{b_n,k}\}$ satisfies (1.9), we obtain

$$a_{b_n,b_n} \le \sum_{k=b_n}^{\infty} |\Delta a_{b_n,k}| \le \sum_{k=\ell}^{\infty} |\Delta a_{b_n,k}| \le \frac{K}{\ell^{r+1+\delta}} \sum_{j=1}^{\ell} j^{r+1} a_{b_n,j}, \tag{2.5}$$

$$a_{b_n,c_n+1+\ell} \le \sum_{k=c_n+1+\ell}^{\infty} |\Delta a_{b_n,k}| \le \sum_{k=\ell}^{\infty} |\Delta a_{b_n,k}| \le \frac{K}{\ell^{r+1+\delta}} \sum_{j=1}^{\ell} j^{r+1} a_{b_n,j}, \tag{2.6}$$

and

$$\begin{aligned} \sum_{k=\ell}^{b_n-c_n-2} |\Delta a_{b_n,c_n+1+k}| &\le \sum_{k=\ell}^{\infty} |\Delta a_{b_n,c_n+1+k}| \\ &\le K a_{b_n,c_n+1+\ell} \le \frac{K}{\ell^{r+1+\delta}} \sum_{j=1}^{\ell} j^{r+1} a_{b_n,j}. \end{aligned} \tag{2.7}$$

Now, using (2.4), (2.5), (2.6), and (2.7), we get

$$|\mathbb{K}_{c_n,0}^{b_n}(t)| = \sum_{k=0}^{\ell} a_{b_n,c_n+1+k} + \mathcal{O}\left(\frac{1}{\ell^{r+1+\delta}} \sum_{j=1}^{\ell} j^{r+1} a_{b_n,j} \right),$$

which proves (2.1).

Since $s = 0$, $\sin\beta \ge \frac{2\beta}{\pi}$ for $\beta \in (0, \pi/2]$, and $\{a_{b_n,k}\}$ satisfies (1.10), we have

$$\begin{aligned} |\mathbb{K}_{c_n,0}^{b_n}(t)| &\le \frac{\pi}{t} \left[a_{b_n,c_n+1} + \sum_{k=0}^{b_n-c_n-2} |\Delta a_{b_n,c_n+1+k}| + a_{b_n,b_n} \right] \\ &\le \frac{\pi}{t} \left[a_{b_n,c_n+1} + \sum_{k=c_n+1}^{b_n-1} |\Delta a_{b_n,k}| + a_{b_n,b_n} \right] \\ &\le \frac{\pi}{t} \left[a_{b_n,c_n+1} + \frac{K}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} + a_{b_n,b_n} \right]. \end{aligned} \tag{2.8}$$

Note that condition (1.10) for $b_n \ge c_n + 1$, guarantees the inequality

$$a_{b_n,c_n+1} - a_{b_n,b_n} \le \sum_{k=c_n+1}^{b_n-1} |\Delta a_{b_n,k}| \le \frac{K}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j}$$

or

$$a_{b_n,c_n+1} \le \frac{K}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} + a_{b_n,b_n},$$

and along with (2.8) imply

$$|\mathbb{K}_{c_n,0}^{b_n}(t)| \le \frac{2\pi(K+1)}{t} \left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} + a_{b_n,b_n} \right),$$

which proves (2.2).

The proof is completed. ∎

3. Main Results

At first, we prove the following.

Theorem 3.1. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and let $\{a_{n,k}\}$ satisfy conditions (1.6) and (1.10). If $\omega(t)$ satisfies (1.3) and (1.4), then*

$$\|T^{c,b}_{n,A_0}(f) - f\| = O\left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} H(a_{b_n,b_n}) + a_{b_n,b_n} H(a_{b_n,b_n})\right).$$

Proof. We will use the equality

$$T^{c,b}_{n,A_0}(f;x) - f(x) = \frac{1}{\pi}\int_0^\pi \psi_x(t) \sum_{k=c_n+1}^{b_n} a_{b_n,k} \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} dt = \frac{1}{\pi}\int_0^\pi \frac{\psi_x(t)}{2\sin\frac{t}{2}} \mathbb{K}^{b_n}_{c_n,0}(t) dt,$$

where $\psi_x(t) = f(x+t) + f(x-t) - 2f(x)$ and

$$\mathbb{K}^{b_n}_{c_n,0}(t) = \sum_{k=0}^{b_n-c_n-1} a_{b_n,c_n+1+k} \sin\left(c_n + k + \frac{3}{2}\right)t.$$

Since $|\psi_x(t)| \leq 2\omega(t)$, then we can write

(3.1)
$$\begin{aligned}\|T^{c,b}_{n,A_0}(f) - f\| &\leq \frac{1}{\pi}\int_0^\pi \frac{\omega(t)}{\sin\frac{t}{2}} |\mathbb{K}^{b_n}_{c_n,0}(t)| dt \\ &= \frac{1}{\pi}\int_0^{a_{b_n,b_n}} \frac{\omega(t)}{\sin\frac{t}{2}} |\mathbb{K}^{b_n}_{c_n,0}(t)| dt + \frac{1}{\pi}\int_{a_{b_n,b_n}}^\pi \frac{\omega(t)}{\sin\frac{t}{2}} |\mathbb{K}^{b_n}_{c_n,0}(t)| dt := R_1 + R_2.\end{aligned}$$

It is obvious that condition (1.6) provides the estimate

$$\left|\mathbb{K}^{b_n}_{c_n,0}(t)\right| \leq \sum_{k=0}^{b_n-c_n-1} a_{b_n,c_n+1+k} = 1.$$

Whence, using $\sin\frac{t}{2} \geq \frac{t}{\pi}$, $t \in [0,\pi]$, and Lemma 2.1, we get

$$R_1 = O(1)\int_0^{a_{b_n,b_n}} \frac{\omega(t)}{t} dt = O\left(a_{b_n,b_n} H(a_{b_n,b_n})\right). \tag{3.2}$$

Now, using condition (1.10) and (2.2) of Lemma 2.3, we also have

(3.3)
$$\begin{aligned}R_2 &= O\left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} + a_{b_n,b_n}\right) \int_0^{a_{b_n,b_n}} \frac{\omega(t)}{t^2} dt \\ &= O\left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} H(a_{b_n,b_n}) + a_{b_n,b_n} H(a_{b_n,b_n})\right).\end{aligned}$$

Finally, inserting (3.2) and (3.3) into (3.1), we obtain the required estimate.
The proof is completed. ∎

Remark 3.1. Because of (1.8), we conclude that Theorem 3.1 also holds true, if $\{a_{b_n,k}\}$ satisfies

$$\sum_{k=1}^{m-1} |a_{b_n,k} - a_{b_n,k+1}| \leq K a_{b_n,m}$$

instead of condition (1.10).

Let us suppose that $\mathbb{G}$ is a subset of $\mathbb{N}$ and consider $\mathbb{G}$ as the range of a strictly increasing sequence of positive integers, say $\mathbb{G} = \{\lambda(n)\}_1^\infty$. The following transformation one can find in [18] and [19]:

$$\tau^{\lambda}_{n,A_0}(f;x) := \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} s_k(f;x), \quad (n = 0, 1, \dots),$$

where $\lambda(0) := 0$, $a_{\lambda(n),k} \geq 0$, $k = 0, 1, \dots, \lambda(n)$, and we agree $a_{\lambda(n),0}$ to be 0.

Thus, for $b_n = \lambda(n)$ and $c_n = 0$ in Theorem 3.1, we have:

Corollary 3.2. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and let $\{a_{\lambda(n),k}\}$ satisfy conditions (1.6) and (1.10). If $\omega(t)$ satisfies (1.3) and (1.4), then*

$$\|\tau^{\lambda}_{n,A_0}(f) - f\| = O\left(\frac{1}{(\lambda(n))^{r+1+\delta}} \sum_{j=1}^{\lambda(n)} j^{r+1} a_{\lambda(n),j} H(a_{\lambda(n),\lambda(n)}) + a_{\lambda(n),\lambda(n)} H(a_{\lambda(n),\lambda(n)})\right).$$

Moreover, specifying the transformation $T^{c,b}_{n,A_0}(f;x)$ to be deferred Riesz transformation

$$DR^{c,b}_n(f;x) = \frac{1}{P^{b_n}_{c_n+1}} \sum_{k=c_n+1}^{b_n} p_k s_k(f;x),$$

we have:

Corollary 3.3. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and let $\{p_k\}$ be a non-negative sequence satisfying condition (1.10). If $\omega(t)$ satisfies (1.3) and (1.4), then*

$$\|DR^{c,b}_n(f) - f\| = O\left(\left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} p_j + p_{b_n}\right) \frac{1}{P^{b_n}_{c_n+1}} H\left(\frac{p_{b_n}}{P^{b_n}_{c_n+1}}\right)\right).$$

Furthermore, if we choose the deferred Cesàro transformation

$$DC^{c,b}_n(f;x) = \frac{1}{b_n - c_n} \sum_{k=c_n+1}^{b_n} s_k(f;x)$$

instead of the transformation $T^{c,b}_{n,A_0}(f;x)$, we get:

Corollary 3.4. *Let $r \in \mathbb{N} \cup \{0\}$ and $0 < \delta \leq 1$. If $\omega(t)$ satisfies (1.3) and (1.4), then*

$$\|DC^{c,b}_n(f) - f\| = O\left(\left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} + 1\right) \frac{1}{b_n - c_n} H\left(\frac{1}{b_n - c_n}\right)\right).$$

Theorem 3.5. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$, $\{a_{b_n,k}\}$ satisfy (1.6) and (1.10), and $\omega(t)$ satisfies (1.3). Then*

$$\|T^{c,b}_{n,A_0}(f) - f\| = O\left(\omega\left(\frac{\pi}{b_n}\right)\right) + O\left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} H\left(\frac{\pi}{b_n}\right) + a_{b_n,b_n} H\left(\frac{\pi}{b_n}\right)\right).$$

If, in addition, $\omega(t)$ satisfies (1.4) then

$$\|T^{c,b}_{n,A_0}(f) - f\| = O\left(\frac{1}{b_n^{r+1+\delta}} \sum_{j=1}^{b_n} j^{r+1} a_{b_n,j} H\left(\frac{\pi}{b_n}\right) + a_{b_n,b_n} H\left(\frac{\pi}{b_n}\right)\right).$$

Proof. First of all, we can write

$$\begin{aligned}\|T^{c,b}_{n,A_0}(f)-f\| &\leq \frac{1}{\pi}\int_0^{\pi}\frac{\omega(t)}{\sin\frac{t}{2}}|\mathbb{K}^{b_n}_{c_n,0}(t)|dt \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{b_n}}\frac{\omega(t)}{\sin\frac{t}{2}}|\mathbb{K}^{b_n}_{c_n,0}(t)|dt+\frac{1}{\pi}\int_{\frac{\pi}{b_n}}^{\pi}\frac{\omega(t)}{\sin\frac{t}{2}}|\mathbb{K}^{b_n}_{c_n,0}(t)|dt := J_1+J_2.\end{aligned} \tag{3.4}$$

Using (1.6) and $\left|\sin\left(c_n+k+\frac{3}{2}\right)t\right| \leq \left(c_n+k+\frac{3}{2}\right)t$, we find that

$$J_1 = O(c_n+b_n)\int_0^{\frac{\pi}{b_n}}\omega(t)dt = O\left(\omega\left(\frac{\pi}{b_n}\right)\right). \tag{3.5}$$

Moreover, (1.10) and Lemma 2.3 (2.2), imply

$$\begin{aligned}J_2 &= O\left(\frac{1}{b_n^{r+1+\delta}}\sum_{j=1}^{b_n} j^{r+1}a_{b_n,j}+a_{b_n,b_n}\right)\int_0^{\frac{\pi}{b_n}}\frac{\omega(t)}{t^2}dt \\ &= O\left(\frac{1}{b_n^{r+1+\delta}}\sum_{j=1}^{b_n} j^{r+1}a_{b_n,j}H\left(\frac{\pi}{b_n}\right)+a_{b_n,b_n}H\left(\frac{\pi}{b_n}\right)\right).\end{aligned} \tag{3.6}$$

Consequently, inserting (3.5) and (3.6) into (3.4), we obtain the first conclusion of our theorem.

Now, we are going to prove second conclusion of our theorem. We use Lemma 2.2 in

$$J_1 = O(c_n+b_n)\int_0^{\frac{\pi}{b_n}}\omega(t)dt$$

to find that

$$J_1 = O(c_n+b_n)\int_0^{\frac{\pi}{b_n}}\omega(t)dt = O\left(\frac{\pi}{b_n}H\left(\frac{\pi}{b_n}\right)\right). \tag{3.7}$$

Despite this, condition (1.10), for $k\geq c_n+1$, implies

$$a_{b_n,k} \leq \frac{K}{b_n^{r+1+\delta}}\sum_{j=1}^{b_n} j^{r+1}a_{b_n,j}+a_{b_n,b_n},$$

and thus

$$\begin{aligned}1=\sum_{k=c_n+1}^{b_n}a_{b_n,k} \quad &\leq \quad (b_n-c_n)\left(\frac{K}{b_n^{r+1+\delta}}\sum_{j=1}^{b_n} j^{r+1}a_{b_n,j}+a_{b_n,b_n}\right) \\ &\Longrightarrow \quad \frac{1}{b_n}\leq \frac{K}{b_n^{r+1+\delta}}\sum_{j=1}^{b_n} j^{r+1}a_{b_n,j}+a_{b_n,b_n}.\end{aligned}$$

So, J_1 (from (3.7)) can be majored as follows

$$J_1 = O\left(\left(\frac{1}{b_n^{r+1+\delta}}\sum_{j=1}^{b_n} j^{r+1}a_{b_n,j}+a_{b_n,b_n}\right)H\left(\frac{\pi}{b_n}\right)\right). \tag{3.8}$$

Finally, (3.8), (3.6) and (3.4) imply the second conclusion of our theorem.

The proof is completed. ∎

Theorem 3.6. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and let $\{a_{n,k}\}$ satisfy conditions (1.6) and (1.9). Then*

$$\|T^{c,b}_{n,A_0}(f) - f\| = O\left(\omega\left(\frac{\pi}{b_n}\right) + \sum_{m=1}^{b_n-1}\left(A_{c_n,b_n;m+1} + m^{r+\delta}\sum_{j=1}^{m} j^{r+1}a_{b_n,j}\right)\frac{1}{m}\omega\left(\frac{\pi}{m}\right)\right).$$

Proof. We already have proved (Theorem 3.5, relation (3.5)) that

$$J_1 = O\left(\omega\left(\frac{\pi}{b_n}\right)\right). \tag{3.9}$$

We use Lemma 2.3 with (2.1) to obtain

(3.10)

$$\begin{aligned}
J_2 &= \frac{1}{\pi}\int_{\frac{\pi}{b_n}}^{\pi} \frac{\omega(t)}{\sin\frac{t}{2}}|\mathbb{K}^{b_n}_{c_n,0}(t)|dt \\
&= \mathcal{O}\left(\int_{\frac{\pi}{b_n}}^{\pi} t^{-1}\omega(t)A_{c_n,b_n;\ell}dt\right) + \mathcal{O}\left(\int_{\frac{\pi}{b_n}}^{\pi} \frac{\omega(t)}{t\ell^{r+1+\delta}}\sum_{j=1}^{\ell} j^{r+1}a_{b_n,j}dt\right) \\
&= \mathcal{O}\left(\sum_{m=1}^{b_n-1}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} t^{-1}\omega(t)A_{c_n,b_n;\ell}dt\right) + \mathcal{O}\left(\sum_{m=1}^{b_n-1}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{\omega(t)}{t\ell^{r+1+\delta}}\sum_{j=1}^{\ell} j^{r+1}a_{b_n,j}dt\right) \\
&= \mathcal{O}\left(\sum_{m=1}^{b_n-1}\frac{1}{m}\omega\left(\frac{\pi}{m}\right)A_{c_n,b_n;m+1}\right) + \mathcal{O}\left(\sum_{m=1}^{b_n-1}\frac{1}{m}\omega\left(\frac{\pi}{m}\right)\sum_{j=1}^{m} j^{r+1}a_{b_n,j}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{1}{\ell^{r+1+\delta}}dt\right) \\
&= \mathcal{O}\left(\sum_{m=1}^{b_n-1}\frac{1}{m}\omega\left(\frac{\pi}{m}\right)A_{c_n,b_n;m+1}\right) + \mathcal{O}\left(\sum_{m=1}^{b_n-1}m^{r+\delta-1}\omega\left(\frac{\pi}{m}\right)\sum_{j=1}^{m} j^{r+1}a_{b_n,j}\right) \\
&= \mathcal{O}\left(\sum_{m=1}^{b_n-1}\frac{1}{m}\omega\left(\frac{\pi}{m}\right)\left(A_{c_n,b_n;m+1} + m^{r+\delta}\sum_{j=1}^{m} j^{r+1}a_{b_n,j}\right)\right).
\end{aligned}$$

Consequently, (3.4) with (3.9) and (3.10) imply the conclusion of our theorem.
The proof is completed. ∎

Theorem 3.7. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and let $\{a_{n,k}\}$ satisfy conditions (1.6) and (1.9). If $\omega(t)$ satisfies conditions (1.3) and (1.4), then*

$$\|T^{c,b}_{n,A_0}(f) - f\| = O\left(a_{b_n,b_1}H(a_{b_n,b_1})\right).$$

Proof. First of all, relation (3.1) can be written as follows

(3.11)

$$\begin{aligned}
\|T^{c,b}_{n,A_0}(f) - f\| &\leq \frac{1}{\pi}\int_0^{\pi}\frac{\omega(t)}{\sin\frac{t}{2}}|\mathbb{K}^{b_n}_{c_n,0}(t)|dt \\
&= \frac{1}{\pi}\int_0^{a_{b_n,1}}\frac{\omega(t)}{\sin\frac{t}{2}}|\mathbb{K}^{b_n}_{c_n,0}(t)|dt + \frac{1}{\pi}\int_{a_{b_n,1}}^{\pi}\frac{\omega(t)}{\sin\frac{t}{2}}|\mathbb{K}^{b_n}_{c_n,0}(t)|dt := R^1_1 + R^1_2.
\end{aligned}$$

Similar to relation (3.2) we can write (using Lemma 2.1)

$$R^1_1 = O(1)\int_0^{a_{b_n,1}}\frac{\omega(t)}{t}dt = O\left(a_{b_n,1}H(a_{b_n,1})\right). \tag{3.12}$$

The following we have already obtained (at relation (2.8) for $c_n = 0$)

$$|\mathbb{K}^{b_n}_{0,0}(t)| \leq \frac{\pi}{t}\left[a_{b_n,1} + \sum_{k=1}^{b_n-1}|\Delta a_{b_n,k}| + a_{b_n,b_n}\right] \leq \frac{\pi}{t}\left[a_{b_n,1} + \sum_{k=1}^{\infty}|\Delta a_{b_n,k}| + a_{b_n,b_n}\right]. \tag{3.13}$$

Since $\{a_{n,k}\}$ satisfies condition (1.9), than we get

$$\sum_{k=1}^{\infty}|\Delta a_{b_n,k}| \leq K a_{b_n,1},$$

$$a_{b_n,b_n} \leq \sum_{k=b_n}^{\infty}|\Delta a_{b_n,k}| \leq \sum_{k=1}^{\infty}|\Delta a_{b_n,k}| \leq K a_{b_n,1},$$

and, thus (3.13) takes its form

$$|\mathbb{K}^{b_n}_{c_n,0}(t)| = |\mathbb{K}^{b_n}_{0,0}(t)| = \mathcal{O}\left(\frac{a_{b_n,1}}{t}\right). \tag{3.14}$$

Whence, (3.14) and (1.3) enable us to obtain

$$R_2^1 = O\left(a_{b_n,b_1}\right)\int_0^{a_{b_n,b_1}} \frac{\omega(t)}{t^2}dt = O\left(a_{b_n,b_1}H(a_{b_n,b_1})\right). \tag{3.15}$$

Subsequently, (3.12), (3.15) and (3.11) imply the requested conclusion.
The proof is completed. ∎

Remark 3.2. Because of (1.7), we conclude that Theorem 3.7 also holds true, if $\{a_{n,k}\}$ satisfies

$$\sum_{k=m}^{\infty}|a_{b_n,k} - a_{b_n,k+1}| \leq K a_{b_n,m}$$

instead of condition (1.9).

Similarly, for $b_n = \lambda(n)$ and $a_n = 0$, Theorem 3.7 implies the following:

Corollary 3.8. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and let $\{a_{\lambda(n),k}\}$ satisfy conditions (1.6) and (1.9). If $\omega(t)$ satisfies conditions (1.3) and (1.4), then*

$$\|\tau^{\lambda}_{n,A_0}(f) - f\| = O\left(a_{\lambda(n),b_1}H(a_{\lambda(n),b_1})\right).$$

Also, if we choose $T^{c,b}_{n,A_0}(f;x)$ to be the deferred Nörlund transformation

$$DN^{c,b}_n(f;x) = \frac{1}{P_0^{b_n-c_n-1}}\sum_{k=c_n+1}^{b_n} p_{b_n-k}s_k(f;x),$$

then we get:

Corollary 3.9. *Let $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and let $\{p_k\}$ satisfies condition and (1.9). If $\omega(t)$ satisfies conditions (1.3) and (1.4), then*

$$\|DN^{c,b}_n(f) - f\| = O\left(\frac{p_{b_n-b_1}}{P_0^{b_n-c_n-1}}H\left(\frac{p_{b_n-b_1}}{P_0^{b_n-c_n-1}}\right)\right).$$

4. CONCLUSION

In this paper we have introduced the deferred matrix means which contain the deferred Nörlund and the deferred Riesz appearing recently in mathematical research and the deferred Cesàro means introduced about ninety years ago. Using these new means of Fourier series of a continuous function and a new class of sequences, we have determined the degree of approximation of such function via its modulus of continuity and a positive mediate function.

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