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THE RAFU REMAINDER IN TAYLOR'S FORMULA

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ABSTRACT. This work is about the remainder in Taylor's formula. Specifically, the RAFU remainder is studied. Its mathematical expression is given. Some examples are shown. Different ways to obtain this remainder are developed.

Key words and phrases: RAFU functions; RAFU method; RAFU approximation; RAFU remainder.

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1. INTRODUCTION

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem.

Theorem 1. *f is a function k times differentiable in $[a,b]$, then for all $x \in [a,b]$ we have*

$$f(x) = T_{k-1}\left(f;a,x\right) + R_k\left(f;a,x\right)$$

where $T_{k-1}\left(f;a,x\right)$ is Taylor's polynomial of degree $k-1$ of f at $x=a$, i.e.,

$$T_{k-1}\left(f;a,x\right) = \sum_{i=0}^{k-1} \frac{f^{(i)}(a)\left(x-a\right)^i}{i!}$$

and $R_k\left(f;a,x\right)$ is the Taylor's remainder of order k, for some α_x between a and x, i.e.,

$$R_k\left(f;a,x\right) = \frac{f^{(k)}(\alpha_x)\left(x-a\right)^k}{k!}$$

There is a wide range of results on the topic related to the remainder in Taylor's formula from different perspectives. Some of them about estimates of the remainder as [1, 4, 5, 8], others about different forms of the remainder [6] and others study the asymptotic behavior of the remainder term of the formula [7].

In [2], the called RAFU remainder was defined as a sequence uniformly convergent to the Taylor's remainder in any closed interval. But no more was said about it. Moreover, the slow convergence speed of the sequence to the function $R_k\left(f;a,x\right)$ in the interval $[a,b]$ has been criticized for some authors.

In this work, we will devote Section 2 to improve the speed of convergence of the mentioned sequence to the function $R_k\left(f;a,x\right)$. The mathematical expressions of the RAFU remainder for the cases of $R_2\left(f;a,x\right)$ and $R_3\left(f;a,x\right)$ and some examples are shown in Section 3. Given that the sequence uniformly convergent to the function $R_k\left(f;a,x\right)$ given in [2] depends on some values of $f^{(k)}(x)$ at some points belonging to $[a,b]$, several remarks have been done related to this problem in practice. In Section 4 we construct the sequence uniformly convergent to $R_k\left(f;a,x\right)$ from sample means, local averages, linear combinations and approximate values of the data $f^{(k)}(x_p)$ which serve to define the RAFU remainder. In Section 5 we give the expression of the RAFU remainder of a function f from numerical approximations of its first and the second derivatives. The case of non-uniformly spaced data is studied in Section 6. Section 7 is for concluding remarks.

2. MAIN RESULT

Let f be an arbitrary function defined in $[a,b]$ and let $P=\{a=x_0,...,x_n=b\}$ be a partition of $[a,b]$ for each natural n. The RAFU method on approximation is an approximation procedure to the function f by a sequence $\left(C_n\right)_n$of radical continuous functions defined by the formula

$$C_n(x) = f(x_1) + \sum_{i=2}^{n}[f(x_i)-f(x_{i-1})]\cdot F_{n,p}\left(x_{i-1},x\right) \tag{2.1}$$

being

$$F_{n,p}\left(x_i,x\right) = \frac{\sqrt[2n^p+1]{x_i-a} + \sqrt[2n^p+1]{x-x_i}}{\sqrt[2n^p+1]{b-x_i} + \sqrt[2n^p+1]{x_i-a}}$$

$i=1$, ..., $n-1$ with $p \geq 1$ a natural number. For details about RAFU approximation, we refer the reader to [2, 3].

With this notation the following result can be established.

Theorem 2. *Let f be a function k times continuously differentiable in $[a,b]$, then there exists a sequence $(H_n)_n$ defined in $[a,b]$ such that for each $j = 0, ..., k$*

$$\left\|f^{(j)} - H_n^{(j)}\right\| \leq \left[\frac{2\,(M-m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)\right](b-a)^{k-j} \tag{2.2}$$

being $n \geq 2$, $\|.\|$ the uniform norm, M and m the maximum and the minimum of $f^{(k)}$ in $[a,b]$ respectively, $\omega\left(f^{(k)}, \frac{b-a}{n}\right)$ its modulus of continuity and

$$H_n(x) = \sum_{i=0}^{k-1} f^{(i)}(a)\frac{(x-a)^i}{i!} + G_n(x)$$

where $G_n(x) = \int_a^x G_n'(t)dt$, $G_n'(x) = \int_a^x G_n''(t)dt$,..., $G_n^{(k-1)}(x) = \int_a^x C_n(t)dt$ and

$$C_n(x) = f^{(k)}(x_1) + \sum_{i=2}^{n}[f^{(k)}(x_i) - f^{(k)}(x_{i-1})] \cdot F_{n,2}\left(x_{i-1}, x\right) \tag{2.3}$$

being $x_i = a + ih$, $i = 0$, ...,n and $h = \frac{b-a}{n}$.

Proof 1. *Theorem 1 ([2], p. 220), establishes that*

$$\left\|f^{(j)} - H_n^{(j)}\right\| \leq \left[\frac{M-m}{\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)\right](b-a)^{k-j}$$

for the functions $F_{n,1}\left(x_i, x\right)$, $i = 0$, ...,n and for all $j = 0, ..., k$ ([2], pp. $227-228$).

On the other hand, by Theorem 2.6 in [3] applied to the continuous function $f^{(k)}$ we can put that

$$\left\|f^{(k)} - C_n\right\| \leq \frac{2\,(M-m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)$$

Given that

$$\left\|f^{(k)} - H_n^{(k)}\right\| = \left\|f^{(k)} - G_n^{(k)}\right\| = \left\|f^{(k)} - C_n\right\|$$

we proceed like in proof of Theorem 1 in [2].

Although the main statement of Theorem 2 is the assertion of the existence of a sequence of functions H_n defined in $[a,b]$ such that $H_n^{(j)}$ converges uniformly to its respective $f^{(j)}$ in $[a,b]$ for all $j = 0$,..., k, another important consequence can be obtained. More precisely, the following Corollary holds.

Corollary 1. *With the hypothesis of Theorem 2, we have*

$$\left\|R_k\left(f; a, x\right) - G_n(x)\right\| \leq \left[\frac{2\,(M-m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)\right](b-a)^k$$

where $R_k\left(f; a, x\right)$ is Taylor's remainder of f of order k.

Proof 2. *Given that*

$$\left\|R_k\left(f; a, x\right) - G_n(x)\right\| = \left\|f(x) - H_n(x)\right\|$$

and Theorem 2 establishes that

$$\left\|f(x) - H_n(x)\right\| \leq \left[\frac{2\,(M-m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)\right](b-a)^k$$

the proof is complete.

Definition 2.1. Let f be a function k times continuously differentiable in $[a,b]$ and let $(G_n)_n$ be the sequence uniformly convergent to

$$R_k\left(f;a,x\right)=\frac{f^{(k)}(\alpha_x)\left(x-a\right)^k}{k!}$$

in $[a,b]$ defined in Theorem 2. We will say that the function G_n is the RAFU remainder of degree n of the function f.

Remarks.

(1) Note that when $f\in C^{\infty}[a,b]$, there exists one $R_k\left(f;a,x\right)$ for each k. In this case we will denote $(G_{n,k})_n$ to the corresponding sequence uniformly convergent to the function $R_k\left(f;a,x\right)$.

(2) In this paper we are concern about the improvement of the uniform rate of convergence of the sequence $(G_n)_n$ to the function $R_k\left(f;a,x\right)$ in $[a,b]$. In this sense, the values M, m or $\omega\left(f^{(k)},\frac{b-a}{n}\right)$ which appear in (2.2) are only useful to ensure the uniform speed of convergence. From Theorem 2 one deduces that G_n is completely defined from some values of $f^{(k)}$at some points of the interval $[a,b]$. In case we do not know the values of $f^{(k)}(x_p)$ that appear in the expression of G_n we will approach them in different ways as it will be shown below.

3. Two cases as examples

The functions G_n, $n\in\mathbb{N}$, of Theorem 2 are defined by the formulas $G_n(x)=\int_a^x G_n'(t)dt$, $G_n'(x)=\int_a^x G_n''(t)dt$,..., $G_n^{(k-1)}(x)=\int_a^x C_n(t)dt$.

Next, for cases $k=2$ and 3, we obtain the mathematical expression of $G_n(x)$

3.1. The case $R_2\left(f;a,x\right)$.

$$G_n''(x)=C_n(x)=f''(x_1)+\sum_{i=2}^{n}\left[f''(x_i)-f''(x_{i-1})\right]\cdot F_{n,2}\left(x_{i-1},x\right)$$

$$G_n'(x)=\int_a^x C_n(t)dt$$

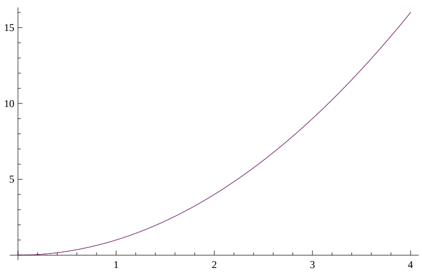
$$=\left[f''(x_1)+\sum_{i=2}^{n}\frac{\left[f''(x_i)-f''(x_{i-1})\right]\cdot\sqrt[2n^2+1]{x_{i-1}-a}}{\sqrt[2n^2+1]{b-x_{i-1}}+\sqrt[2n^2+1]{x_{i-1}-a}}\right](x-a)$$

$$-\left[\sum_{i=2}^{n}\frac{\left[f''(x_i)-f''(x_{i-1})\right]\cdot\sqrt[2n^2+1]{(a-x_{i-1})^{2n^2+2}}}{\sqrt[2n^2+1]{b-x_{i-1}}+\sqrt[2n^2+1]{x_{i-1}-a}}\right]\cdot\frac{2n^2+1}{2n^2+2}$$

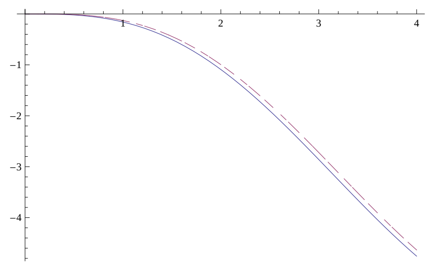
$$+\left[\sum_{i=2}^{n}\frac{\left[f''(x_i)-f''(x_{i-1})\right]\cdot\sqrt[2n^2+1]{(x-x_{i-1})^{2n^2+2}}}{\sqrt[2n^2+1]{b-x_{i-1}}+\sqrt[2n^2+1]{x_{i-1}-a}}\right]\cdot\frac{2n^2+1}{2n^2+2}$$

$$G_n(x)=\int_a^x G_n'(t)dt=\int_a^x\int_a^x C_n(t)dt$$

$$=\left[f''(x_1)+\sum_{i=2}^{n}\frac{\left[f''(x_i)-f''(x_{i-1})\right]\cdot\sqrt[2n^2+1]{x_{i-1}-a}}{\sqrt[2n^2+1]{b-x_{i-1}}+\sqrt[2n^2+1]{x_{i-1}-a}}\right]\frac{(x-a)^2}{2}$$



(A) Function f



(B) Function g

FIGURE 1. Remainder in Taylor's formula and RAFU remainder

$$- \left[\sum_{i=2}^{n} \frac{\left[f''(x_i) - f''(x_{i-1})\right] \cdot \sqrt[2n^2+1]{(a - x_{i-1})^{2n^2+2}}}{\sqrt[2n^2+1]{b - x_{i-1}} + \sqrt[2n^2+1]{x_{i-1} - a}} \right] \cdot \frac{2n^2+1}{2n^2+2} \cdot (x-a)$$

$$+ \left[\sum_{i=2}^{n} \frac{\left[f''(x_i) - f''(x_{i-1})\right] \cdot \left[\sqrt[2n^2+1]{(x - x_{i-1})^{4n^2+3}} - \sqrt[2n^2+1]{(a - x_{i-1})^{4n^2+3}} \right]}{\sqrt[2n^2+1]{b - x_{i-1}} + \sqrt[2n^2+1]{x_{i-1} - a}} \right]$$

$$\cdot \frac{(2n^2+1)^2}{(2n^2+2) \cdot (4n^2+3)}$$

Example 1. (1) *Let* $f(x) = 1 + x + x^2$ *be defined in* $[0,4]$*, we can put*

$$f(x) = T_1(f;0,x) + R_2(f;0,x) = 1 + x + \frac{f''(\alpha_x) \cdot x^2}{2!}$$

with $R_2(f;0,x) = \frac{f''(\alpha_x) \cdot x^2}{2!} = x^2$*. Suppose known the values* $f''(x_i)$ *with* $x_i = a + ih$*,* $i = 0, ..., n$ *and* $h = \frac{4}{n}$*. For* $n = 10$ *in Figure* $1a$) *we can check that* $R_2(f;0,x) = G_{10}(x)$ *in* $[0,4]$

(2) *Let* $g(x) = sin\ x$ *be defined in* $[0,4]$*, we can put*

$$g(x) = T_1(g;0,x) + R_2(g;0,x) = 0 + x + \frac{g''(\alpha_x) \cdot x^2}{2!}$$

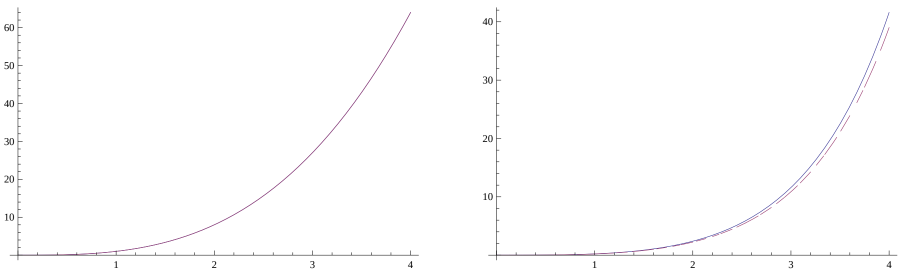
with $R_2(g;0,x) = \frac{g''(\alpha_x) \cdot x^2}{2!} = sin\ x - x$*. Suppose known the values* $g''(x_i)$ *with* $x_i = a + ih$*,* $i = 0, ..., n$ *and* $h = \frac{4}{n}$*. For* $n = 30$*, in Figure* $1b$) *we show* $R_2(g;0,x)$ *(solid line) and* $G_{30}(x)$ *(dashed line) in* $[0,4]$

3.2. **The case** $R_3(f;a,x)$ **.**

$$G_n^{(3)}(x) = C_n(x) = f^{(3)}(x_1) + \sum_{i=2}^{n} \left[f^{(3)}(x_i) - f^{(3)}(x_{i-1})\right] \cdot F_{n,2}(x_{i-1}, x)$$

$$G_n(x) = \int_a^x \int_a^x \int_a^x C_n(t)dt$$

$$= \left[f^{(3)}(x_1) + \sum_{i=2}^{n} \frac{\left[f^{(3)}(x_i) - f^{(3)}(x_{i-1})\right] \cdot \sqrt[2n^2+1]{x_{i-1} - a}}{\sqrt[2n^2+1]{b - x_{i-1}} + \sqrt[2n^2+1]{x_{i-1} - a}} \right] \frac{(x-a)^3}{3!}$$



(A) Function f (B) Function g

FIGURE 2. Remainder in Taylor's formula and RAFU remainder

$$-\left[\sum_{i=2}^{n}\frac{\left[f^{(3)}(x_i)-f^{(3)}(x_{i-1})\right]\cdot\sqrt[2n^2+1]{(a-x_{i-1})^{2n^2+2}}}{\sqrt[2n^2+1]{b-x_{i-1}}+\sqrt[2n^2+1]{x_{i-1}-a}}\right]\cdot\frac{2n^2+1}{2n^2+2}\cdot\frac{(x-a)^2}{2}$$

$$-\left[\sum_{i=2}^{n}\frac{\left[f^{(3)}(x_i)-f^{(3)}(x_{i-1})\right]\cdot\sqrt[2n^2+1]{(a-x_{i-1})^{4n^2+3}}}{\sqrt[2n^2+1]{b-x_{i-1}}+\sqrt[2n^2+1]{x_{i-1}-a}}\right]\cdot\frac{\left(2n^2+1\right)^2\cdot(x-a)}{(2n^2+2)\cdot(4n^2+3)}$$

$$+\left[\sum_{i=2}^{n}\frac{\left[f^{(3)}(x_i)-f^{(3)}(x_{i-1})\right]\cdot\left[\sqrt[2n^2+1]{(x-x_{i-1})^{6n^2+4}}-\sqrt[2n^2+1]{(a-x_{i-1})^{6n^2+4}}\right]}{\sqrt[2n^2+1]{b-x_{i-1}}+\sqrt[2n^2+1]{x_{i-1}-a}}\right]$$

$$\cdot\frac{\left(2n^2+1\right)^3}{(2n^2+2)\cdot(4n^2+3)\cdot(6n^2+4)}$$

Example 2. (1) *Let $f(x)=1+x+x^2+x^3$ be defined in $[0,4]$, we can put*

$$f(x)=T_2\left(f;0,x\right)+R_3\left(f;0,x\right)=1+x+\frac{x^2}{2}+\frac{f^{(3)}(\alpha_x)\cdot x^3}{3!}$$

with $R_3\left(f;0,x\right)=\frac{f^{(3)}(\alpha_x)\cdot x^3}{3!}=\frac{x^2}{2}+x^3$. Suppose known the values $f''(x_i)$ with $x_i=a+ih$, $i=0,...,n$ and $h=\frac{4}{n}$. For $n=10$ in Figure $2a)$ we can check that $R_3\left(f;0,x\right)=G_{10}(x)$ in $[0,4]$

(2) *Let $g(x)=e^x$ be defined in $[0,4]$, we can put*

$$g(x)=T_2\left(g;0,x\right)+R_3\left(g;0,x\right)=1+x+\frac{x^2}{2}+\frac{g^{(3)}(\alpha_x)\left(x-0\right)^3}{3!}$$

with $R_3\left(g;0,x\right)=\frac{g^{(3)}(\alpha_x)(x-0)^3}{3!}=e^x-\left(1+x+\frac{x^2}{2}\right)$. Suppose known the values $g''(x_i)$ with $x_i=a+ih$, $i=0,...,n$ and $h=\frac{4}{n}$. For $n=30$, in Figure $2b)$ we show $R_3\left(g;0,x\right)$ (solid line) and $G_{30}(x)$ (dashed line) in $[0,4]$

4. G_n USING APPROXIMATIONS OF $f^{(k)}(x_p)$

4.1. **Case of sample means .** The functions G_n can be defined from sample means of the data $f^{(k)}(x_p)$, $p=1,...,n$ used in Theorem 2. In fact, the following Proposition is satisfied,

Proposition 1. *If the data $f^{(k)}(x_p)$, $p=1,...,n$ in (2.3) are substituted by $k_p=\frac{f^{(k)}(x_{p1})n_1+...+f^{(k)}(x_{ps})n_s}{n_1+...+n_s}$, $x_{1q}\in[a,x_1]$ or $x_{pq}\in(x_{p-1},x_p]$, $p=2,...,n$, $q=1,...,s$, $n_1+...+n_s\neq 0$, then Corollary 1 holds.*

Proof 3. *The function $f^{(k)}$ is continuous in $[a,b]$. For the values of k_p, $p = 1$, ..., n we know (see* [3]*) that*

$$\left\|f^{(k)} - C_n\right\| \leq \frac{2\,(M-m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)$$

On the other hand, if we proceed like in proof of Theorem 1 *in* [2],

$$\|f - H_n\| \leq \left[\frac{M-m}{\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)\right](b-a)^k$$

Thus, we complete the proof because

$$\|f(x) - H_n(x)\| = \|R_k\,(f; a, x) - G_n(x)\|$$

4.2. **Case of local averages .** When the available data in practice are local averages near a certain x, the functions G_n can also be obtained. Here we consider the special case in which we know data as $\left(\chi_{[-h,h]} \star f^{(k)}\right)(x) = \int_{-\infty}^{+\infty} \chi_{[-h,h]}(y) f^{(k)}(x-y)dy = \int_{x-h}^{x+h} f^{(k)}(z)dz$ where $\star$ denotes the convolution of the functions $\chi_{[-h,h]}$ and $f^{(k)}$. From these data, an analogous assertion to Theorem 2 can be established.

Proposition 2. *If the data $f^{(k)}(x_p)$, $p = 1$, ..., n in (2.3) of Theorem 2 are defined by $k_p = \frac{\int_{\tilde{x}_p-h}^{\tilde{x}_p+h} f^{(k)}(z)dz}{2h}$, with $[\tilde{x}_1 - h, \tilde{x}_1 + h] \subseteq [a, x_1]$ or $[\tilde{x}_p - h, \tilde{x}_p + h] \subseteq (x_{p-1}, x_p]$, $p = 2$,..., n, then Corollary 1 holds.*

Proof 4. *The same proof as Proposition 1.*

4.3. **Case of linear combinations .** The functions G_n can also be obtained from linear combinations of the values $f^{(k)}(x_p)$, $p = 1$, ..., n given in Theorem 2. More exactly,

Proposition 3. *If the values $f^{(k)}(x_p)$, $p = 1$, ..., n in (2.3) are changed by $k_p = \frac{f^{(k)}(\tilde{x}_p) - f^{(k)}(\tilde{x}_{p-1})}{\tilde{x}_p - \tilde{x}_{p-1}} \cdot \left(x'_p - \tilde{x}_{p-1}\right) + f^{(k)}(\tilde{x}_{p-1})$ with $x'_1 \in [\tilde{x}_0, \tilde{x}_1] \subseteq [a, x_1]$ or $x'_p \in [\tilde{x}_{p-1}, \tilde{x}_p] \subseteq (x_{p-1}, x_p]$, $p = 2$,..., n, then Corollary 1 holds.*

Proof 5. *The same proof as Proposition 1.*

4.4. **Case of approximate values .** If what we know in practical applications are approximate values of $f^{(k)}(x_p)$, $p = 1$, ..., n, the function G_n can be found in accordance with to the following result.

Proposition 4. *With the hypothesis of Theorem 2, if the values $f^{(k)}(x_p)$, $p = 1$, ..., n in (2.3) are unknown but we know $f^{(k)}(x_p) + \eta_p$, with $\left|\eta_p\right| < \eta$, $p = 1$,...,n then*

$$\|R_k\,(f; a, x) - G_n(x)\| \leq \left[\frac{2\,(M-m+\eta)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right) + \eta\right](b-a)^k$$

Proof 6. *The function $f^{(k)}$ is continuous in $[a,b]$. For the values $f^{(k)}(x_p) + \eta_p$, with $\left|\eta_p\right| < \eta$, $p = 1$,...,n we know (see* [3]*) that*

$$\left\|f^{(k)} - C_n\right\| \leq \frac{2\,(M-m+\eta)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right) + \eta$$

On the other hand, if we proceed like in proof of Theorem 1 *in* [2],

$$\|f - H_n\| \leq \left[\frac{2\,(M-m+\eta)}{n\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right) + \eta\right](b-a)^k$$

Thus, we complete the proof because

$$\|f(x) - H_n(x)\| = \|R_k\,(f; a, x) - G_n(x)\|$$

5. CASE OF NUMERICAL APPROXIMATION OF THE FIRST AND THE SECOND DERIVATIVES

To define the functions H_n in Theorem 2, the values $f^{(j)}(a)$, $j = 0, 1, ..., k-1$ and $f^{(k)}(x_p)$, $p = 1, ..., n$ are used. Sometimes, this could be difficult or even impossible. For these cases we show two solutions using numerical approximations of the first and second derivatives. If M_1, m_1 and M_2, m_2 are the maximum and the minimum of $f^{(4)}$ and f''in $[a, b]$ respectively, for all $n \geq 2$, it verifies the following results.

Proposition 5. *If a function f has four continuous derivatives in $[a, b]$ and the $f^{(k)}(x_p)$, $p = 1$, ..., n in (2.3) are changed by $k_p = \frac{f(x_p)-2f\left(\frac{x_p+x_{p-1}}{2}\right)+f(x_{p-1})}{\left(\frac{h}{2}\right)^2}$, $p = 1, ..., n$ respectively, then*

$$\|R_k\,(f; a, x) - G_n(x)\| \leq V_n\,(b-a)^2 + \frac{h}{2} M_2\,(b-a)$$

where

$$V_n = \frac{2\,(M_2 - m_2) + \frac{(M_1-m_1)(b-a)^2}{24n^2}}{n\sqrt{n}} + \omega\left(f'', \frac{b-a}{2n}\right) + \frac{(b-a)^2\,M_1}{48n^2}$$

Proof 7. *Theorem 2 ([2], p. 222), established that*

$$\left\|f'' - H_n''\right\| \leq \frac{M_2 - m_2 + \frac{(M_1-m_1)(b-a)^2}{48n^2}}{\sqrt{n}} + \omega\left(f'', \frac{b-a}{2n}\right) + \frac{(b-a)^2\,M_1}{48n^2}$$

for the functions $F_{n,1}\,(x_i, x)$, $i = 0, ...,n$.

If we apply Theorem 2.6 in [3] to the continuous function f'' and we consider the values k_p of the hypothesis of this Proposition, we obtain

$$\left\|f'' - C_n\right\| \leq V_n$$

Given that

$$\left\|f'' - H_n''\right\| = \left\|f'' - G_n''\right\| = \left\|f'' - C_n\right\|$$

we proceed as in proof of Theorem 2 in [2] and we take into account that

$$\|f(x) - H_n(x)\| = \|R_k\,(f; a, x) - G_n(x)\|$$

to complete the proof.

Proposition 6. *If a function f has four continuous derivatives in $[a, b]$ and the $f^{(k)}(x_p)$, $p = 1$, ..., n in (2.3) are changed by $k_p = \frac{f(x_{p+1})-2f(x_p)+f(x_{p-1})}{h^2}$, $p = 1, ..., n-1$ and $k_n = k_{n-1}$, then*

$$\|R_k\,(f; a, x) - G_n(x)\| \leq W_n\,(b-a)^2 + \frac{h}{2} M_2\,(b-a)$$

where

$$W_n = \frac{2\,(M_2 - m_2) + \frac{(M_1-m_1)(b-a)^2}{6n^2}}{n\sqrt{n}} + \omega\left(f'', \frac{b-a}{n}\right) + \frac{(b-a)^2\,M_1}{12n^2}$$

Proof 8. *Corollary 6 ([2], p. 222), established that*

$$\left\|f'' - H_n''\right\| \leq \frac{M_2 - m_2 + \frac{(M_1-m_1)(b-a)^2}{12n^2}}{\sqrt{n}} + \omega\left(f'', \frac{b-a}{n}\right) + \frac{(b-a)^2\,M_1}{12n^2}$$

for the functions $F_{n,1}\,(x_i, x)$, $i = 0, ...,n$.

If we apply Theorem 2.6 *in* [3] *to the continuous function* f'' *and we consider the values* k_p *of the hypothesis of this Proposition, we obtain*

$$\left\| f'' - C_n \right\| \leq W_n$$

Given that

$$\left\| f'' - H_n'' \right\| = \left\| f'' - G_n'' \right\| = \left\| f'' - C_n \right\|$$

we proceed as in proof of Theorem 2 *in* [2] *and we take into account that*

$$\| f(x) - H_n(x) \| = \| R_k(f; a, x) - G_n(x) \|$$

to complete the proof.

Remark.

- In this work we are concerned about the improvement of the uniform convergence speed of the sequence $(G_n)_n$ to the function $R_k(f; a, x)$ in $[a, b]$. In this sense the values M_1, m_1 and M_2, m_2, $\omega\left(f'', \frac{b-a}{n}\right)$ and $\omega\left(f'', \frac{b-a}{2n}\right)$ which appear in these Propositions are only useful for us. For the definition of $G_n$we only need to know the values k_p.

6. CASE OF A NON-UNIFORMLY SPACED DATA

For the case of non-uniformly spaced data the following statement can be established.

Theorem 3. *Let* $P_n = \{a = x_0, x_1, ..., x_{s_n} = b\}$ *be a partition of* $[a, b]$ *with* $\delta(s_n) = \min_{1 \leq j \leq s_n} |x_j - x_{j-1}|$ *and* $\Delta(s_n) = \max_{1 \leq j \leq s_n} |x_j - x_{j-1}|$ *such that* $\frac{3(b-a)}{n^K} \leq \delta(s_n) \leq \Delta(s_n) \leq h$ *being* $h = \frac{b-a}{n}$ *and* $K \geq 2$ *a positive integer. Let* f *be a function* k *times continuously differentiable in* $[a, b]$*, then there exists a sequence* $(H_n)_n$ *defined in* $[a, b]$ *such that for each* $j = 0, ..., k$

$$\| f^{(j)} - H_n^{(j)} \| \leq \left[\frac{2^K (M - m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \Delta(s_n)\right) \right] (b - a)^{k-j}$$

being $n \geq 2$, M *and* m *the maximum and the minimum of* $f^{(k)}$ *in* $[a, b]$ *respectively and* $\omega\left(f^{(k)}, \Delta(s_n)\right)$ *its modulus of continuity,* H_n *as usual and* C_n *as in (2.3).*

Proof 9. *Theorem* 3 *(* [2]*, pp.* $222 - 223$*), established that*

$$\| f^{(j)} - H_n^{(j)} \| \leq \left[\frac{6K}{5} \frac{M - m}{\sqrt{n}} + \omega\left(f^{(k)}, \Delta(s_n)\right) \right] (b - a)^{k-j}$$

for the functions $F_{n,1}(x_i, x)$, $i = 0, ..., n$ *and for all* $j = 0, ..., k$.

On the other hand, Theorem 5.5 *in* [3] *applied to the continuous function* $f^{(k)}$ *established that*

$$\| f^{(k)} - C_n \| \leq \frac{2^K (M - m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \Delta(s_n)\right)$$

Given that

$$\| f^{(k)} - H_n^{(k)} \| = \| f^{(k)} - G_n^{(k)} \| = \| f^{(k)} - C_n \|$$

we proceed as in proof of Theorem 3 *in* [2].

Corollary 2. *With the hypothesis of Theorem 3, we have*

$$\| R_k(f; a, x) - G_n(x) \| \leq \left[\frac{2^K (M - m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \Delta(s_n)\right) \right] (b - a)^k$$

where $R_k(f; a, x)$ *is Taylor's remainder of* f *of order* k.

Proof 10. *Given that*

$$\|R_k(f;a,x) - G_n(x)\| = \|f(x) - H_n(x)\|$$

and Theorem 3 establishes that

$$\|f(x) - H_n(x)\| \leq \left[\frac{2^K(M-m)}{n\sqrt{n}} + \omega\left(f^{(k)}, \Delta(s_n)\right)\right](b-a)^k$$

the proof is complete.

Remark.

- As we have said before, we are concerned about the improvement of the uniform convergence speed of the sequence $(G_n)_n$ to the function $R_k(f;a,x)$ in $[a,b]$. Only in this sense the values M, m, $\omega\left(f^{(k)}, \Delta(s_n)\right)$ and K which appear in this Proposition are useful for us. For the definition of $G_n$we only need to know the values $f^{(k)}(x_p)$.

7. CONCLUDING REMARKS

It is well known that the approximation of the Taylor' polynomial to a function f is a local approximation. We also know that the remainder in Taylor's formula depends on an unknown parameter. In this sense, there is a wide range of results on this topic from different perspectives. As we said at the beginning, some of them about estimates of the remainder [1, 4, 5, 8], others about different forms of the remainder [6] or about the study of the asymptotic behavior of the remainder term of the formula [7].

The RAFU remainder in Taylor's formula published in [2] as a sequence uniformly convergent to the Taylor's remainder in any closed interval was only defined. In this work the speed of convergence has been improved and the mathematical expression of this sequence has been shown. The sequence can be defined from some values of $f^{(k)}(x)$, sample means, local averages, linear combinations or approximate values of them. But also, it can be obtained from numerical approximations of its first and the second derivatives of f and even in case of non-uniformly spaced data.

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