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A NEW METHOD WITH REGULARIZATION FOR SOLVING SPLIT VARIATIONAL INEQUALITY PROBLEMS IN REAL HILBERT SPACES

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ABSTRACT. In this paper, we introduce a new inertial extrapolation method with regularization for approximating solutions of split variational inequality problems in the frame work of real Hilbert spaces. We prove that the proposed method converges strongly to a minimum-norm solution of the problem without using the conventional two cases approach. In addition, we present some numerical experiments to show the efficiency and applicability of the proposed method. The results obtained in this paper extend, generalize and improve several results in this direction.

Key words and phrases: Split variational inequality problem; Split feasibility problem; Inertial iterative scheme; Fixed point problem.

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1. INTRODUCTION

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, C be a nonempty closed convex subset of H and $A : H \to H$ be an operator. The classical Variational Inequality Problem (VIP) is formulated as: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \; \forall \, y \in C. \tag{1.1}$$

The notion of VIP was introduced independently by Stampacchia [26] and Ficher [14, 15] for modeling problems arising from mechanics and for solving Signorini problem. It is well-known that many problems in economics, mathematical sciences, mathematical physics can be formulated as VIP. Censor et al. in [12] extended the concept of VIP (1.1) to the following Split Variational Inequality Problem (SVIP): Find

$$x^* \in C \text{ that solves } \langle A_1 x^*, x - x^* \rangle \geq 0 \; \forall \, x \in C \tag{1.2}$$

such that $y^* = Tx^* \in Q$ solves

$$\langle A_2 y^*, y - y^* \rangle \geq 0 \; \forall \, y \in Q, \tag{1.3}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_1 : H_1 \to H_1, \; A_2 : H_2 \to H_2$ are two operators and $T : H_1 \to H_2$ is a bounded linear operator. When $A_1 = A_2 = 0$, the SVIP reduces to the Split Feasibility Problem (SFP). That is, find

$$x^* \in C \text{ such that } y^* = Tx^* \in Q. \tag{1.4}$$

The concept of SFP was introduced by Censor and Elfving [9] in the framework of finite-dimensional Hilbert spaces. The SFP has found applications in many real-life problems such as image recovery, signal processing, control theory, data compression, computer tomography and so on (see [10, 11] and the references therein). Therefore, it has attracted the attention of a lot of researchers in this direction. For instance, Ceng et al. [6] proposed the following iterative method for solving the SFP:

$$\begin{cases} x_0 = x \in C \\ y_n = (1 - \beta_n) x_n + \beta_n P_C(x_n - \lambda \nabla f_{\alpha_n}(x_n)) \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n) S P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \end{cases} \tag{1.5}$$

where $\nabla f_{\alpha_n} = \alpha_n I + T^*(I - P_Q)T, \; S : C \to C$ is a nonexpansive mapping and the sequences of parameters $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$. The above iterative algorithm is a combination of the regularization method and extragradient method due to Nadezhkina and Takahashi [21]. Under some mild assumptions, they established that the sequence generated by the iterative method converges weakly to a common solution of the SFP and fixed point problem for nonexpansive mapping.

In 2020 Chuasuk and Kaewcharoen [13] proposed the following iterative scheme:

$$\begin{cases} x_0 = H_1 \\ y_n = P_C(x_n - \lambda_n(T^*(I - SP_Q))T + \alpha_n I) x_n) \\ z_n = P_C(x_n - \lambda_n(T^*(I - SP_Q))T + \alpha_n I) y_n) \\ w_n = (1 - \sigma_n) z_n + \sigma_n U z_n \\ s_n = (1 - \beta_n) z_n + \beta_n U w_n \\ x_{n+1} = (1 - \gamma_n) z_n + \gamma_n U s_n, \end{cases} \tag{1.6}$$

where $S : Q \to Q$ is a nonexpansive mapping, $U : C \to C$ is a pseudo-contractive an L-Lipschitzian continuous mapping and the sequences of parameters $\{\sigma_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$. Under some mild assumptions, they established that the sequence generated by the iterative method converges weakly to a common solution of the SFP and fixed point problem for nonexpansive mapping. The above iterative scheme is the combination of an extragradient method with regularization due to a generalized Ishikawa iterative scheme.

As already mentioned, Algorithm (1.5) and Algorithm (1.6) are regularization-type methods with the regularization steps involving $\alpha I + T^*(I - P_Q)T$. Regularization-type methods have been employed in a number of problems, mainly due to its efficiency in solving these problems. For example, let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function, then the minimization problem

$$\min_{x\in C} f(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2$$

is ill-posed (see [29]). To address this problem, Xu [29] considered the following Tikhonov regularized problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2 + \frac{1}{2}\alpha\|x\|,$$

where $\alpha > 0$ is the regularization parameter.

Remark 1.1. The traditional Tikhonov regularization methods are usually used to solve ill-posed optimization problems. Moreover, one of the advantages of regularization methods are their possible strong convergence to minimum-norm solutions to optimization problems (see [5, 6, 7, 29] and the references therein).

Question 1: It is natural to ask if Algorithms (1.5) and (1.6) can be modified to converge strongly to a minimum-norm solution of the SVIP (1.2)-(1.3)?

The inertial extrapolation method has proven to be an effective way for accelerating the rate of convergence of iterative algorithms. The technique is based on a discrete version of a second order dissipative dynamical system [2, 3]. The inertial type algorithms use its two previous iterates to obtain its next iterate [1, 20]. For details on inertia extrapolation, see [4, 22, 23] and the references therein.

Motivated by the research works in this direction, in this paper, we provide an affirmative answer to Question 1 raised above. That is, we propose an inertial regularization method for solving the SVIP (1.2)-(1.3) in real Hilbert spaces. We prove that the method converges strongly to a minimum-norm solution of the problem when the underlying operators are α-inverse strongly monotone operator and Lipschitz continuous monotone operator. Moreover, our method of proof does not rely on the conventional two cases approach for strong convergence. To the best of our knowledge the regularization method is yet to be used to solve the SVIP. Furthermore, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite and finite dimensional Hilbert spaces. We emphasize that one of the novelty of this work is in the use of the regularization approach and in the method of proof of its strong convergence to a minimum-norm solution of the SVIP. The results obtained in this work extend, generalize and improve several results in this direction.

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method and highlight some of its useful features. advantages over other existing algorithms. In Section 4, we establish strong convergence of our method and in Section 5, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces. Lastly in Section 6, we give the conclusion of the paper.

2. PRELIMINARIES

In this section we begin by recalling some known and useful results which are needed in the sequel.

Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$\langle x, y\rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x - y\|^2). \tag{2.1}$$

$$\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x - y\rangle. \tag{2.2}$$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \tag{2.3}$$

Definition 2.1. Let $T : H \to H$ be an operator. Then the operator T is called

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for all $x, y \in H$. If $L = 1$, then T is called nonexpansive;

(b) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0,\ \forall x, y \in H;$$

(c) α-inverse strongly monotone (α-ism) if there exists $\alpha > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2,\ \forall\, x, y \in H.$$

Let C be a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|\ \forall y \in C.$$

The operator P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2,$$

for all $x, y \in H$. Furthermore, P_C is characterized by the property

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

for all $x \in H$ and $y \in C$.

Lemma 2.1. *Let C be nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if $\langle x - z, z - y\rangle\ \forall\, y \in C$.*

It is well-known that the metric projection P_C is firmly nonexpansive, that is,

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$$
$$\Leftrightarrow \|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2\ \forall x, y \in H. \tag{2.4}$$

It is well-known that for any nonexpansive mapping T, the set of fixed points of T is closed and convex. Also, T satisfies the following inequality

$$\langle (x - Tx) - (y - Ty), Ty - Tx\rangle \leq \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2,\ \forall\, x, y \in H. \tag{2.5}$$

Thus, for all $x \in H$ and $x^* \in F(T)$, we have that

$$\langle x - Tx, x^* - Tx\rangle \leq \frac{1}{2}\|Tx - x\|^2,\ \forall\, x, y \in H. \tag{2.6}$$

Definition 2.2. [19] Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. A mapping $T : C \to C$ is said to be demiclosed at 0, if for any sequence $\{x_n\} \subset C$ which converges weakly to x and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0,\ Tx = x$.

Lemma 2.2. [27] *Let T be an α-ism operator, then*

(1) *T is a $\frac{1}{\alpha}$-Lipschitz continuous monotone operator.*
(2) *if $\lambda \in (0, 2\alpha)$, then $(I - \lambda T)$ is a nonexpansive mapping, where I is the identity operator on H.*

Lemma 2.3. [19] *Let C be a closed and convex subset of a Hilbert space H and $T : C \to C$ be nonexpansive mapping with $F(T) \neq \emptyset$. Then, T is demiclosed at 0.*

Lemma 2.4. [25] *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty} \{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.5. [28] *Let $A : H \to H$ be a continuous and monotone operator. Then, x^* is a solution of VIP (1.1) if and only if x^* is a solution of the following problem. Find $x^* \in C$ such that $\langle Ax, x - x^*\rangle \geq 0$ for all $x \in C$.*

3. PROPOSED ALGORITHM

In this section we present our proposed method and highlight some of its important features. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 3.1. *Suppose that the following conditions hold:*

(1) *The sets C and Q are nonempty closed and convex subsets of the real Hilbert spaces H_1 and H_2 respectively.*
(2) *$A_1 : H_1 \to H_1$ is monotone and Lipschitz continuous operator and $A_2 : H_2 \to H_2$ is α-inverse strongly monotone operator.*
(3) *$T : H_1 \to H_2$ is a bounded linear operator.*
(4) *The solution set $\Gamma = \{x \in VI(A_1, C) : Tx \in VI(A_2, Q)\} \neq \emptyset$, where $VI(A_1, C)$ is the solution set for the classical VIP (1.1).*

We present the following iterative algorithm.

Algorithm 3.2. Initialization: *Given $\lambda, \gamma_n > 0$, $\theta_n, \alpha_n, \mu \in (0,1)$, and $\beta_n \subset (b, 1 - \alpha_n)$ for some $b > 0$, for all $n \in \mathbb{N}$. Let $x_0, x_1, \in H$ be arbitrary.*

Iterative steps:
Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\max\{n^2\|x_n - x_{n-1}\|^2, n^2\|x_n - x_{n-1}\|\}}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise} \end{cases}, \tag{3.1}$$

with θ been a positive constant and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$.
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = w_n - \gamma_n(T^*(I - P_Q(I - \eta A_2))T + \alpha_n I)w_n, \tag{3.2}$$

where $\eta \in (0, 2\alpha)$ and

$$\gamma_n \in \left(\epsilon, \frac{\|P_Q(I - \lambda_n A_2) - I)Tw_n\|^2}{\|T^*(P_Q(I - \lambda_n A_2) - I)Tw_n\|^2} - \epsilon\right), \text{ if } P_Q(I - \lambda_n A_2)Tw_n \neq Tw_n \text{ otherwise } \gamma_n = \epsilon, .$$

Step 3. *Compute*

$$v_n = P_C(u_n - \lambda_n A_1 u_n) \tag{3.3}$$

$$y_n = u_n - \tau_n b_n, \tag{3.4}$$

where $b_n = u_n - v_n - \lambda_n(A_1 u_n - A_1 v_n)$; $\tau_n = \frac{\langle u_n - v_n, b_n\rangle}{\|b_n\|^2}$ if $b_n \neq 0$; otherwise $\tau_n = 0$; and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - v_n\|}{\|A_1 u_n - A_1 v_n\|},\ \lambda_n\right\}, & \text{if } A_1 u_n \neq A_1 v_n, \\ \lambda_n, & \text{otherwise} \end{cases}. \tag{3.5}$$

Step 4. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n y_n. \tag{3.6}$$

Remark 3.1.

(1) A notable advantage of this method (Algorithm 3.2) is that $\{x_n\}$ converges strongly to a minimum-norm solution of the SVIP. This is very desirable in optimization theory.
(2) The choice of the stepsize $\{\gamma_n\}$ used in Algorithm 3.2 does not require the prior knowledge of the operator norm $\|T\|$ which is very difficult to find in practice. In addition, the stepsize $\{\lambda_n\}$ is self adaptive.
(3) As we shall see in our convergence analysis, we do not use the popular two cases method usually used in numerous papers to guarantee strong convergence. Thus the techniques and ideas employed in our strong convergence analysis are new for solving the problem considered in this paper.
(4) In Algorithm 3.2, it is easy to compute step 1 since the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. It is also easy to see from (3.1) that $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

    Indeed, since, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, which means that $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, we have that $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, it implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

(5) It is easy to see in (3.5) that $\lambda_{n+1} \leq \lambda_n$ for all $n \in \mathbb{N}$. More so, since A_1 is L-Lipschitz continuous, we obtain in the case when $Au_n \neq Av_n$ that

$$\frac{\mu\|u_n - v_n\|}{\|A_1 u_n - A_1 v_n\|} \geq \frac{\mu\|u_n - v_n\|}{L\|u_n - v_n\|} = \frac{\mu}{L},$$

    which follows that $\lambda_n \geq \min\{\lambda_1, \frac{\mu}{L}\}$ for all $n \in \mathbb{N}$. This gives that the limit of $\{\lambda_n\}$ exists and $\lim\limits_{n\to\infty} \lambda_n \geq \min\{\lambda_1, \frac{\mu}{L}\} > 0$.

4. CONVERGENCE ANALYSIS

In this section we establish strong convergence result of our proposed method.

Lemma 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Then, under Assumption 3.1, we have that $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$ and since $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$, for all $n \in \mathbb{N}$. Then from **Step** 2, we have

$$\begin{aligned}
\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\
&\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\leq \|x_n - p\| + \alpha_n N_1. \qquad (4.1)
\end{aligned}$$

Using (2.1) and (2.2), we obtain

$$\begin{aligned}
\|u_n - p\|^2 &= \|w_n + \gamma_n(T^*(P_Q(I-\eta A_2) - I)T - \alpha_n I)w_n - p\|^2 \\
&= \|w_n - p\|^2 + \|\gamma_n(T^*(P_Q(I-\eta A_2) - I)T - \alpha_n I)w_n\|^2 \\
&+ 2\langle w_n - p, \gamma_n(T^*(P_Q(I-\eta A_2) - I)T - \alpha_n I)w_n\rangle \\
&\leq \|w_n - p\|^2 + \gamma_n^2\|T^*P_Q(I-\eta A_2) - I)Tw_n\|^2 \\
&+ 2\langle \gamma_n\alpha_n w_n, \gamma_n(T^*(P_Q(I-\eta A_2) - I)T - \alpha_n I)w_n\rangle \\
&+ 2\langle w_n - p, \gamma_n T^*(P_Q(I-\eta A_2) - I)Tw_n\rangle + 2\langle w_n - p, -\gamma_n\alpha_n w_n\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2\|T^*(P_Q(I-\eta A_2) - I)Tw_n\|^2 \\
&+ 2\gamma_n\langle w_n - p, T^*(P_Q(I-\eta A_2) - I)Tw_n\rangle \\
&- \gamma_n\alpha_n\langle 2(u_n - p) + \gamma_n\alpha_n w_n, w_n\rangle. \qquad (4.2)
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\langle w_n - p, T^*(P_Q(I-\eta A_2) - I)Tw_n\rangle \\
&= \langle Tw_n - Tp, P_Q(I-\eta A_2)Tw_n - Tw_n\rangle \\
&= \langle Tw_n + P_Q(I-\eta A_2)Tw_n - P_Q(I-\eta A_2)Tw_n - Tw_n + Tw_n - Tp, \\
&P_Q(I-\eta A_2)Tw_n - Tw_n\rangle \\
&= \langle P_Q(I-\eta A_2)Tw_n - Tp, P_Q(I-\eta A_2)Tw_n - Tw_n\rangle \\
&- \|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2 \\
&= \frac{1}{2}[\|P_Q(I-\eta A_2)Tw_n - Tp\|^2 + \|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2 \\
&- \|Tw_n - Tp\|^2] - \|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2 \\
&\leq \frac{1}{2}\|Tw_n - Tp\|^2 - \frac{1}{2}\|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2 - \frac{1}{2}\|Tw_n - Tp\|^2 \\
&= -\frac{1}{2}\|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2. \qquad (4.3)
\end{aligned}$$

Substituting (4.3) into (4.2), we have

$$
\begin{aligned}
\|u_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 \\
&- \gamma_n \|P_Q(I - \eta A_2)Tw_n - Tw_n\|^2 - \gamma_n \alpha_n \langle 2(u_n - p) + \gamma_n \alpha_n w_n, w_n \rangle \\
&\leq \|w_n - p\|^2 + \gamma_n^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 \\
&- \gamma_n(\gamma_n + \epsilon) \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 - \gamma_n \alpha_n \langle 2(u_n - p) + \gamma_n \alpha_n w_n, w_n \rangle \\
&= \|w_n - p\|^2 - \gamma_n [\epsilon \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 + \alpha_n \langle 2(u_n - p) + \gamma_n \alpha_n w_n, w_n \rangle] \qquad (4.4) \\
&\leq \|w_n - p\|^2,
\end{aligned}
$$

this implies that

$$\|u_n - p\| \leq \|w_n - p\|. \tag{4.5}$$

Since $v_n = P_C(u_n - \lambda_n A_1 u_n)$ and $p \in VI(A_1, C) \subset C$, then by the characterization of P_C, we have

$$\langle v_n - p, v_n - u_n + \lambda_n A_1 u_n \rangle \leq 0.$$

Using the monotonicity of A_1, we obtain

$$
\begin{aligned}
\langle v_n - p, b_n \rangle &= \langle v_n - p, u_n - v_n - \lambda_n A_1 u_n \rangle + \lambda_n \langle v_n - p, A_1 v_n \rangle \\
&\geq \lambda_n \langle v_n - p, A_1 v_n \rangle \\
&= \lambda_n \langle v_n - p, A_1 v_n - A_1 p \rangle + \lambda_n \langle v_n - p, A_1 p \rangle \geq 0.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\langle u_n - p, b_n \rangle &= \langle u_n - v_n, b_n \rangle + \langle v_n - p, b_n \rangle \\
&\geq \langle u_n - v_n, b_n \rangle. \qquad (4.6)
\end{aligned}
$$

Hence from **Step** 3 and (4.6), we have

$$
\begin{aligned}
\|y_n - p\|^2 &= \|u_n - \tau_n b_n - p\|^2 \\
&= \|u_n - p\|^2 + \tau_n^2 \|b_n\|^2 - 2\tau_n \langle u_n - p, b_n \rangle \\
&\leq \|u_n - p\|^2 + \tau_n^2 \|b_n\|^2 - 2\tau_n \langle u_n - v_n, b_n \rangle \\
&\leq \|u_n - p\|^2 + \tau_n^2 \|b_n\|^2 - 2\tau_n^2 \|b_n\|^2 \\
&= \|u_n - p\|^2 - \|\tau_n b_n\|^2 \\
&\leq \|u_n - p\|^2, \qquad (4.7)
\end{aligned}
$$

this implies that

$$\|y_n - p\| \leq \|w_n - p\|. \tag{4.8}$$

In addition we observe that

$$\begin{aligned}
&\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|^2\\
&=(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|y_n-p\|^2\\
&+2(1-\alpha_n-\beta_n)\beta_n\langle x_n-p,y_n-p\rangle\\
&\leq(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|y_n-p\|^2\\
&+2(1-\alpha_n-\beta_n)\beta_n\|x_n-p\|\|y_n-p\|\\
&\leq(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|y_n-p\|^2\\
&+(1-\alpha_n-\beta_n)\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)\beta_n\|y_n-p\|^2\\
&=(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n\|y_n-p\|^2\\
&\leq(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n\|w_n-p\|^2\\
&\leq(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n[\|x_n-p\|+\alpha_nN_1]^2\\
&=(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n\|x_n-p\|^2\\
&+2(1-\alpha_n)\beta_n\alpha_n\|x_n-p\|N_1+(1-\alpha_n)\beta_n\alpha_n^2N_1^2\\
&\leq(1-\alpha_n)^2\|x_n-p\|^2+2(1-\alpha_n)\alpha_n\|x_n-p\|N_1+\alpha_n^2N_1^2\\
&=[(1-\alpha_n)\|x_n-p\|+\alpha_nN_1]^2, \qquad (4.9)
\end{aligned}$$

this implies that

$$\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|\leq(1-\alpha_n)\|x_n-p\|+\alpha_nN_1. \qquad (4.10)$$

Lastly, we have

$$\begin{aligned}
\|x_{n+1}-p\|&=\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)-\alpha_np\|\\
&\leq\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|+\alpha_n\|p\|\\
&\leq(1-\alpha_n)\|x_n-p\|+\alpha_nN_1+\alpha_n\|p\|\\
&=(1-\alpha_n)\|x_n-p\|+\alpha_n(N_1+\|p\|)\\
&\leq\max\{\|x_n-p\|,N_1+\|p\|\}\\
&\quad\vdots\\
&\leq\max\{\|x_1-p\|,N_1+\|p\|\}. \qquad (4.11)
\end{aligned}$$

Thus, $\{x_n\}$ is bounded. ∎

Lemma 4.2. *Let Assumption 3.1 hold and let* $\{x_n\}$ *be a sequence generated by Algorithm 3.2. Assume that the subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *converges weakly to a point* x^**, and* $\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=\lim_{k\to\infty}\|u_{n_k}-v_{n_k}\|=0$*, then,* $x^*\in\Gamma$.

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $x^*\in H_1$. It is easy to see that

$$\|w_{n_k}-x_{n_k}\|=\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_n-1}\|\to 0 \text{ as } k\to\infty. \qquad (4.12)$$

It follows that

$$\|u_{n_k}-x_{n_k}\|\leq\|u_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.13)$$

Since T is a bounded linear operator, it follows from (4.12) that $\{Tw_{n_k}\}$ converges weakly to $Tx^* \in Q \subset H_2$. Also, by (4.13), we obtain that u_{n_k} converges weakly to x^*. In addition, we have

$$\|v_{n_k} - x_{n_k}\| \leq \|v_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{4.14}$$

From (4.4), we have that

$$\begin{aligned}\|u_n - p\|^2 &\leq \|w_n - p\|^2 - \gamma_n \epsilon \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 \\ &\leq \|w_n - p\|^2 - \epsilon^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2,\end{aligned} \tag{4.15}$$

which implies that

$$\begin{aligned}\epsilon^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_{n_k}\|^2 &\leq \|w_{n_k} - p\|^2 - \|u_{n_k} - p\|^2 \\ &\leq \|w_{n_k} - u_{n_k}\|^2 + 2\|u_{n_k} - p\|\|w_{n_k} - u_{n_k}\|,\end{aligned} \tag{4.16}$$

thus, we have that

$$\lim_{k\to\infty} \|T^*(P_Q(I - \eta A_2) - I)Tw_{n_k}\| = 0. \tag{4.17}$$

More so, from (4.4), we have

$$\begin{aligned}\|u_n - p\| &\leq \|w_n - p\|^2 + \gamma_n^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 - \gamma_n \|P_Q(I - \eta A_2)Tw_n - Tw_n\|^2 \\ &\leq \|w_n - p\|^2 + \gamma_n^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 - \epsilon \|P_Q(I - \eta A_2)Tw_n - Tw_n\|^2,\end{aligned} \tag{4.18}$$

which implies that

$$\begin{aligned}&\epsilon \|P_Q(I - \eta A_2)Tw_{n_k} - Tw_{n_k}\|^2 \\ &\leq \|w_{n_k} - p\|^2 - \|u_{n_k} - p\|^2 + \gamma_n^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_{n_k}\|^2 \\ &\leq \|w_{n_k} - u_{n_k}\|^2 + 2\|u_{n_k} - p\|\|w_{n_k} - u_{n_k}\| + \gamma_n^2 \|T^*(P_Q(I - \eta A_2) - I)Tw_{n_k}\|^2,\end{aligned} \tag{4.19}$$

which implies that

$$\lim_{k\to\infty} \|P_Q(I - \eta A_2)Tw_{n_k} - Tw_{n_k}\| = 0. \tag{4.20}$$

Using Lemma 2.3 and (4.20), we have that

$$Tx^* \in F(P_Q(I - \eta A_2)) \Rightarrow Tx^* \in VI(A_2, Q). \tag{4.21}$$

In addition since $v_{n_k} = P_C(u_{n_k} - \lambda_{n_k} A_1 u_{n_k})$, we obtain

$$\langle u_{n_k} - \lambda_{n_k} A_1 u_{n_k} - v_{n_k}, v - v_{n_k} \rangle \leq 0 \ \forall v \in C. \tag{4.22}$$

Then,

$$\begin{aligned}\langle u_{n_k} - v_{n_k}, v - v_{n_k} \rangle &\leq \lambda_{n_k} \langle A_1 u_{n_k}, v - v_{n_k} \rangle \\ &\leq \lambda_{n_k} \langle A_1 u_{n_k}, u_{n_k} - v_{n_k} \rangle + \lambda_{n_k} \langle A_1 u_{n_k}, v - u_{n_k} \rangle \ \forall \, v \in C.\end{aligned} \tag{4.23}$$

Now fix $v \in C$ and take limit as $n \to \infty$ in (4.23), since $\|u_{n_k} - v_{n_k}\| \to 0$ and $\liminf \lambda_{n_k} > 0$, we have

$$0 \leq \liminf_{k\to\infty} \langle A_1 u_{n_k}, v - u_{n_k} \rangle \ \forall v \in C. \tag{4.24}$$

Since A_1 is monotone, we then have

$$\langle A_1 v, v - u_{n_k} \rangle \geq \langle A_1 u_{n_k}, v - u_{n_k} \rangle \ \forall \, v \in C. \tag{4.25}$$

Taking liminf of both sides, we have

$$\liminf_{k\to\infty} \langle A_1 v, v - u_{n_k} \rangle \geq \liminf_{k\to\infty} \langle A_1 u_{n_k}, v - u_{n_k} \rangle \ \forall \, v \in C. \tag{4.26}$$

More so since $\{u_{n_k}\}$ converges weakly to x^*, the it follows from (4.24) and (4.26) that

$$\langle A_1 v, v - x^* \rangle = \liminf_{k\to\infty} \langle A_1 v, v - u_{n_k} \rangle \geq 0. \tag{4.27}$$

Thus, using Lemma 2.5, we have that $x^* \in VI(A_1, C)$. Using this fact and (4.21), we have that $x^* \in \Gamma$. ∎

Theorem 4.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then, under the Assumption 3.1, if $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $\|p\| = \min\{\|x^*\| : x^* \in \Gamma\}$.*

Proof. Let $p \in \Gamma$. To start with, observe that

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n \|x_{n-1} - p\| \|x_n - p\| + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n N_1] \\
&\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2, \qquad (4.28)
\end{aligned}$$

for some $N_2 > 0$. ∎

Also,

$$\begin{aligned}
\|(1-\beta_n)x_n + \beta_n y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(y_n - p)\|^2 \\
&= (1-\beta_n)^2 \|x_n - p\|^2 + \beta_n^2 \|y_n - p\|^2 + 2(1-\beta_n)\beta_n \langle x_n - p, y_n - p \rangle \\
&\leq (1-\beta_n)^2 \|x_n - p\|^2 + \beta_n^2 \|w_n - p\|^2 + 2(1-\beta_n)\beta_n \|x_n \\
&- p\| \|y_n - p\| \\
&\leq (1-\beta_n)^2 \|x_n - p\|^2 + \beta_n^2 \|w_n - p\|^2 + (1-\beta_n)\beta_n \|x_n - p\|^2 \\
&+ (1-\beta_n)\beta_n \|y_n - p\|^2 \\
&\leq (1-\beta_n)^2 \|x_n - p\|^2 + \beta_n^2 \|w_n - p\|^2 + (1-\beta_n)\beta_n \|x_n - p\|^2 \\
&+ (1-\beta_n)\beta_n \|w_n - p\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n \|w_n - p\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n [\|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2] \\
&\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2. \qquad (4.29)
\end{aligned}$$

Hence we have that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)[(1-\beta_n)x_n+\beta_n y_n-p]-[\beta_n\alpha_n(x_n-y_n)+\alpha_n p]\|^2 \\
&\le (1-\alpha_n)^2\|(1-\beta_n)x_n+\beta_n y_n-p\|^2-2\langle\beta_n\alpha_n(x_n-y_n)+\alpha_n p, x_{n+1}-p\rangle \\
&\le (1-\alpha_n)^2\|(1-\beta_n)x_n+\beta_n y_n-p\|^2+2\langle\beta_n\alpha_n(x_n-y_n), p-x_{n+1}\rangle \\
&+2\alpha_n\langle p, p-x_{n+1}\rangle \\
&\le (1-\alpha_n)[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2]+2\alpha_n\beta_n\|x_n-y_n\|\|x_{n+1}-p\| \\
&+2\alpha_n\langle p, p-x_{n+1}\rangle \\
&\le (1-\alpha_n)\|x_n-p\|^2+2\alpha_n\beta_n\|x_n-y_n\|\|x_{n+1}-p\|+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|N_2 \\
&+2\alpha_n\langle p, p-x_{n+1}\rangle \\
&= (1-\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\|x_n-y_n\|\|x_{n+1}-p\|+\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|N_2 \\
&+2\langle p, p-x_{n+1}\rangle] \\
&= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\delta_n \qquad (4.30)
\end{aligned}
$$

where $\delta_n := 2\beta_n\|x_n-y_n\|\|x_{n+1}-p\|+\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|N_2+2\langle p, p-x_{n+1}\rangle$. According to Lemma 2.4, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\delta_{n_k}\le 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\}\ge 0. \tag{4.31}$$

To establish that $\limsup_{k\to\infty}\delta_{n_k}\le 0$, we suppose that for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ such that (4.31) holds. Then,

$$
\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\} \\
&=\liminf_{k\to\infty}\{(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)\} \\
&\ge 0. \qquad (4.32)
\end{aligned}
$$

Now using **Step 4** , we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n-\beta_n)x_n+\beta_n y_n-p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)-\alpha_n p\|^2 \\
&\le \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|^2+\alpha_n^2\|p\|^2 \\
&-2\alpha_n\langle(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p), p\rangle \\
&\le \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|^2+\alpha_n M \\
&\le (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|y_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M \\
&\le (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M \\
&\le \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M \qquad (4.33)
\end{aligned}
$$

for some $M > 0$. This implies from (4.32) that

$$\begin{aligned}\limsup_{k\to\infty}[(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|y_{n_k}-x_{n_k}\|^2] &\leq \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\\ &\quad+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+\alpha_{n_k}M]\\ &\leq -\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\leq 0, \end{aligned} \tag{4.34}$$

which gives

$$\lim_{k\to\infty}\|y_{n_k}-x_{n_k}\|=0. \tag{4.35}$$

Similarly using **Step 4,** (4.4), (4.7), (4.28), (4.35) and (4.33), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n[\|u_n-p\|^2-\|\tau_n b\|^2]\\ &\quad-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M\\ &\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2-\beta_n\|u_n-y_n\|^2\\ &\quad-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M\\ &\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-\beta_n\|u_n-y_n\|^2\\ &\quad-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M \end{aligned} \tag{4.36}$$

This implies from (4.32) that

$$\begin{aligned}\limsup_{k\to\infty}[\beta_{n_k}\|u_{n_k}-y_{n_k}\|^2] &\leq \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\\ &\quad+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\ &\quad-(1-\alpha_{n_k}-\beta_{n_k})\beta_n\|y_{n_k}-x_{n_k}\|^2+\alpha_{n_k}M]\\ &\leq -\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\leq 0, \end{aligned} \tag{4.37}$$

which gives

$$\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0. \tag{4.38}$$

Now, observe that

$$\begin{aligned}\langle u_{n_k}-v_{n_k},b_{n_k}\rangle &= \langle u_{n_k}-v_{n_k},u_{n_k}-v_{n_k}-\lambda_{n_k}(A_1u_{n_k}-A_1v_{n_k})\rangle\\ &= \|u_{n_k}-v_{n_k}\|^2-\langle u_{n_k}-v_{n_k},\lambda_{n_k}(A_1u_{n_k}-A_1v_{n_k})\rangle\\ &\geq \|u_{n_k}-v_{n_k}\|^2-\lambda_{n_k}\|u_{n_k}-v_{n_k}\|\|A_1u_{n_k}-A_1v_{n_k}\|\\ &\geq \|u_{n_k}-v_{n_k}\|^2-\frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}}\|u_{n_k}-v_{n_k}\|^2\\ &= (1-\frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}})\|u_{n_k}-v_{n_k}\|^2 \end{aligned} \tag{4.39}$$

which implies that

$$\begin{aligned}\|u_{n_k}-v_{n_k}\|^2 &\le \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\langle u_{n_k}-v_{n_k}, b_{n_k}\rangle\\ &= \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\tau_{n_k}\|b_{n_k}\|^2\\ &= \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\tau_{n_k}\|b_{n_k}\|\|u_{n_k}-v_{n_k}-\lambda_{n_k}(A_1u_{n_k}-A_1v_{n_k})\|\\ &\le \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\|u_{n_k}-y_{n_k}\|[\|u_{n_k}-v_{n_k}\|+\lambda_{n_k}\|A_1v_{n_k}-A_1u_{n_k}\|]\\ &= \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}(1+\frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}})\|u_{n_k}-y_{n_k}\|\|u_{n_k}-v_{n_k}\|. \end{aligned} \tag{4.40}$$

Using (4.38), we have that

$$\lim_{k\to\infty}\|u_{n_k}-v_{n_k}\|=0. \tag{4.41}$$

Using **Step 4,** (3.2), (4.4), (4.35) and (4.33), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|u_n-p\|^2\\ &-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_nM\\ &\le (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2\\ &-\epsilon^2\beta_n\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2\\ &-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_nM\\ &\le \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-\epsilon^2\beta_n\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2\\ &-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_nM \end{aligned} \tag{4.42}$$

for some $M>0$. This implies from (4.32)

$$\begin{aligned}&\limsup_{k\to\infty}[\epsilon^2\beta_{n_k}\|T^*(P_Q(I-\eta A_2)-I)Tw_{n_k}\|^2]\\ &\le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\ &-(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|y_{n_k}-x_{n_k}\|^2+\alpha_{n_k}M]\\ &\le -\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\le 0, \end{aligned} \tag{4.43}$$

which gives

$$\lim_{k\to\infty}\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2=0. \tag{4.44}$$

Using a similar approach as in (4.42) and (4.18), we have that

$$\lim_{k\to\infty}\|(P_Q(I-\eta A_2)-I)Tw_n\|^2=0. \tag{4.45}$$

Using (4.44) and our hypothesis, we have

(4.46)

$$\|u_{n_k}-w_{n_k}\|=\|w_{n_k}+\gamma_nT^*(P_Q(I-\eta A_2-I)Tw_n-\alpha_nI)w_{n_k}-w_{n_k}\|\to 0 \text{ as } k\to\infty.$$

It is easy to see that, as $k \to \infty$, we have

$$\|w_{n_k} - x_{n_k}\| = \theta_{n_k}||x_{n_k} - x_{n_k-1}|| = \alpha_{n_k} \cdot \frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| \to 0. \tag{4.47}$$

In addition, we have that

$$\|w_{n_k} - y_{n_k}\| \le \|w_{n_k} - x_{n_k}\| + \|x_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{4.48}$$

$$\|u_{n_k} - x_{n_k}\| \le \|u_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{4.49}$$

$$\|v_{n_k} - x_{n_k}\| \le \|v_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{4.50}$$

From the Algorithm 3.2 and (4.35), observe that

$$\begin{aligned} \|x_{n_k+1} - y_{n_k}\| &= \|(1-\alpha_n - \beta_n)x_{n_k} + \beta_n y_{n_k} - y_{n_k}\| \\ &\le (1-\alpha_{n_k} - \beta_{n_k})\|x_{n_k} - y_{n_k}\| + \beta_{n_k}\|y_{n_k} - y_{n_k}\| + \alpha_{n_k}\|y_{n_k}\| \to 0 \text{ as } k \to \infty \end{aligned} \tag{4.51}$$

Using (4.51) and (4.35), it is easy to see that

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{4.52}$$

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to x^* such that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \lim_{j\to\infty}\langle p, p - x_{n_{k_j}}\rangle = \langle p, p - x^*\rangle. \tag{4.53}$$

Also, we obtain from (4.46), (4.41) and Lemma 4.2 that $x^* \in \Gamma$. Hence, since $p = P_\Omega 0$, we have obtain from (4.53) that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \langle p, p - x^*\rangle \le 0, \tag{4.54}$$

which implies that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k+1}\rangle \le 0, \tag{4.55}$$

Using using our assumption, (4.35) and (4.55), we have that $\limsup_{k\to\infty}\delta_{n_k} := 2\beta_n\|x_n - y_n\|\|x_{n+1} - p\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle \le 0$. Thus, the last part of Lemma 2.4 is achieved. Hence, we have that $\lim\limits_{n\to\infty}\|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p \in \Gamma$.

5. NUMERICAL EXAMPLES

In this section we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces.

Example 5.1. *Let $H_1 = H_2 = \ell_2$ be the linear space whose elements consists of all 2-summable sequence of scalars $(x_1, x_2, \ldots, x_j, \ldots)$, i.e.,*

$$\ell_2 = \left\{ \bar{x} = (x_1, x_2, \ldots, x_j, \ldots) \quad \textit{and} \quad \sum_{j=1}^{\infty}|x_j|^2 < \infty \right\}$$

with inner product $\langle\cdot,\cdot\rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle \bar{x}, \bar{y}\rangle = \sum_{j=1}^{\infty} x_j y_j$ and norm $\|x\|_2 := \left(\sum_{j=1}^{\infty}|x_j|^2\right)^{\frac{1}{2}}$, where $\bar{x} = \{x_j\} \in \ell_2$ and $\bar{y} = \{y_j\} \in \ell_2$. Let C be defined by $C = \{x \in \ell_2 : \langle a, x\rangle = b\}$ where $a = (3, 5, 3, 0, \ldots, 0, \ldots)$ and $b = 4$ and $Q := \{x \in \ell_2 : \langle c, x\rangle \ge d\}$ where $c = (3, 1, 0, 0, ;0, \ldots)$ and $d = 3$. Thus, we have

$$P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|_2^2}\right\} a + \bar{x},$$

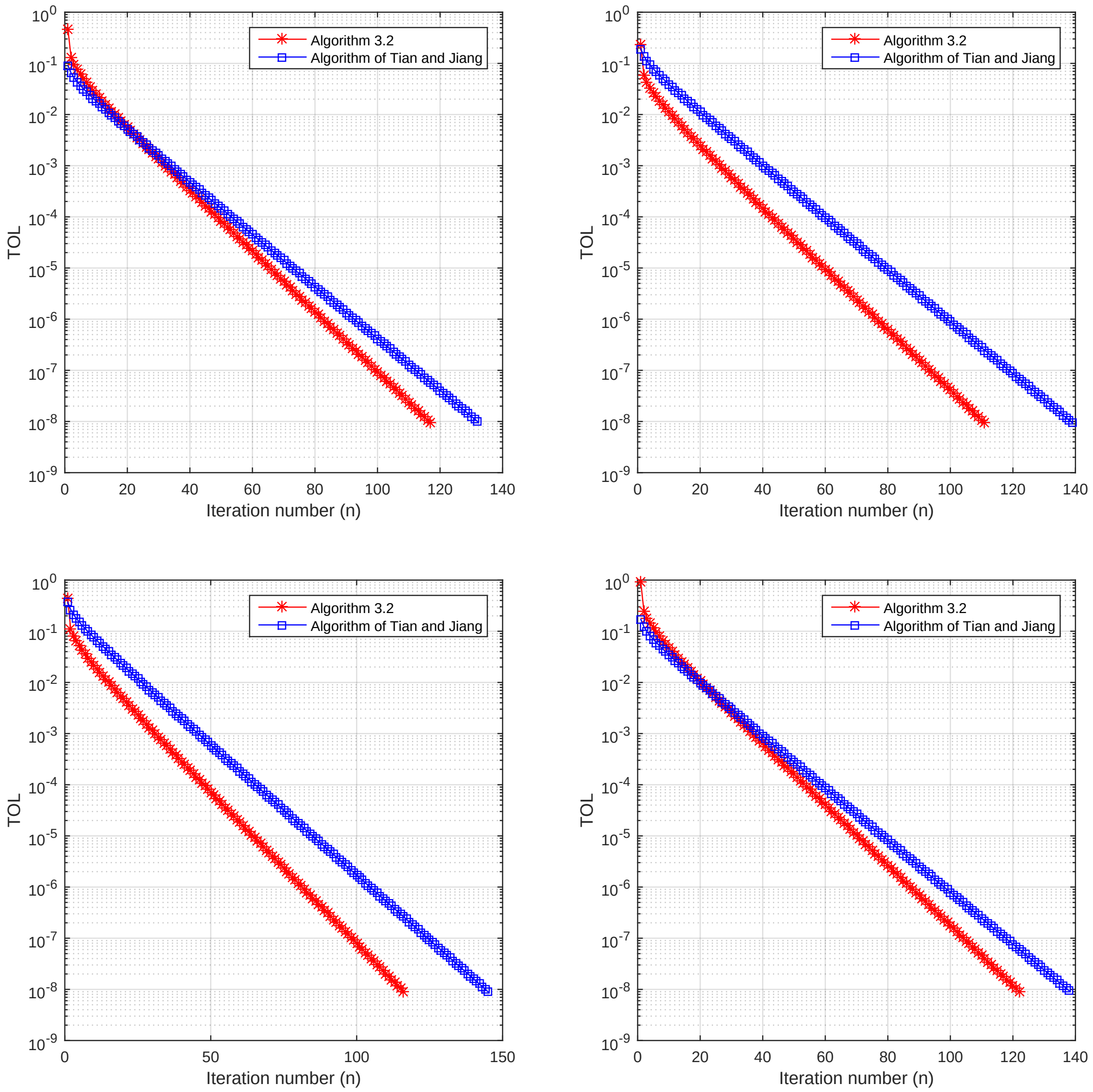

Figure 1: Example 5.1, Top Left: Case I; Top Right: Case II; Bottom Left: case III; Bottom Right: Case IV.

and

$$P_Q(\bar{x}) = \frac{d - \langle c, \bar{x}\rangle}{\|c\|_2^2} c + \bar{x}.$$

Let $T : \ell_2 \to \ell_2$ *be defined by* $T\bar{x} = 5\bar{x}$, *thus* T *is a bounded linear operator. Suppose* $A_1 : \ell_2 \to \ell_2$ *be defined by* $A_1\bar{x} = (3x_1, 3x_2, \dots, 3x_j, \dots)$ *and* $A_2 : \ell_2 \to \ell_2$ *be defined by* $A_2\bar{x} = \left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_j}{2}, \dots\right)$. *It is easy to see that* A_1 *and monotone and Lipschitz continuous and* A_2 *is inverse strongly monotone. We choose* $\gamma_n = 2, \lambda_1 = 1, \mu = 0.5, \theta_n = \overline{\theta}, \alpha_n = \frac{1}{5n+2}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}, \beta_n = \frac{1}{2} - \alpha_n$, *for all* $n \in \mathbb{N}$. *It is easy to verify that all hypothesis of Theorem 4.3 are satisfied. We implement our algorithm for different values of* x_0, x_1 *as follows.*

Case I: $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \dots), x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10} \dots);$
Case II: $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10} \dots); x_0 = (1, \frac{1}{2}, \frac{1}{3}, \dots),$
Case III: $x_1 = (1, \frac{1}{4}, \frac{1}{8}, \dots), x_0 = (2, 1, \frac{1}{8}, \dots);$
Case IV: $x_1 = x_0 = (2, 1, \frac{1}{8}, \dots); x_0 = (1, \frac{1}{4}, \frac{1}{9}, \dots).$

Example 5.2. *Let $H_1 = H_2 = L_2([0,1])$ be equipped with the inner product*

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \; \forall \, x, y \in L_2([0,1]) \text{ and } \|x\|^2 := \int_0^1 |x(t)|^2 dt \; \forall x, y, \in L_2([0,1]).$$

Let $A_1, A_2 : L_2([0,1]) \to L_2([0,1])$ be defined by

$$A_1 x(t) = \int_0^1 \left(x(t) - \left(\frac{2tse^{t+s}}{e\sqrt{e^2-1}} \right) \cos x(s) \right) ds + \frac{2te^t}{e\sqrt{e^2-1}}, \; x \in L_2([0,1]) \text{ and}$$

$$A_2 x(t) = \max\{0, \frac{x(t)}{2}\}, \; t \in [0,1].$$

It is easy to see that A_1 is Lipschitz continuous and monotone and A_2 is inverse strongly monotone on $L_2([0,1])$. Let $T : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Tx(s) = \int_0^1 K(s,t)x(t)dt \; \forall \, x \in L_2([0,1]),$$

where K is a continuous real-valued function defined on $[0,1] \times [0,1]$. Thus, T is a bounded linear operator with adjoint

$$T^* x(s) = \int_0^1 K(t,s)x(t)dt \; \forall \, x \in L_2([0,1]).$$

Let C be defined by $C = \{x \in L_2 : \langle a, x\rangle = b\}$ where $a \neq 0$ and $b = 2$ and $Q := \{x \in L_2 : \langle c, x\rangle \geq d\}$ where $c \neq 0$ and $d = 4$. Thus, we have

$$P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x},$$

and

$$P_Q(\bar{x}) = \frac{d - \langle c, \bar{x}\rangle}{\|c\|^2} c + \bar{x}.$$

We choose $\gamma_n = 2, \lambda_1 = 1, \mu = 0.5, \theta_n = \overline{\theta}, \alpha_n = \frac{1}{5n+2}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}, \beta_n = \frac{1}{2} - \alpha_n$, for all $n \in \mathbb{N}$. It is easy to verify that all hypothesis of Theorem 4.3 are satisfied. We implement our algorithm for different values of x_0, x_1 as follows.

Case I: $x_0(t) = 2t^2 + t + 2, \; x_1(t) = t;$

Case II: $x_0(t) = 2t^2 + e^{2t} + 1, \; x_1(t) = 3t^3 + 3;$

Case III: $x_0(t) = t + 2, \; x_1(t) = \cos(t);$

Case IV: $x_0(t) = \cos(t) + 2t^2 + 4, \; x_1(t) = 2t + 2 + e^t.$

6. CONCLUSION

A new inertial regularization method for solving the SVIP (1.2)-(1.3) is proposed, we establish strong converge to a minimum-norm solution of the problem in two real Hilbert spaces. The main advantage of this method is the combination of both the inertial extrapolation step and the regularization method, which has not been used to solve the SVIP (1.2)-(1.3). In addition, our method uses a simple self-adaptive stepsize that is generated at each iteration, which allows it to be easily implemented without the prior knowledge of the operator norm as well as the Lipschitz constant. Finally, we present some numerical experiments to establish the applicability and efficiency of our method. The results obtain in this paper is new in solving the SVIP (1.2)-(1.3).

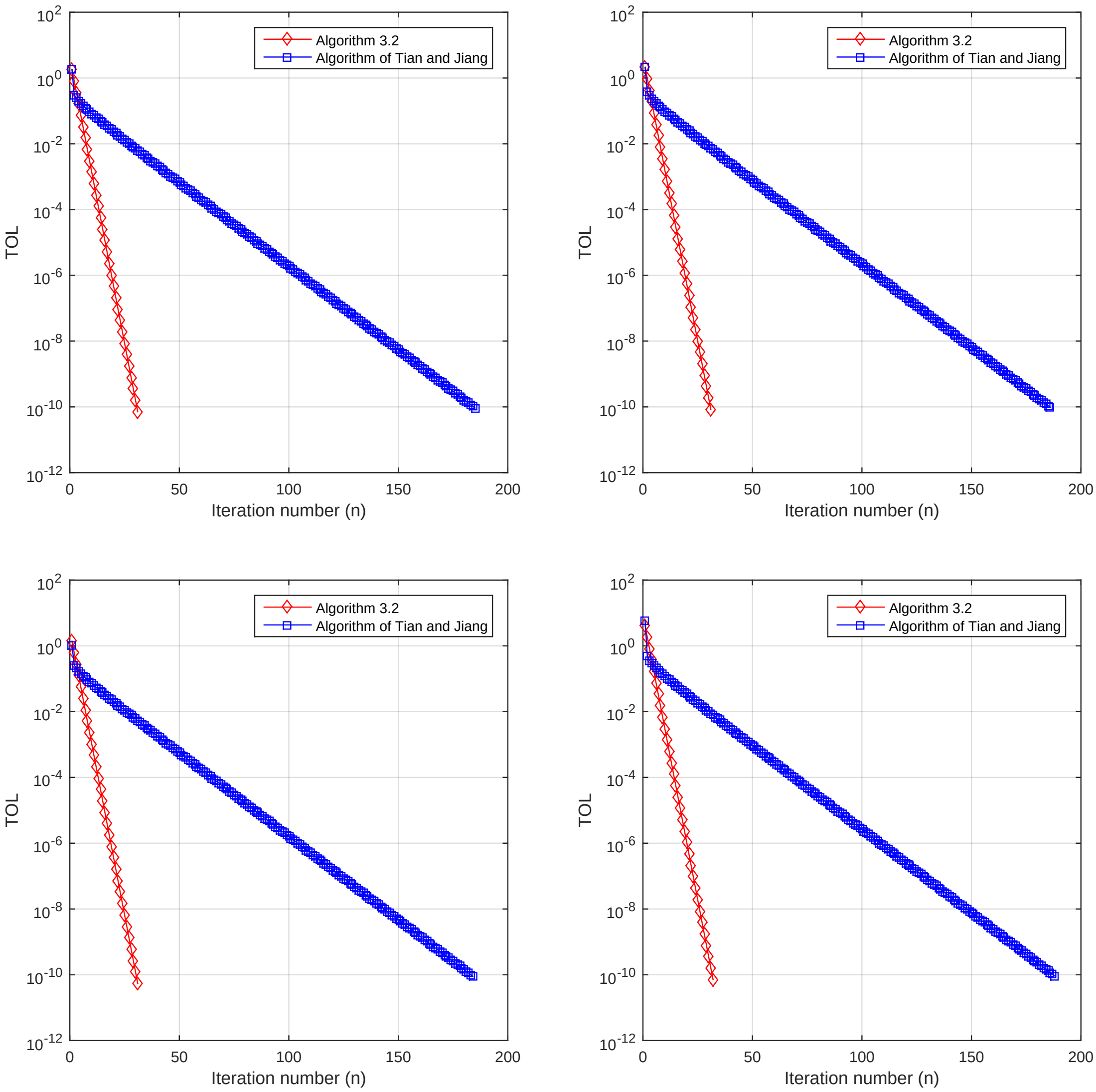

Figure 2: Example 5.1, Top Left: Case I; Top Right: Case II; Bottom Left: case III; Bottom Right: Case IV.

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