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A CARATHEODORY'S APPROXIMATE SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS UNDER THE HÖLDER CONDITION

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ABSTRACT. In this paper, based on the theorem of the uniqueness of the solution of the stochastic differential equation, the convergence possibility of the Caratheodory's approximate solution was studied by approximating the unique solution. To obtain this convergence theorem, we used a Hölder condition and a weakened linear growth condition. Furthermore, The auxiliary theorems for the existence and continuity of the Caratheodory's approximate solution were investigated as a prerequisite.

Key words and phrases: Hölder inequality; Moment inequality; Stachurska's inequality; Hölder condition.

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1. INTRODUCTION

Nonlinear stochastic differential equation has come to play an important role in many branches of natural and applied science where more and more researcher have encountered stochastic differential equations(short for SDEs). See the references to this [2]- [9] and [10]-[16]. Also, the problems of the approximate solution to the SDEs has become an important field of study because the solution of the SDEs does not have an explicit expression except for linear cases as well as the question of the existence of stochastic integral part in the equations. See the references to this [6], [11], and [12].

Xuerong Mao [11] had established the existence and uniqueness theorems and discussed the properties of the solution for the SDEs in his book. He had introduced the stochastic differential equations studied by previous researchers;

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \tag{1.1}$$

on the closed interval $[t_0, T], t_0 \leq T$. And he obtained that if Lipschiz condition and linear growth condition hold, then the SDEs (1.1) had a unique solution $x(t)$, moreover, $x(t) \in \mathcal{M}^2([t_0, T]; R^{d\times m})$ which means that we denoted by $\mathcal{M}^2$ the family of processes $\{f(t)\}$ in $\mathcal{L}^p$ such that $E\int_{t_0}^T |f(t)|^2 dt < \infty$.

However, the Lipschiz condition etc. only guarantee the existence and uniqueness of the solution and, in general, the solution does not have an explicit expression except the linear case which were discussed in previous researchers. See the references to this [11]. In practice, we therefore often seek the approximate solution rather than the accurate solution.

Especially, Bae at al.[2] obtained that if two condition (1.2) and (1.3) hold: For any $y, z \in R^d$ and $t \in [t_0, T]$, we assume that

$$|f(y,t) - f(z,t)|^2 \vee |g(y,t) - g(z,t)|^2 \leq \overline{K}|y - z|^{2\alpha} \tag{1.2}$$

where $\overline{K}$ is a positive constant and $0 < \alpha \leq 1$ is a constant. For any $t \in [t_0, T]$ it follows that $f(0,t), g(0,t) \in \mathcal{L}^2([t_0, T])$ it follows that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K \tag{1.3}$$

where K is a positive constant, then there exists a unique solution $x(t)$ to equation(1.1) and the solution belongs to $\mathcal{M}^2([t_0, T]; R^d)$. In the this paper, by using the Picard iteration procedure, authors established the theorem on the existence and uniqueness of the solution for d-dimensional stochastic differential equation. As the by-product, authors also obtained the Picard approximate solution for the equation and following Theorem 1.1 which gives an estimate on the difference, called the error, between the approximate and the accurate solution.

Theorem 1.1. *Assume that (1.2) and (1.3) hold. Let $x(t)$ be the unique solution $x(t)$ of equation (1.1) and $x_n(t)$ be the Picard iteration. Then*

$$E\left(\sup_{t_0 \leq t \leq T} |x_n(t) - x(t)|^2\right) \leq \gamma_1 \exp\left(2M(T - t_0)\right) \tag{1.4}$$

for all $n \geq 1$.

In practice, given the error $\epsilon > 0$, one can determine n for left-hand side of (1.4) to be less than ϵ, and then compute $x_0(t), x_1(t), \cdots, x_n(t)$ by the Picard iteration. According to Theorem 1.1, we have

$$E\left(\sup_{t_0 \leq t \leq T} |x_n(t) - x(t)|^2\right) \leq \epsilon.$$

So we can use $x_n(t)$ as the approximate solution the equation(1.1). The disadvantage of the Picard approximations is that one needs to compute $x_0(t), x_1(t), \cdots, x_{n-1}(t)$ in order to compute $x_n(t)$, and this will involve a lot of calculations on stochastic integrals. More efficient ways in this direction are Caratheodory's approximation procedure and Cauchy-Maruyama's.

Motivated by [5], [6], [11], and [13], one of the objectives of this paper is to get one proof to Caratheodory's approximation procedure for given SDEs. The other objective of this paper is to estimate on how fast the Caratheodory's approximation iterations $x_n(t)$ convergence the unique solution $x(t)$ of the SDEs.

2. PRELIMINARY

Let $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Assume that $B(t)$ is an m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$.

Consider the d-dimensional stochastic differential equation of Itô type

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \quad on\ t_0 \leq t \leq T. \tag{2.1}$$

with initial value $x(t_0) = x_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s) \quad on\ t_0 \leq t \leq T. \tag{2.2}$$

First, let us define the solution of the stochastic differential equations.

Definition 2.1. ([11]) An R^d-valued stochastic process $\{x(t)\}_{t_0\leq t\leq T}$ is called a solution of equation (2.1) if it has the following properties:
(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;
(ii) $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$ and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; R^{d\times m})$;
(iii) equation (2.1) holds for every $t \in [t_0, T]$ with probability 1.
A solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$is indistinguishable from $\{x(t)\}$, that is

$$P\{x(t) = \bar{x} \ \ for\ all\ t_0 \leq t \leq T\} = 1.$$

For the convenience of the reader, we state following lemmas.

Lemma 2.1. ([1, 11]) *(Hölder's inequality) If* $\frac{1}{p} + \frac{1}{q} = 1$ *for any* $p, q > 1$, $f \in \mathcal{L}^p$, *and* $g \in \mathcal{L}^q$, *then* $fg \in \mathcal{L}^1$ *and* $\int_a^b fgdx \leq \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^q dx\right)^{\frac{1}{q}}$.

Lemma 2.2. ([11])*(moment inequality) Let* $p \geq 2$. *Let* $f \in \mathcal{M}^2([0, T]; R^{d\times m})$ *such that*

$$E\int_0^T |f(s)|^p ds < \infty.$$

Then

$$E\left|\int_0^T f(s)dB(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |f(s)|^p ds.$$

Lemma 2.3. ([11])*(moment inequality) If* $p \geq 2$, $f \in \mathcal{M}^2\left([0,T]; R^{d\times m}\right)$ *such that*

$$E\int_0^T |f(s)|^p ds < \infty,$$

then

$$E\left(\sup_{0\leq t\leq T}\left|\int_0^t f(s)dB(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{p}} E\int_0^T |f(s)|^p ds.$$

Lemma 2.4. ([1]) *(Stachurska's inequality) Let* $x(t)$ *and* $y(t)$ *be nonnegative continuous functions for* $t \geq \alpha$ *, and let* $x(t) \leq a(t) + b(t)\int_\alpha^t y(s)x^p(s)ds$, $t \in J = [\alpha, \beta)$, *where* $\frac{a}{b}$ *is nondecreasing function and* $0 < p < 1$. *Then*

$$x(t) \leq a(t)\left(1-(p-1)\left[\frac{a(t)}{b(t)}\right]^{p-1}\int_\alpha^t y(s)b^p(s)ds\right)^{\frac{-1}{p-1}}.$$

In order to attain the approximate solution of equation (2.1) with initial data, we propose the following assumptions:

(H1) (Hölder condition) For any $\varphi, \psi \in R^d$ and $t \in [t_0, T]$, we assume that

$$|f(\varphi,t) - f(\psi,t)|^2 \vee |g(\varphi,t) - g(\psi,t)|^2 \leq K|\varphi-\psi|^{2\alpha},$$

where K is a positive constant and $0 < \alpha < 1$ is a constant.

(H2) (Weakened linear growth condition) For any $t \in [t_0, T]$, it follows that $f(0,t), g(0,t) \in L^2$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K_1,$$

where K_1 is a positive constant.

3. MAIN RESULTS

In order to obtain an approximate solution to SDEs, let us now give a definition of Caratheodory's approximate solutions.

For every integer $n \geq 1$, define $x_n(t) = x_0$ for $t_0 - 1 \leq t \leq t_0$ and

$$x_n(t) = x_0 + \int_{t_0}^t f(x_n(s-1/n), s)ds + \int_{t_0}^t g(x_n(s-1/n), s)dB(s) \tag{3.1}$$

for $t_0 \leq t \leq T$. Note that for $t_0 \leq t \leq t_0 + 1/n$, $x_n(t)$ can be computed by

$$x_n(t) = x_0 + \int_{t_0}^t f(x_0, s)ds + \int_{t_0}^t g(x_0, s)dB(s);$$

then for $t_0 + 1/n < t \leq t_0 + 2/n$,

$$x_n(t) = x_n(t_0+1/n) + \int_{t_0+1/n}^t f(x_n(s-1/n), s)ds + \int_{t_0+1/n}^t g(x_n(s-1/n), s)dB(s),$$

and so on. In other words, $x_n(t)$ can be computed step-by-step on the intervals $[t_0, t_0 + 1/n], (t_0 + 1/n, t_0 + 2/n], \ldots$.

We need to prepare two lemmas in order to establish one of the main results.

Lemma 3.1. *Let* $f : R^d \times [t_0, T] \to R^d$ *and* $g : R^d \times [t_0, T] \to R^{d\times m}$ *be both Borel measurable. Consider the* d*-dimensional Caratheodory's approximate solutions (3.1). Assume that there exist two constants* K *and* K_1 *such that*

(i) *(Hölder condition) For any* $\varphi, \psi \in R^d$ *and* $t \in [t_0, T]$, *we assume that*

$$|f(\varphi,t) - f(\psi,t)|^2 \vee |g(\varphi,t) - g(\psi,t)|^2 \leq K|\varphi - \psi|^{2\alpha}, \tag{3.2}$$

where K *is a positive constant and* $0 < \alpha < 1$ *is a constant.*

(ii) *(Weakened linear growth condition) For any* $t \in [t_0, T]$, *it follows that* $f(0,t), g(0,t) \in L^2$ *such that*

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K_1, \tag{3.3}$$

where K_1 *is a positive constant.*

Then, for all $n \geq 1$, *we have*

$$\begin{aligned} &\sup_{t_0<t\leq T} E|x_n(t)|^2 \\ &\leq C_1 := a(T)\Big\{1 - (\alpha-1)\big[a(T)\big]^{\alpha-1}\big[6K(T-t_0+1)\big]\big(T-t_0\big)\Big\}^{-1/(\alpha-1)}, \end{aligned} \tag{3.4}$$

where $a(T) = 3E|x_0|^2 + 6K_1(T-t_0)(T-t_0+1)$.

Proof. Fix $n \geq 1$ arbitrarily. It is easy to see from the definition of $x_n(t)$ and condition (3.2) and (3.3) that $\{x_n(t)\}_{t_0 \leq t \leq T} \in \mathcal{M}^2((t_0,T]; R^d)$. From (3.1), using the elementary inequality $(y+z+w)^2 \leq 3(y^2+z^2+w^2)$, we have following for $t_0 \leq t \leq T$

$$|x_n(t)|^2 \leq 3|x_0|^2 + 3\Big|\int_{t_0}^t f(x_n(s-1/n),s)ds\Big|^2 + 3\Big|\int_{t_0}^t g(x_n(s-1/n),s)dB(s)\Big|^2.$$

By Hölder's inequality, we can derive that

$$|x_n(t)|^2 \leq 3|x_0|^2 + 3(T-t_0)\int_{t_0}^t |f(x_n(s-1/n),s)|^2 ds + 3\Big|\int_{t_0}^t g(x_n(s-1/n),s)dB(s)\Big|^2.$$

Taking the expectation on both sides and using the Lemma 2.2, we have following

$$\begin{aligned} &E|x_n(t)|^2 \\ &\leq 3E|x_0|^2 + 3(T-t_0)E\int_{t_0}^t |f(x_n(s-1/n),s)|^2 ds + 3E\int_{t_0}^t |g(x_n(s-1/n),s)|^2 ds. \end{aligned}$$

By the condition (3.2) and (3.3), we obtain

$$\begin{aligned} &E|x_n(t)|^2 \\ &\leq 3E|x_0|^2 + 6K_1(T-t_0)(T-t_0+1) + 6K(T-t_0+1)E\int_{t_0}^t |x_n(s-1/n)|^{2\alpha} ds. \end{aligned}$$

Consequently

$$\sup_{t_0\leq s\leq t} E|x_n(s)|^2 \leq a(T) + 6K(T-t_0+1)\int_{t_0}^t E\sup_{t_0\leq r\leq s}|x_n(r)|^{2\alpha} ds,$$

where $a(T) = 3E|x_0|^2 + 6K_1(T-t_0)(T-t_0+1)$. An application of the Stachurska's inequality (Lemma2.4) implies that

$$\begin{aligned} &\sup_{t_0\leq s\leq t} E|x_n(s)|^2 \\ &\leq a(T)\Big\{1 - (\alpha-1)\Big[\frac{a(T)}{6K(T-t_0+1)}\Big]^{\alpha-1}\int_{t_0}^t [3K(T-t_0+1)]^\alpha ds\Big\}^{-1/(\alpha-1)} \\ &\leq a(T)\Big\{1 - (\alpha-1)\big[a(T)\big]^{\alpha-1}\big[6K(T-t_0+1)\big]\big(t-t_0\big)\Big\}^{-1/(\alpha-1)} \end{aligned}$$

for all $t_0 \leq t \leq T$. In particular, the required inequality follows (3.4) follows when $t = T$. The proof is complete. ∎

Lemma 3.2. *Under the Hölder condition (3.2) and weakened linear growth condition (3.3), for all $n \geq 1$ and $t_0 \leq s < t \leq T$ with $t - s \leq 1$, the Caratheodory's approximate solution has the property*

$$E|x_n(t) - x_n(s)|^2 \leq C_2(t-s) \tag{3.5}$$

where $C_2 = 8K_1 + 8KC_1^\alpha$ and C_1 is defined in Lemma 3.1

Proof. From (3.1), we have following for $t_0 \leq s < t \leq T$

$$x_n(t) - x_n(s) = \int_s^t f(x_n(r-1/n), r)dr + \int_s^t g(x_n(r-1/n), r)dB(r).$$

Using the elementary inequality $(y+z)^2 \leq 2(y^2+z^2)$, we have following

$$|x_n(t) - x_n(s)|^2 \leq 2\Big|\int_s^t f(x_n(r-1/n), r)dr\Big|^2 + 2\Big|\int_s^t g(x_n(r-1/n), r)dB(r)\Big|^2.$$

By Hölder's inequality, we can derive that

$$|x_n(t) - x_n(s)|^2 \leq 2(t-s)\int_s^t |f(x_n(r-1/n), r)|^2 dr + 2\Big|\int_s^t g(x_n(r-1/n), r)dB(r)\Big|^2.$$

Taking the expectation on both sides and using the Lemma 2.2, we have following

$$E|x_n(t) - x_n(s)|^2 \leq 2(t-s)E\int_s^t |f(x_n(r-1/n), r)|^2 dr + 2E\int_s^t |g(x_n(r-1/n), r)|^2 dr.$$

By the condition (3.2) and (3.3), we obtain

$$\begin{aligned}
&E|x_n(t) - x_n(s)|^2 \\
&\leq 4(t-s)\Big[E\int_s^t K|x_n(r-1/n)|^{2\alpha}dr + K_1(t-s)\Big] \\
&\quad +4\Big[E\int_s^t K|x_n(r-1/n)|^{2\alpha}dr + K_1(t-s)\Big] \\
&= 4K_1(t-s)[t-s+1] + 4[t-s+1]E\int_s^t K|x_n(r-1/n)|^{2\alpha}dr \\
&\leq 8K_1(t-s) + 8E\int_s^t K|x_n(r-1/n)|^{2\alpha}dr.
\end{aligned}$$

Consequently

$$E|x_n(t) - x_n(s)|^2 \leq 8K_1(t-s) + 8K\int_s^t E\sup_{t_0\leq k\leq r}|x_n(k)|^{2\alpha}dr.$$

Hence, an application of Lemma3.1 implies that

$$E|x_n(t) - x_n(s)|^2 \leq 8K_1(t-s) + 8KC_1^\alpha(t-s)$$

for all $t_0 \leq s < t \leq T$. The proof is complete. ∎

Theorem 3.3. *Assume that the Hölder condition (3.2) and the weakened linear growth condition (3.3), hold. Let $x(t)$ be the unique solution of the d-dimensional stochastic differential equation (2.1) of Itô type and $x_n(t)$ be the Caratheodory's approximate solution. Then, for all $n \geq 1$, such that*

$$E\Big(\sup_{t_0\leq t\leq T}|x_n(t)-x(t)|^2\Big)\leq C_3\frac{1}{n^\alpha} \tag{3.6}$$

where $0<\alpha<1$, $C_3 = b(T)\{1-(\alpha-1)[b(T)]^{\alpha-1}[4K(T-t_0+1)](t-t_0)\}^{-1/(\alpha-1)}$, $b(T) = 4K(T-t_0+4)C_2^\alpha$, *and* C_2 *is defined in Lemma 3.2*

Proof. From the definition of $x(t)$ and $x_n(t)$, it is not difficult to derive that

$$\begin{aligned}&x_n(t)-x_n(s)\\&=\int_{t_0}^t[f(x_n(s-1/n),s)-f(x(s),s)]ds+\int_{t_0}^t[g(x_n(s-1/n),s)-g(x(s),s)]dB(s).\end{aligned}$$

Using the elementary inequality $(y+z)^2\leq 2(y^2+z^2)$ and Hölder's inequality, we have following

$$\begin{aligned}&\sup_{t_0\leq r\leq t}|x_n(r)-x(r)|^2\\&\leq 2(t-t_0)\int_{t_0}^t|f(x_n(s-1/n),s)-f(x(s),s)|^2ds\\&+2\sup_{t_0\leq r\leq t}\Big|\int_{t_0}^r[g(x_n(s-1/n),s)-g(x(s),s)]dB(s)\Big|^2.\end{aligned}$$

Taking the expectation on both sides and using the Lemma 2.3, we have following

$$\begin{aligned}&E\Big(\sup_{t_0\leq r\leq t}|x_n(r)-x(r)|^2\Big)\\&\leq 4(t-t_0)E\int_{t_0}^t[|f(x_n(s),s)-f(x_n(s-1/n),s)|^2+|f(x_n(s),s)-f(x(s),s)|^2ds\\&+16E\int_{t_0}^t[|g(x_n(s),s)-g(x_n(s-1/n),s)|^2+|g(x_n(s),s)-g(x(s),s)|^2]ds.\end{aligned}$$

By the condition (3.2) and (3.3), we obtain

$$\begin{aligned}&E\Big(\sup_{t_0\leq r\leq t}|x_n(r)-x(r)|^2\Big)\\&\leq 4K(t-t_0+4)E\int_{t_0}^t|x_n(s)-x_n(s-1/n)|^{2\alpha}ds\\&+4K(t-t_0+4)E\int_{t_0}^t\sup_{t_0\leq r\leq s}|x_n(r)-x(r)|^{2\alpha}ds.\end{aligned}$$

But, an application of Lemma3.2 implies that

$$E|x_n(s)-x_n(s-1/n)|^2\leq C_2\frac{1}{n}$$

if $s\geq t_0+1/n$, otherwise if $t_0\leq s<t_0+1/n$, we have

$$E|x_n(s)-x_n(s-1/n)|^2=E|x_n(s)-x_n(t_0)|^2\leq C_2(s-t_0)\leq C_2\frac{1}{n}$$

Therefore, it follows from the above inequality that

$$E\Big(\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2\Big)$$
$$\le 4K(t-t_0+4)C_2^{\alpha}\Big(\frac{1}{n}\Big)^{\alpha}+4K(t-t_0+4)\int_{t_0}^{t}\Big(E\sup_{t_0\le r\le s}|x_n(r)-x(r)|^2\Big)^{\alpha}ds.$$

An application of the Stachurska's inequality (Lemma2.4) implies that

$$E\Big(\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2\Big)$$
$$\le b(t)\Big(\frac{1}{n}\Big)^{\alpha}\Big\{1-(\alpha-1)\big[b(t)\big]^{\alpha-1}\big[4K(t-t_0+1)\big]\big(t-t_0\big)\Big\}^{-1/(\alpha-1)},$$

where $b(t)=4K(t-t_0+4)C_2^{\alpha}$. In particular, the required inequality follows (3.6) follows when $t=T$. The proof is complete. ∎

In other words, the authors in ([13]) established the existence and uniqueness theorem using the condition (3.7) and (3.8). Moreover, under quite general conditions, we are still able to show that the Caratheodory's approximate solutions converge to the unique solution of equation (2.1). This is described as follows.

Lemma 3.4. *Let $f: R^d\times[t_0,T]\to R^d$ and $g: R^d\times[t_0,T]\to R^{d\times m}$ be both Borel measurable. Consider the d-dimensional Caratheodory's approximate solutions (3.1). Assume that there exist a constant K_1 such that*

(i) *For any $\varphi,\psi\in R^d$ and $t\in[t_0,T]$, we assume that*

(3.7) $$|f(\varphi,t)-f(\psi,t)|^2\vee|g(\varphi,t)-g(\psi,t)|^2\le\kappa(|\varphi-\psi|^{2\alpha}),$$

where $0<\alpha<1$ and $\kappa(\cdot)$ is a concave non-decreasing function from R_+ to R_+ such that $\kappa(0)=0$, $\kappa(u)>0$ for $u>0$.

(ii) *For any $t\in[t_0,T]$, it follows that $f(0,t),g(0,t)\in L^2$ such that*

(3.8) $$|f(0,t)|^2\vee|g(0,t)|^2\le K_1,$$

where K_1 is a positive constant.

Then, for all $n\ge 1$, we have

(3.9) $$\sup_{t_0<t\le T}E|x_n(t)|^2$$
$$\le C_4:=d(T)\Big\{1-(\alpha-1)\big[d(T)\big]^{\alpha-1}\big[6K(T-t_0+1)\big]\big(T-t_0\big)\Big\}^{-1/(\alpha-1)},$$

where $d(T)=a(T)+6a(T-t_0)(T-t_0+1)$ and $a(T)$ is defined in Lemma 3.1.

Proof. Fix $n\ge 1$ arbitrarily. It is easy to see from the definition of $x_n(t)$ and condition (3.7) and (3.8) that $\{x_n(t)\}_{t_0\le t\le T}\in\mathcal{M}^2((t_0,T];R^d)$. From (3.1), using the elementary inequality $(y+z+w)^2\le 3(y^2+z^2+w^2)$, we have following for $t_0\le t\le T$

$$|x_n(t)|^2\le 3|x_0|^2+3\Big|\int_{t_0}^{t}f(x_n(s-1/n),s)ds\Big|^2+3\Big|\int_{t_0}^{t}g(x_n(s-1/n),s)dB(s)\Big|^2.$$

By Hölder's inequality, we can derive that

$$|x_n(t)|^2\le 3|x_0|^2+3(T-t_0)\int_{t_0}^{t}|f(x_n(s-1/n),s)|^2ds+3\Big|\int_{t_0}^{t}g(x_n(s-1/n),s)dB(s)\Big|^2.$$

Taking the expectation on both sides and using the Lemma 2.2, we have following

$$\begin{aligned} &E|x_n(t)|^2 \\ &\leq 3E|x_0|^2 + 3(T-t_0)E\int_{t_0}^t |f(x_n(s-1/n),s)|^2 ds + 3E\int_{t_0}^t |g(x_n(s-1/n),s)|^2 ds. \end{aligned}$$

By the condition (3.7) and (3.8), we obtain

$$\begin{aligned} &E|x_n(t)|^2 \\ &\leq a(t) + 6(t-t_0+1)E\int_{t_0}^t \kappa(|x_n(s-1/n)|^{2\alpha})ds, \end{aligned}$$

where $a(t) = 3E|x_0|^2 + 6K_1(t-t_0)(t-t_0+1)$. Since $\kappa(\cdot)$ is concave and nondecreasing, there must exit a positive number a such that

$$\kappa(u) \leq a(1+u) \tag{3.10}$$

on $u \geq 0$. Consequently

$$\sup_{t_0 \leq s \leq t} E|x_n(s)|^2 \leq d(t) + 6a(t-t_0+1)\int_{t_0}^t E \sup_{t_0 \leq r \leq s} |x_n(r)|^{2\alpha} ds,$$

where $d(t) = a(t) + 6a(T-t_0)(T-t_0+1)$. An application of the Stachurska's inequality (Lemma2.4) implies that

$$\begin{aligned} &\sup_{t_0 \leq s \leq t} E|x_n(s)|^2 \\ &\leq d(T)\Big\{1-(\alpha-1)\Big[\frac{d(T)}{6a(T-t_0+1)}\Big]^{\alpha-1}\int_{t_0}^t [6a(T-t_0+1)]^\alpha ds\Big\}^{-1/(\alpha-1)} \\ &\leq d(T)\Big\{1-(\alpha-1)\big[d(T)\big]^{\alpha-1}\big[6a(T-t_0+1)\big]\big(t-t_0\big)\Big\}^{-1/(\alpha-1)} \end{aligned}$$

for all $t_0 \leq t \leq T$. In particular, the required inequality follows (3.9) follows when $t = T$. The proof is complete. ∎

Lemma 3.5. *Under the condition (3.7) and weakened linear growth condition (3.8), for all* $n \geq 1$ *and* $t_0 \leq s < t \leq T$ *with* $t - s \leq 1$, *the Caratheodory's approximate solution has the property*

$$E|x_n(t) - x_n(s)|^2 \leq C_5(t-s), \tag{3.11}$$

where $C_5 = 8K_1 + 8a + 8aC_4^\alpha$, a *is positive constant, and* C_4 *is defined in Lemma 3.4.*

Proof. From (3.1), using the elementary inequality $(y+z)^2 \leq 2(y^2+z^2)$, we have following

$$|x_n(t) - x_n(s)|^2 \leq 2\Big|\int_s^t f(x_n(r-1/n),r)dr\Big|^2 + 2\Big|\int_s^t g(x_n(r-1/n),r)dB(r)\Big|^2.$$

By Hölder's inequality, we can derive that

$$|x_n(t) - x_n(s)|^2 \leq 2(t-s)\int_s^t |f(x_n(r-1/n),r)|^2 dr + 2\Big|\int_s^t g(x_n(r-1/n),r)dB(r)\Big|^2.$$

Taking the expectation on both sides and using the Lemma 2.2, we have following

$$\begin{aligned}&E|x_n(t)-x_n(s)|^2\\&\le 4(t-s)E\int_s^t[|f(x_n(r-1/n),r)-f(0,r)|^2+|f(0,r)|^2]dr\\&+4E\int_s^t|g(x_n(r-1/n),r)-g(0,r)|^2+|g(0,r)|^2]dr.\end{aligned}$$

By the condition (3.7) and (3.8), we obtain

$$\begin{aligned}&E|x_n(t)-x_n(s)|^2\\&\le 4(t-s)\Big[E\int_s^t\kappa(|x_n(r-1/n)|^{2\alpha})dr+K_1(t-s)\Big]\\&\quad+4\Big[E\int_s^t\kappa(|x_n(r-1/n)|^{2\alpha})dr+K_1(t-s)\Big]\\&=4K_1(t-s)[t-s+1]+4[t-s+1]E\int_s^t\kappa(|x_n(r-1/n)|^{2\alpha})dr\\&\le 8K_1(t-s)+8E\int_s^t\kappa(|x_n(r-1/n)|^{2\alpha})dr.\end{aligned}$$

From the definition of $\kappa(\cdot)$ and the inequality (3.10), we have following that

$$E|x_n(t)-x_n(s)|^2\le 8K_1(t-s)+8a(t-s)+8K\int_s^t E\sup_{t_0\le k\le r}|x_n(k)|^{2\alpha}dr.$$

Hence, an application of Lemma3.4 implies that

$$E|x_n(t)-x_n(s)|^2\le[8K_1+8a+8aC_4^{\alpha}](t-s)$$

for all $t_0\le s<t\le T$. The proof is complete. ∎

Theorem 3.6. *Let $x(t)$ be the unique solution of the d-dimensional stochastic differential equation (2.1) of Itô type and $x_n(t)$ be the Caratheodory's approximate solution. Assume that the weakened linear growth condition (3.8) and the following weakened Hölder condition hold. For any $\varphi,\psi\in R^d$ and $t\in[t_0,T]$, we assume that*

$$|f(\varphi,t)-f(\psi,t)|^2\vee|g(\varphi,t)-g(\psi,t)|^2\le\kappa(|\varphi-\psi|^{2\alpha}),\tag{3.12}$$

where $0<\alpha<1$ and $\kappa(\cdot)$ is a concave non-decreasing function from R_+ to R_+ such that $\kappa(0)=0$, $0<\kappa(u)\le\beta u$ for $u>0$ and a constant $\beta(>0)$. Then, for all $n\ge1$, such that

$$E\Big(\sup_{t_0\le t\le T}|x_n(t)-x(t)|^2\Big)\le C_6\frac{1}{n^{\alpha}},\tag{3.13}$$

where $0<\alpha<1$, $C_6=h(T)\{1-(\alpha-1)[h(T)]^{\alpha-1}[4\beta(t-t_0+1)](t-t_0)\}^{-1/(\alpha-1)}$, $h(T)=4\beta(T-t_0+4)C_5^{\alpha}$, and C_5 is defined in Lemma 3.5.

Proof. From the definition of $x(t)$ and $x_n(t)$, using the elementary inequality $(y+z)^2 \le 2(y^2 + z^2)$ and Hölder's inequality, we have following

$$\begin{aligned}
&\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2 \\
&\le 2(t-t_0)\int_{t_0}^{t}|f(x_n(s-1/n),s)-f(x(s),s)|^2ds \\
&+2\sup_{t_0\le r\le t}\Big|\int_{t_0}^{r}[g(x_n(s-1/n),s)-g(x(s),s)]dB(s)\Big|^2.
\end{aligned}$$

Taking the expectation on both sides and using the Lemma 2.3, we have following

$$\begin{aligned}
&E\Big(\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2\Big) \\
&\le 4(t-t_0)E\int_{t_0}^{t}[|f(x_n(s),s)-f(x_n(s-1/n),s)|^2+|f(x_n(s),s)-f(x(s),s)|^2ds \\
&+16E\int_{t_0}^{r}[|g(x_n(s),s)-g(x_n(s-1/n),s)|^2+|g(x_n(s),s)-g(x(s),s)|^2]ds.
\end{aligned}$$

By the condition (3.8) and (3.12), we obtain

$$\begin{aligned}
&E\Big(\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2\Big) \\
&\le 4(t-t_0+4)E\int_{t_0}^{t}\kappa(|x_n(s)-x_n(s-1/n)|^{2\alpha})ds \\
&+4(t-t_0+4)E\int_{t_0}^{r}\kappa(|x_n(r)-x(r)|^{2\alpha})ds.
\end{aligned}$$

From the definition of $\kappa(\cdot)$, we have following that

$$\begin{aligned}
&E\Big(\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2\Big) \\
&\le 4\beta(t-t_0+4)E\int_{t_0}^{t}|x_n(s)-x_n(s-1/n)|^{2\alpha}ds \\
&+4\beta(t-t_0+4)E\int_{t_0}^{r}|x_n(r)-x(r)|^{2\alpha}ds.
\end{aligned}$$

But, an application of Lemma3.5 implies that

$$E|x_n(s)-x_n(s-1/n)|^2 \le C_5\frac{1}{n}$$

if $s \ge t_0 + 1/n$, otherwise if $t_0 \le s < t_0 + 1/n$, we have

$$E|x_n(s)-x_n(s-1/n)|^2 = E|x_n(s)-x_n(t_0)|^2 \le C_5(s-t_0) \le C_5\frac{1}{n}.$$

Therefore, it follows from the above inequality that

$$\begin{aligned}
&E\Big(\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2\Big) \\
&\le 4\beta(t-t_0+4)C_5^{\alpha}\Big(\frac{1}{n}\Big)^{\alpha}+4\beta(t-t_0+4)\int_{t_0}^{t}\Big(E\sup_{t_0\le r\le s}|x_n(r)-x(r)|^2\Big)^{\alpha}ds.
\end{aligned}$$

An application of the Stachurska's inequality (Lemma2.4) implies that

$$E\Big(\sup_{t_0\le r\le t}|x_n(r)-x(r)|^2\Big)$$
$$\le h(t)\Big(\frac{1}{n}\Big)^{\alpha}\Big\{1-(\alpha-1)\big[h(t)\big]^{\alpha-1}\big[4\beta(t-t_0+1)\big]\big(t-t_0\big)\Big\}^{-1/(\alpha-1)},$$

where $h(t)=4\beta(t-t_0+4)C_5^{\alpha}$. In particular, the required inequality follows (3.13) follows when $t=T$. The proof is complete. ∎

4. CONCLUSION

Using the Hölder's condition and weakened linear growth condition, in the Theorem 3.3, we have shown that the Caratheodory's approximate solution $x^n(t)$ converge to the unique solution $x(t)$ of equation (2.1) for rational number α. In practice, given the error $\epsilon>0$, one can let n^α be a large number than C_3/ϵ and then compute $x_n(t)$ over the intervals $[t_0,t_0+1/n],(t_0+1/n,t_0+2/n],\cdots$, step by step. Theorem 3.3 guarantees that this $x_n(t)$ is closed enough to the accurate solution $x(t)$ in the sense

$$E\Big(\sup_{t_0\le t\le T}|x_n(t)-x(t)|^2\Big)<\epsilon.$$

In the Theorem 3.6, using the weakened Hölder's condition and weakened linear growth condition, we have shown that a dynamic movement relationship between the Caratheodory's approximate solution $x^n(t)$ and the unique solution $x(t)$ of equation (2.1). In other words, Theorem 3.6 guarantees that this $x_n(t)$ is closed enough to the accurate solution $x(t)$ in the same sense in Theorem 3.3.

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