



A NEW RELAXED COMPLEX-VALUED b -METRIC TYPE AND FIXED POINT RESULTS

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ABSTRACT. In this paper, we study the existence and uniqueness of fixed point in complex valued b -metric spaces and introduce a new relaxed α, β Complex-valued b -metric type by relaxing the triangle inequality and determine whether the fixed point theorems are applicable in these spaces.

Key words and phrases: b -metric; Complex-valued b -metric space; Fixed point theorem.

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1. INTRODUCTION

The concept of a b -metric was initiated from the contributions of Bourbaki [2] and Bakhtin [1]. Czerwik [3] gave an axiom which was weaker than the triangular inequality and formally defined a b -metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [4] discussed some kind of relaxation in the triangular inequality and called this new distance measure a non-linear elastic pattern matching. In 2011, A. Azzam, B. Fisher and M. Khan introduced the notion of a complex valued metric space and called the complex-valued metric space as an extension of the classical metric space and proved some common fixed point theorems, [5]. In a similar way various authors have studied and proved the fixed point results for mappings satisfying different types of contractive conditions in the framework of complex-valued metric spaces,[6]. In 2013, Rao, et al. introduced the concept of a complex-valued b -metric space which is a generalization of the concept of a complex-valued metric space, [8] and subsequent to that A.A Mukheimer obtained common fixed point results, [7]. In this paper, we generalize the concept of a complex-valued b -metric and prove the common fixed results satisfying certain expressions in this new space.

2. PRELIMINARIES

Let \mathbb{C} be the set of complex numbers and if $z_1, z_2 \in \mathbb{C}$ then define a partial ordering \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \iff \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2)$$

Futhermore, if $z_3 \in \mathbb{C}$, we obtain that following:

- (i) If $0 \preceq z_1 \not\preceq z_2$ then $|z_1| < |z_2|$
- (ii) If $z_1 \preceq z_2$ and $z_2 \preceq z_3$ then $z_1 \preceq z_3$
- (iii) If $a, b \in \mathbb{R}$ and $a \leq b$ then $az \preceq bz$ for all $z \in \mathbb{C}$

Definition 2.1. Let X be a non-empty set. A function $d : X \times X \rightarrow \mathbb{C}$ is a complex-valued b -metric on X , [8], if there exists a real number $\alpha \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

- (i) $0 \preceq d(x, y)$ and $d(x, y) = 0 \iff x = y$
- (ii) $d(x, y) = d(y, x)$
- (iii) $d(x, y) \preceq \alpha [d(x, z) + d(z, y)]$

The pair (X, d) is a called a complex-valued b -metric space.

Definition 2.2. Let X be a non-empty set. A function $\rho : X \times X \rightarrow \mathbb{C}$ is a generalized α, β complex-valued b -metric on X if there exists real numbers $\alpha, \beta \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

- (i) $0 \preceq \rho(x, y)$ and $\rho(x, y) = 0 \iff x = y$
- (ii) $\rho(x, y) = \rho(y, x)$
- (iii) $\rho(x, y) \preceq \alpha \rho(x, z) + \beta \rho(z, y)$

The pair (X, ρ) is a called a α, β complex-valued b -metric space.

The following example justifies the generalization found in the definition.

Example 2.1. Let $X = (1, 3)$ and let $\rho : X \times X \rightarrow \mathbb{C}$ be a function defined by

$$\rho(x, y) = \begin{cases} e^{|x-y|} + ie^{|x-y|}, & \text{if } x \neq y \\ 0, & \text{iff } x = y. \end{cases}$$

To show that the example is a generalized α, β complex-valued b -metric, we only need to verify the α, β triangle inequality:

For $x \neq y, z \in X$ and $\theta \in (0, 1)$

$$\begin{aligned} \rho(x, y) &\preceq (1+i)e^{|x-z|+|z-y|} \\ &= (1+i)e^{\theta|x-z|+(1-\theta)|z-y|}e^{(1-\theta)|x-z|+\theta|z-y|} \\ &\preceq \sup_{x,y,z \in X} e^{\theta|x-z|+(1-\theta)|z-y|} ((1-\theta)(1+i)e^{|x-z|} + \theta(1+i)e^{|z-y|}) \\ &\preceq (1-\theta)e^2(1+i)e^{|x-z|} + \theta(1+i)e^2e^{|z-y|} \\ &= (1-\theta)e^2\rho(x, z) + \theta e^2\rho(z, y). \end{aligned}$$

For $\theta = \frac{1}{3}$ we have constants $\alpha = \frac{2}{3}e^2$ and $\beta = \frac{1}{3}e^2$.

One introduces a topology on a generalized α, β b -metric space (X, ρ) in the usual way. The open ball $B(x, \epsilon)$ with centre $x \in X$ and radius $0 \prec \epsilon \in \mathbb{C}$ is given by

$$B(x, \epsilon) = \{y \in X : \rho(x, y) \prec \epsilon\}$$

A subset A of X is open if for every $x \in A$ there is a number $0 \prec \epsilon \in \mathbb{C}$ such that $B(x, \epsilon) \subseteq A$.

Definition 2.3. Let (X, ρ) be a generalized α, β complex-valued b -metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then:

- (i) The sequence $\{x_n\}$ converges to $x \in X$, if for every $0 \prec \epsilon \in \mathbb{C}$ then there is $N \in \mathbb{N}$ such that $\rho(x_n, x) \prec \epsilon$. The sequence $\{x_n\}$ converges to $x \in X \iff |\rho(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$, [8].
- (ii) The sequence $\{x_n\}$ is a Cauchy in (X, ρ) if for every $\epsilon \in \mathbb{C}$ there is $N \in \mathbb{N}$ such that $\rho(x_n, x_{n+m}) \prec \epsilon$, where $m \in \mathbb{N}$. The sequence $\{x_n\}$ is a Cauchy in $(X, \rho) \iff |\rho(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$, where $m \in \mathbb{N}$, [8].
- (iii) The space (X, ρ) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

3. FIXED POINT THEOREM FOR GENERALIZED α, β COMPLEX-VALUED b -METRIC SPACES

As a consequence of (iii) of definition 2.2, the α, β triangle inequality, we get for $n, m \in \mathbb{N}$ with $m > n$

$$\begin{aligned} \rho(x_n, x_m) &\preceq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_m) \\ (3.1) \quad &\preceq \alpha\rho(x_n, x_{n+1}) + \beta[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_m)] \end{aligned}$$

Successively applying the α, β triangle inequality, we obtain

$$(3.2) \quad \rho(x_n, x_m) \preceq \alpha \sum_{i=0}^{m-n-2} \beta^i \rho(x_{n+i}, x_{n+i+1}) + \beta^{m-n-1} \rho(x_{m-1}, x_m).$$

This theorem is a generalization of the fixed point theorem studied by Mishra, et al., in [9].

Theorem 3.1. Let (X, ρ) be a complete generalized α, β complex-valued b -metric space and $T : X \rightarrow X$ a mapping such that

$$\rho(Tx, Ty) \preceq a\rho(x, Tx) + b\rho(y, Ty) + c\rho(x, y)$$

for all $x, y \in X$, where a, b , and c are non-negative real numbers satisfying $a + \beta(b + c) < 1$, then T has a unique fixed point.

Proof. We begin by proving that for $x_0 \in X$, the sequence $\{x_n\}$ generated by the recursive formula $x_n = Tx_{n-1} = T^n x_0$ is a Cauchy sequence in X . For $n \in \mathbb{N}$ we obtain

$$(3.3) \quad \begin{aligned} \rho(x_{n+2}, x_{n+1}) &= \rho(Tx_{n+1}, Tx_n) \\ &\leq a\rho(x_{n+1}, Tx_{n+1}) + b\rho(x_n, Tx_n) + c\rho(x_{n+1}, x_n) \\ &\leq a\rho(x_{n+2}, x_{n+1}) + b\rho(x_{n+1}, x_n) + c\rho(x_{n+1}, x_n) \end{aligned}$$

$$(3.4) \quad \leq a\rho(x_{n+2}, x_{n+1}) + (b + c)\rho(x_{n+1}, x_n)$$

$$(1 - a)\rho(x_{n+2}, x_{n+1}) \leq (b + c)\rho(x_{n+1}, x_n)$$

$$(3.5) \quad \rho(x_{n+2}, x_{n+1}) \leq \left(\frac{b + c}{1 - a} \right) \rho(x_{n+1}, x_n)$$

If we let $\gamma = \left(\frac{b+c}{1-a} \right)$ then repeated use of (3.5) we get

$$(3.6) \quad \rho(x_{n+2}, x_{n+1}) \leq \gamma^{n+1} \rho(x_1, x_0).$$

Using (3.2) and (3.6) for $k \in \mathbb{N}$, we get

$$(3.7) \quad \begin{aligned} \rho(x_n, x_{n+k+1}) &\leq \alpha \sum_{i=0}^{k-1} \beta^i \rho(x_{n+i}, x_{n+i+1}) + \beta^k \rho(x_{n+k}, x_{n+k+1}) \\ &\leq \alpha \sum_{i=0}^{k-1} \beta^i \gamma^{n+i} \rho(x_0, x_1) + \beta^k \gamma^{n+k} \rho(x_0, x_1) \\ &= \gamma^n \left[\alpha \sum_{i=0}^{k-1} \beta^i \gamma^i + \beta^k \gamma^k \right] \rho(x_0, x_1) \\ &= \gamma^n \left[\alpha \frac{1 - \beta^k \gamma^k}{1 - \beta \gamma} + \beta^k \gamma^k \right] \rho(x_0, x_1) \\ &= \frac{\gamma^n}{(1 - \beta \gamma)} \left[\alpha - \beta^k \gamma^k (\alpha + \beta \gamma - 1) \right] \rho(x_0, x_1) \\ &< \gamma^n \frac{\alpha}{(1 - \beta \gamma)} \rho(x_0, x_1). \end{aligned}$$

Now,

$$(3.8) \quad |\rho(x_n, x_{n+k+1})| \leq \gamma^n \frac{\alpha}{(1 - \beta \gamma)} \rho(x_0, x_1)$$

Since $a + \beta(b + c) < 1$ for $\beta \geq 1$ then $\beta \gamma < 1$ and $\gamma < 1$. Taking the limit $n \rightarrow \infty$ we get $\gamma^n \rightarrow 0$, which implies that $|\rho(x_n, x_{n+k+1})| \rightarrow 0$ as $n \rightarrow \infty$ thus the sequence $\{x_n\}$ is a Cauchy sequence. Since X is a complete α, β complex-valued b -metric space the sequence converges to $x^* \in X$. We show that x^* is a fixed point of T . Using the α, β triangle inequality, we have

$$(3.9) \quad \begin{aligned} \rho(x^*, Tx^*) &\leq \alpha\rho(x^*, x_{n+1}) + \beta\rho(x_{n+1}, Tx^*) \\ &\leq \alpha\rho(x^*, x_{n+1}) + \beta\rho(Tx_n, Tx^*) \\ &\leq \alpha\rho(x^*, x_{n+1}) + \beta[a\rho(x_n, Tx_n) + b\rho(x^*, Tx^*) + c\rho(x_n, x^*)] \end{aligned}$$

$$(3.10) \quad \begin{aligned} (1 - b\beta)\rho(x^*, Tx^*) &\preceq \alpha\rho(x^*, x_{n+1}) + a\beta\rho(x_n, Tx_n) + c\beta\rho(x_n, x^*) \\ \rho(x^*, Tx^*) &\preceq \frac{1}{(1 - b\beta)} [\alpha\rho(x^*, x_{n+1}) + a\beta\rho(x_n, x_{n+1}) + c\beta\rho(x_n, x^*)] \end{aligned}$$

Taking the absolute value of both sides, we get

$$(3.11) \quad \begin{aligned} |\rho(x^*, Tx^*)| &\leq \frac{1}{(1 - b\beta)} |[\alpha\rho(x^*, x_{n+1}) + a\beta\rho(x_n, x_{n+1}) + c\beta\rho(x_n, x^*)]| \\ &\leq \frac{1}{(1 - b\beta)} |\alpha\rho(x^*, x_{n+1})| + a\beta\gamma^n |\rho(x_0, x_1)| + c\beta |\rho(x_n, x^*)| \end{aligned}$$

Since x_n converges to x^* , taking limit $n \rightarrow \infty$ implies that $|\rho(x^*, Tx^*)| \rightarrow 0$ which yields $x^* = Tx^*$. To show uniqueness of the fixed point. Assume that there exists $x^{**} \in X$ such that $Tx^{**} = x^{**}$. Then

$$(3.12) \quad \rho(x^{**}, x^*) = \rho(Tx^{**}, Tx^*) \preceq a\rho(x^{**}, Tx^{**}) + b\rho(x^*, Tx^*) + c\rho(x^*, x^{**})$$

which implies that $\rho(x^{**}, x^*) \preceq c\rho(x^*, x^{**})$. Taking the absolute value of both sides, we get $|\rho(x^{**}, x^*)| \leq c|\rho(x^*, x^{**})|$. This implies that $\rho(x^*, x^{**}) = 0$. Thus $x^* = x^{**}$. ■

Theorem 3.2. Let (X, ρ) be a complete generalized complex-valued α, β b -metric space and let $T : X \rightarrow X$ be a mapping such that

$$(3.13) \quad \rho(Tx, Ty) \preceq a\rho(x, Ty) + b\rho(y, Tx)$$

for every $x, y \in X$, where a, b are non-negative constants with $\beta b < \frac{1}{1+\alpha}$. Then T has a fixed point in X and has a unique fixed point if $a + b < 1$.

Proof. Let $x_0 \in X$ be fixed then consider the sequence generated by the formula $x_n = Tx_{n-1} = T^n x_0$. Let $n \in \mathbb{N}$ then we get

$$(3.14) \quad \begin{aligned} \rho(x_{n+2}, x_{n+1}) &= \rho(Tx_{n+1}, Tx_n) \\ &\preceq a\rho(x_{n+1}, Tx_n) + b\rho(x_n, Tx_{n+1}) \\ &= a\rho(x_{n+1}, x_{n+1}) + b\rho(x_n, x_{n+2}) \\ &= b\rho(x_n, x_{n+2}) \\ &\preceq b\alpha\rho(x_n, x_{n+1}) + b\beta\rho(x_{n+1}, x_{n+2}) \\ (1 - b\beta)\rho(x_{n+2}, x_{n+1}) &\preceq b\alpha\rho(x_n, x_{n+1}) \\ \rho(x_{n+2}, x_{n+1}) &\preceq \frac{b\alpha}{1 - b\beta}\rho(x_n, x_{n+1}) \end{aligned}$$

Letting $\gamma = \frac{b\alpha}{1 - b\beta}$ and repeated use of (3.14) yields

$$(3.15) \quad \begin{aligned} \rho(x_{n+1}, x_{n+2}) &\preceq \gamma\rho(x_n, x_{n+1}) \\ &\preceq \gamma^2\rho(x_{n-1}, x_n) \\ &\vdots \\ &\preceq \gamma^{n+1}\rho(x_0, x_1) \end{aligned}$$

Let $m \in \mathbb{N}$ then

$$\begin{aligned}
\rho(x_n, x_{n+m}) &\preceq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m}) \\
&\preceq \alpha\rho(x_n, x_{n+1}) + \beta[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_{n+m})] \\
&\preceq \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1}) + \beta^{m-1}\rho(x_{n+m-1}, x_{n+m}) \\
&\preceq \alpha\gamma^n\rho(x_0, x_1) + \alpha\beta\gamma^{n+1}\rho(x_0, x_1) + \cdots + \alpha\gamma^{n+m-2}\beta^{m-2}\rho(x_0, x_1) + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1) \\
&\preceq \alpha\gamma^n\rho(x_0, x_1) [1 + \beta\gamma + \cdots + \gamma^{m-2}\beta^{m-2}] + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1) \\
&\preceq \alpha\gamma^n\rho(x_0, x_1) \left[\frac{1-(\beta\gamma)^{m-1}}{1-\beta\gamma} \right] + \beta^{m-1}\gamma^{n+m-1}\rho(x_0, x_1) \\
&\preceq \alpha \frac{\gamma^n}{1-\beta\gamma} \rho(x_0, x_1) [\alpha - (\beta\gamma)^{m-1}(\alpha - 1 + \beta\gamma)] \\
&\prec \alpha \frac{\gamma^n}{1-\beta\gamma} \rho(x_0, x_1)
\end{aligned}$$

Since $\beta b < \frac{1}{1+\alpha}$ then $0 < \gamma < \frac{1}{\beta}$ and $\beta\gamma < 1$. Taking the limit $n \rightarrow \infty$, we get $\gamma^n \rightarrow 0$. This implies that $|\rho(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$. The sequence $\{x_n\}$ is a Cauchy sequence in X . Since (X, ρ) is a complete complex-valued b -metric space then $\{x_n\}$ converges to $x^* \in X$.

We show that x^* is a fixed point of T .

$$\begin{aligned}
\rho(x^*, Tx^*) &\preceq \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*) \\
&= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*) \\
&\preceq \alpha\rho(x^*, x_n) + \beta[a\rho(x_{n-1}, Tx^*) + b\rho(Tx_{n-1}, Tx^*)] \\
&= \alpha\rho(x^*, x_n) + \beta a\rho(x_{n-1}, Tx^*) + \beta b\rho(x_n, x^*) \\
&\preceq (\alpha + \beta b)\rho(x_n, x^*) + \beta a\alpha\rho(x_{n-1}, x^*) + a\beta^2\rho(x^*, Tx^*) \\
(3.16) \quad [1 - a\beta^2]\rho(x^*, Tx^*) &\preceq (\alpha + \beta b)\rho(x_n, x^*) + a\beta\alpha\rho(x_{n-1}, x^*)
\end{aligned}$$

Since $\{x_n\}$ converges to x^* we get $|\rho(x^*, x_n)| \rightarrow 0$ as $n \rightarrow \infty$, and taking the absolute value of both sides of (3.16) we obtain $|\rho(x^*, Tx^*)| \leq 0$ thus $\rho(x^*, Tx^*) = 0$, which implies that $Tx^* = x^*$.

To prove the uniqueness of the fixed point we assume that there $x^*, x^{**} \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Now

$$\begin{aligned}
(3.17) \quad \rho(x^*, x^{**}) &= \rho(Tx^*, Tx^{**}) \\
(3.18) \quad &\preceq a\rho(x^*, Tx^{**}) + b\rho(x^{**}, Tx^*) \\
(3.19) \quad &= a\rho(x^*, x^{**}) + b\rho(x^{**}, x^*) \\
(3.20) \quad &= (a + b)\rho(x^*, x^{**})
\end{aligned}$$

Thus we get $|\rho(x^*, x^{**})| \leq |(a + b)| |\rho(x^*, x^{**})|$. Since $a + b < 1$ we get $|\rho(x^*, x^{**})| = 0$ thus $x^* = x^{**}$.

■

Author Kir et. al., studied the following fixed point theorem in b -metric spaces and we generalized the result into a α, β complex-valued b -metric spaces, [10].

Theorem 3.3. Let (X, ρ) be a α, β complex-valued b -metric space and let $T : X \rightarrow X$ be a mapping such that

$$\rho(Tx, Ty) \preceq \lambda[\rho(x, Tx) + \rho(y, Ty)],$$

where $\lambda \in [0, \frac{1}{2})$, for all $x, y \in X$. Then T has a unique fixed point.

Proof. We begin by showing that for $x_0 \in X$ fixed, the sequence $\{x_n\}$ where $x_n = Tx_{n-1} = T^n x_0$ is a Cauchy sequence in (X, ρ) .

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
 \rho(x_{n+2}, x_{n+1}) &= \rho(Tx_{n+1}, Tx_n) \\
 &\preceq \lambda[\rho(x_{n+1}, Tx_{n+1}) + \rho(x_n, Tx_n)] \\
 &= \lambda[\rho(x_{n+1}, x_{n+2}) + \rho(x_n, Tx_n)] \\
 (1 - \lambda)\rho(x_{n+2}, x_{n+1}) &\preceq \lambda\rho(x_{n+1}, x_n) \\
 (3.21) \quad \rho(x_{n+2}, x_{n+1}) &\preceq \frac{\lambda}{(1-\lambda)}\rho(x_{n+1}, x_n)
 \end{aligned}$$

Repeated use of (3.21), for $n \in \mathbb{N}$, we get

$$\rho(x_{n+2}, x_{n+1}) \preceq \left(\frac{\lambda}{1-\lambda}\right)^{n+1} \rho(x_1, x_0).$$

For $m, n \in X$

$$\begin{aligned}
 &\rho(x_n, x_{n+m}) \\
 &\preceq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m}) \\
 &\preceq \alpha\rho(x_n, x_{n+1}) + \beta\alpha\rho(x_{n+1}, x_{n+2}) + \beta^2\rho(x_{n+2}, x_{n+m}) \\
 &\preceq \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1}) \\
 &\quad + \beta^{m-1}\rho(x_{n+m-1}, x_{n+m}) \\
 &\preceq \alpha\left(\frac{\lambda}{1-\lambda}\right)^n \rho(x_0, x_1) + \alpha\beta\left(\frac{\lambda}{1-\lambda}\right)^{n+1} \rho(x_0, x_1) + \cdots + \alpha\beta^{m-2}\left(\frac{\lambda}{1-\lambda}\right)^{n+m-2} \rho(x_0, x_1) \\
 &\quad + \beta^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+m-1} \rho(x_0, x_1) \\
 &= \alpha\left(\frac{\lambda}{1-\lambda}\right)^n \rho(x_0, x_1) \left[1 + \beta\left(\frac{\lambda}{1-\lambda}\right) + \cdots + \beta^{m-2}\left(\frac{\lambda}{1-\lambda}\right)^{m-2}\right] + \beta^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+m-1} \rho(x_0, x_1)
 \end{aligned}$$

Since $\lambda \in [0, \frac{1}{2})$ implies that $0 \leq \frac{\lambda}{1-\lambda} < 1$. Taking the absolute value of both sides we get $|\rho(x_n, x_m)| \rightarrow 0$ as $n \rightarrow \infty$. It follows that the sequence $\{x_n\}$ is a Cauchy sequence in (X, ρ) . Since (X, ρ) is complete there exists a $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} \rho(x_n, x^*) = 0.$$

We now show that x^* is a fixed point of the mapping T .

$$\begin{aligned}
 \rho(x^*, Tx^*) &\preceq \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*) \\
 &= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*) \\
 &\preceq \alpha\rho(x^*, x_n) + \beta\lambda\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda\rho(x^*, Tx^*) \\
 (3.22) \quad [1 - \beta\lambda]\rho(x^*, Tx^*) &\preceq \alpha\rho(x^*, x_n) + \beta\lambda\rho(x_{n-1}, x_n)
 \end{aligned}$$

Taking the absolute value of both sides of (3.22) and taking $n \rightarrow \infty$, we obtain $\rho(x^*, Tx^*) = 0$. This implies that x^* is a fixed point of T .

To prove the uniqueness of the fixed point we assume that there $x^*, x^{**} \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Now

$$\begin{aligned}
 \rho(x^*, x^{**}) &= \rho(Tx^*, Tx^{**}) \\
 &\preceq \lambda[\rho(x^*, Tx^*) + \rho(x^{**}, Tx^{**})] \\
 &= 0
 \end{aligned}$$

Thus we get $|\rho(x^*, x^{**})| \leq 0$, which implies that $x^* = x^{**}$. Hence x^* is a unique fixed point.



4. CONCLUSION

In this paper we have presented a relaxed α, β complex-valued b -metric and proved some fixed point results for this new class of a generalized metric. The generalization may bring a wider applications of fixed point results. We have shown that for complex-valued b -metric spaces the mappings that have fixed points have fixed points in the generalized space.

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