Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 2, Art. 5, 19 pp.
AJMAA

LIE GROUP THEORETIC APPROACH OF ONE-DIMENSIONAL BLACK-SCHOLES EQUATION

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Received 4 March, 2020; accepted 24 March, 2021; published 16 July, 2021.

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ABSTRACT. This study discusses the Lie Symmetry Analysis of Black-Scholes equation via a modified local one-parameter transformations. It can be argued that the transformation of the Black-Scholes equation is firstly obtained by means of riskless rate. Thereafter, the corresponding determining equations to the reduced equation are found. Furthermore, new symmetries of the Black-Scholes equation are constructed and lead to invariant solutions.

Key words and phrases: Group theoretic approach; Lie symmetry; Invariant solution; Black-Scholes equation.

2010 *Mathematics Subject Classification.* Primary 34C14, 34M55.

ISSN (electronic): 1449-5910

MB Matadi thanks the research office of the University of Zululand for its continued support.

1. INTRODUCTION

Over the past 40 years, several techniques have been employed in studying and solving the Black-Scholes (BS) model. These include among others, numerical, stability, analytical and approximation techniques, and many more. Near the end of the 20th century, an important technique of *Lie Theory of Symmetry groups* was used to study the BS model. The Lie's theory of symmetry groups is named after Marius Sophus Lie, the great Norwegian mathematician who lived near the end of the 19th century. Sophus Lie applied the theory of symmetry groups to differential equations. He unified all the methods of differential equations and deduced that they can all be characterized by his theory of *Lie groups* [15]. *Lie groups* are mathematical objects that depicts properties of groups as they are known in group theory. The idea behind *Lie group* theory is to apply suitable transformations of independent and dependent variables to obtain a *lie symmetries*. This process results into differential equations with reduced orders compared to the original equation [16, 1]. Transformations of this kind are known as *infinitesimal transformations*. The important feature of *Lie groups* is the concept of *infinitesimal generators*. Infinitesimal generators are obtained by solving a *symmetry condition* of the symmetry group [5, 14]. Ultimately, working from the infinitesimal generators symmetries of differential equations can be generated. Lie provided a very comprehensive classification of differential equations. According to this classification, all parabolic equations admitting the symmetry group of highest order reduce to the heat conduction equation [15, 13]. This is where the theory of *Lie group* analysis connects with the BS model, as it is also transformable into the heat equation. Lie's theory have thus proved useful in facilitating analysis for option pricing using BS model.

In this paper, we demonstrate how the theory of Lie symmetry analysis can be used to find invariant solutions of the Black-Scholes option pricing model, under constant volatility. The transformation of the BC equation is obtained from the use of a new independent variable. However, we employ the technique of Masebe and Manale [7] to find the corresponding determining equation, and to construct new symmetries of the reduced equation that lead to invariant solution.

This paper is organised as follows. In Section 2, the standard BS model is formulated and it assumes a complete system of markets, which is to say that every good in a market is tradeable. A heuristic background of the concepts underlying the Manale's formula and Lie symmetry analysis are introduced in Section 3. In Section 5, the modified local one-parameter transformations is presented and used to calculate invariant solutions of equation BC equation. The numerical solutions is performed and presented graphically in Section 6.

2. THE BLACK-SCHOLES MODEL DERIVATION

The process of pricing options can either be discrete or continuous. An example of a discrete-time pricing model is the Binomial model, which is a model that assumes two tradeable assets, in which one is risky (e.g stock) and the other one is risk-less (e.g bond). The binomial model is strongly linked to probability theory. The prominent result in the binomial model is what is called *The risk-neutral valuation*. This result states that the discounted price of the stock is a martingale, i.e. it satisfies the martingale property in that the expected value of the future stock price equals the price of the stock today under some probability measure called *Risk-neutral probability*. This study is confined only to continuous model, and looks at the particular model for pricing option by means of the Black-Scholes model.

The derivation of the BS equation is quite traditional and basic in finance, in the derivation, some details are skipped and only important features are considered. The price of the stock is traditionally modelled by the following stochastic differential equation.

$$dS = rSdt + \sigma SdW \tag{2.1}$$

where

- r is riskless rate
- S is the price of an underlying asset (the stock)
- W is the Brownian motion term, and
- σ is the volatility of the stock

In the classical case, the volatility σ is considered to be constant, but in real world situation, it is stochastic in nature and needs to be modelled using stochastic differential equations. If V denotes the value of an European option, then we can invoke the Ito's lemma, which states that V evolves according to the equation:

$$dV = \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\sigma S \frac{\partial V}{\partial S}\right) dW \tag{2.2}$$

At this stage, the *self-financing* portfolio denoted by Π is constructed. This portfolio consists of one option and a certain amount of stock, which is denoted by Δ. Furthermore, the portfolio is usually assumed to be riskless, which is to say it is insensitive to changes in the price of the stock. So this portfolio looks like the following:

$$\Pi(t) = V + \Delta S. \tag{2.3}$$

Taking the total derivative with respect to time we obtain

$$d\Pi = dV + \Delta dS. \tag{2.4}$$

Combining equation (2.2) and (2.4) we obtain

$$d\Pi = \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Delta rS\right) dt + \left(\sigma S \frac{\partial V}{\partial S} + \Delta \sigma S\right) dW. \tag{2.5}$$

Since the portfolio is riskless the second term of (2.5) must be zero because it contains random term dW, that is.

$$\left(\sigma S \frac{\partial V}{\partial S} + \Delta \sigma S\right) dW = 0. \tag{2.6}$$

Equation (2.6) gives:

$$\Delta = -\frac{\partial V}{\partial S}. \tag{2.7}$$

This means that

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{2.8}$$

Since Π also depends on the risk-free rate r it follows that Π must earn interest at the rate r, which means $d\Pi = r\Pi dt$ [2, 3, 13]. Therefore, putting this result in equation (2.7) we obtain:

$$
\begin{aligned}
d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \\
&= r\left(V - \frac{\partial V}{\partial S} S\right) dt \qquad (2.9)
\end{aligned}
$$

Dropping the dt terms we get the Black-Scholes differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \qquad (2.10)$$

3. BACKGROUND

3.1. **The Rational behind Manale's Formula.** Manale's formula (see [6]) results from solving the second-order ordinary differential equation of the form

$$a_0\ddot{y} + b_0 + c_0\dot{y} = 0 \qquad (3.1)$$

for y where x is the independent variable and a_0, b_0, and c_0 are parameters. Normally this equation is solved by using Euler's formulas which are

$$
y = \begin{cases}
e^{-}\dfrac{b_0}{2a_0}x\left(Ae^{-\hat{\omega}x} + Be^{\hat{\omega}x}\right), & b_0^2 > 4a_0c_0 \\
A + Bx & b_0^2 = 4a_0c_0 \\
e^{-\frac{b_0}{2a_0}x}\left[A\cos(\hat{\omega}x)\right] & \\
+Be^{-}\dfrac{b_0}{2a_0}x\left[\sin(\hat{\omega}x)\right], & b_0^2 < 4a_0c_0
\end{cases}
$$

where $\hat{\omega} = \sqrt{\frac{b_0^2 - 4a_0c_0}{2a_0}}$. But for the case when $b_0 = c_0$ equation (3.1) will not reduce to $y = A + Bx$. To obtain the exact solution corresponding to this case let

$$y = \beta(x)z(x), \qquad (3.2)$$

which means

$$
\begin{aligned}
\dot{y} &= \dot{\beta}z + \beta\dot{z} \\
\ddot{y} &= \ddot{\beta}z + 2\dot{\beta}\dot{z} + \beta\ddot{z}.
\end{aligned}
$$

Plugging back to (3.1) gives

$$a_0(\ddot{y} = \ddot{\beta}z + 2\dot{\beta}\dot{z} + \beta\ddot{z}) + b_0(\dot{\beta}z + \beta\dot{z}) + c_0\beta z.$$

Rearranging gives

$$a_0\beta\ddot{z} + (2a_0\dot{\beta} + b_0\ddot{z}) + (a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta)z = 0. \qquad (3.3)$$

For $b_0 = c_0 = 0$ the function $\beta(x)$ must satisfy $2a_0\dot{\beta} + b_0\beta$. That is

$$\beta = C_{00}e^{-\frac{b_0}{2a_0}x},$$

where C_{00} is a constant, and

$$
\begin{aligned}
\ddot{z} &= -\frac{a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta}{a_0\beta} \\
&= \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right). \qquad (3.4)
\end{aligned}
$$

Since $\ddot{z} = \dot{z}dz/dx$ we then have

$$\frac{\dot{z}^2}{2} = \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right)\frac{z^2}{2} + C_{01} \tag{3.5}$$

where C_{01} is a constant. This means

$$\dot{z} = \frac{dz}{\sqrt{\left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)\frac{z^2}{2} + 2C_{01}}} = dx. \tag{3.6}$$

Dividing both sides by $\frac{1}{\sqrt{\frac{-b_0^2-4a_0c_0}{4_0^2}}}$ gives

$$\frac{dz}{\sqrt{\frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}} - z^2}} = \sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}dx. \tag{3.7}$$

Which gives

$$z = \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}}\sin\left(\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}x + C_{02}\right), \tag{3.8}$$

where C_{02} is a constant. We then get the expression for y, which is

$$y = C_{00}e^{-\frac{b_0}{2a_0}}\frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}}\sin\left(\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}x + C_{02}\right). \tag{3.9}$$

If we set

$$\overline{\omega} = \sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}},$$

we get

$$y = C_00e^{-\frac{b_0}{2a_0}x}\frac{2C_{01}}{\overline{\omega}}\sin(\overline{\omega}x + C_{02}). \tag{3.10}$$

Applying the compound formula for Sine gives

$$y = C_{00}e^{-\frac{b_0}{2a_0}x}2C_{01}\left[\frac{\sin(C_{02})}{\overline{\omega}} + \cos(C_{02})\frac{\sin(\overline{\omega}x)}{\overline{\omega}}\right]. \tag{3.11}$$

For a trivial case $\ddot{y} = 0$ we must have

$$\sin(C_{02}) = C_{03}\sin(\overline{\omega})$$
$$\cos(C_{02}) = C_{04}\cos(\overline{\omega})$$

with

$$C_{03}^2 + C_{04}^2 = 1 \tag{3.12}$$

and finally we obtain

$$y = C_{00}e^{-\frac{b_0}{2a_0}x}2C_{01}\frac{C_{03}\sin(\overline{\omega})\cos(\overline{\omega}x)}{\overline{\omega}} + C_{00}e^{-\frac{b_0}{2a_0}x}2C_{01}\frac{C_{04}\sin(\overline{\omega}x)}{\overline{\omega}}. \tag{3.13}$$

This formula is known as the Manale's formula. It was obtained by Jacob Manale (see [6]) of the University of South Africa in 2014, who applied this formula to uncover more symmetries of the heat equation. This formula was applied by Masebe and Manale (see [7]) into the Black-Scholes equation and it led to new symmetries.

3.2. **The Rational behind Lie Point Symmetries of Partial Differential Equations.** The BS equation (2.10) is a partial differential equation that consists of one dependent variable V and two independent variables t and S. This section shows how to deal with PDEs of this kind. The result obtained here will be used in subsequent sections to find Lie symmetries for equation (2.10). In the case of two dependent variables, the point transformation are of the form

$$\Gamma : (x, t, u) \mapsto (\hat{x}(x, t, u)), \hat{t}(x, t, u), \hat{u}(x, t, u). \tag{3.14}$$

Therefore, the goal is to seek point symmetries of the form [11, 10]

$$\begin{aligned} \hat{x} &= x + \varepsilon\xi(x, t, u) + O(\varepsilon^2) \\ \hat{t} &= t + \varepsilon\tau(x, t, u) + O(\varepsilon^2) \\ \hat{u} &= u + \varepsilon\eta(x, t, u) + O(\varepsilon^2). \end{aligned} \tag{3.15}$$

The corresponding infinitesimal generator is given by [12, 9]

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u. \tag{3.16}$$

The first and the second prolongations of (3.16) are

$$X^{[1]} = \xi\partial_x + \tau\partial_t + \eta\partial_u + \eta_x^{(1)}\partial_{u_x} + \eta_t^{(1)}\partial_{u_t} \tag{3.17}$$

$$X^{[2]} = X^{[1]} + \eta_{xx}^2\partial_{xx} + \eta_{xt}^{(2)}\partial_{u_{xt}} + \eta_{tt}^{(2)}\partial_{u_{tt}}. \tag{3.18}$$

Define the following PDE as

$$\triangle(x, t, u, u_x, u_t, u_{xx}, u_{tt},) = 0. \tag{3.19}$$

The symmetry condition is given by

$$\triangle[\hat{x}, \hat{t}, \hat{u}, \hat{u_x}, \hat{u}_t, \hat{u_{xx}}, \hat{u_{tt}}] = 0 \tag{3.20}$$

when (3.19) holds. This symmetry condition can be differentiated with respect to the parameter ϵ at $\epsilon = 0$ to obtain the linearised symmetry condition [5, 11]

$$X^{[2]} \triangle [\hat{x}, \hat{t}, \hat{u}, \hat{u_x}, \hat{u}_t, \hat{u_{xx}}, \hat{u_{tt}}] = 0 \tag{3.21}$$

when

$$\triangle[x, t, u, u_x, u_t, u_{xx}, u_{tt}] = 0. \tag{3.22}$$

4. LIE SYMMETRY ANALYSIS OF THE ONE-DIMENSIONAL BLACK-SCHOLES EQUATION

In this section we discuss the Symmetry Analysis approach of the Black-Scholes (BC) equation via a modified local one-parameter transformations. The reduced form of BC equation is obtained from the use of new independent variable to find the corresponding determining equations. In consequence of this, the construction of new symmetries is obtained and lead to invariant solutions.

4.1. **New Symmetries of the Black-Scholes Equation Using Manale's Formula.** The classical Black-Scholes equation is given by:

$$V_t + \frac{1}{2}A^2S^2V_{SS} + BSV_r - CV = 0 \tag{4.1}$$

with constant coefficients A, B and C, where A is not allowed to be zero. It is traditional to define $D = B - \frac{A^2}{2}$ [4]. The symmetry analysis of (4.1) as outlined in [7] follows below. Considering the following change of variables.

$$V_S = \frac{\partial V}{\partial S} \tag{4.2}$$

and

$$SV_S = V_{\ln S} = \frac{\partial V}{\partial \ln S}. \tag{4.3}$$

By letting

$$r = \ln S \tag{4.4}$$

$$r_S = \frac{1}{S} \tag{4.5}$$

$$S_r = S. \tag{4.6}$$

Now, substituting equation (4.4) into equation (4.3) we obtain

$$SV_S = V_r. \tag{4.7}$$

Recall that

$$V_{SS} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right). \tag{4.8}$$

Multiplying both sides of equation (4.8) by S^2 gives

$$\begin{aligned} S^2V_{SS} &= S^2\frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) \\ &= S\left\{\frac{\partial}{\frac{\partial S}{S}}\left(\frac{\partial V}{\partial S}\right)\right\} \\ &= S\left\{\frac{\partial}{\partial \ln S}\left(\frac{\partial V}{\partial S}\right)\right\}. \end{aligned} \tag{4.9}$$

Then from equation (4.4), we have

$$S^2V_{SS} = S\left\{\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial S}\right)\right\}. \tag{4.10}$$

Now, from equation (4.3) we have

$$\begin{aligned}
S\left\{\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial S}\right)\right\} &= S\left\{\frac{\partial}{\partial r}\left(\frac{1}{S}\frac{\partial V}{\partial \ln S}\right)\right\} \\
&= S\left\{\frac{\partial}{\partial r}\left(\frac{1}{S}\frac{\partial V}{\partial r}\right)\right\} \\
&= S\left\{-\frac{1}{S^2}\frac{\partial r}{\partial S}\frac{\partial V}{\partial r} + \frac{1}{S}\frac{\partial^2 V}{\partial r^2}\right\} \\
&= S\left\{-\frac{1}{S}\frac{\partial r}{\partial S}\frac{\partial V}{\partial r} + \frac{1}{S}\frac{\partial^2 V}{\partial r^2}\right\} \\
&= -\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}. \qquad (4.11)
\end{aligned}$$

Therefore,

$$S^2 V_{SS} = V_{rr} - V_r. \qquad (4.12)$$

By substituting equation (4.7) and (4.12) into (4.1), the BS equation is transformed into

$$V_t + \frac{1}{2}A^2 V_{rr} + DV_r - CV = 0. \qquad (4.13)$$

4.2. **New Symmetries of Black-Scholes Equation** (4.13)**.** The first and second prolongations of equation (4.13) are:

$$X^{[1]} = \xi^1(t,r)\partial_t + \xi^2(t,r)\partial_r + \eta(t,r)\partial_V \qquad (4.14)$$

$$X^{[2]} = X^[1] + \eta_t^{(1)}\partial_t + \eta_r^{(1)}\partial_{V_r} + \eta_{rr}^{(2)}\partial_{V_{rr}}. \qquad (4.15)$$

This leads to the symmetry condition:

$$X^{[2]}[V_t + \frac{1}{2}A^2 V_{rr} + DV_r - CV]|_{V_{rr}=-\frac{2}{A^2}[V_t+DV_r-CV]} = 0. \qquad (4.16)$$

In terms of the point transformations up to first and second derivatives, the symmetry condition becomes

$$\eta_t^{(1)} + \frac{1}{2}A^2\eta_{rr}^{(2)} + D\eta_r^{(1)} - C\eta = 0, \qquad (4.17)$$

when

$$V_{rr} = \left(-\frac{2}{\sigma^2}\right)[V_t + DV_r - CV]. \qquad (4.18)$$

We therefore obtained the following functions

$$\eta = fV + g \qquad (4.19)$$

$$\eta_r^{(1)} = g_r + f_r V + [f - \xi_r^2]V_r - \xi_r^1 V_t \qquad (4.20)$$

$$\eta_t^{(1)} = g_t + f_t V + [f - \xi_t^1]V_t - \xi_t^2 V_r \qquad (4.21)$$

$$\begin{aligned}\eta_{rr}^{(2)} &= g_{rr} + f_{rr}V + [2f_r - \xi_{rr}^2]V_r - \xi_{rr}^1 V_t \\ &+ [f - 2\xi_r^2]V_{rr}] - 2\xi_r^1 V_{tr}\end{aligned} \qquad (4.22)$$

Substituting the functions $\eta_r^{(1)}$, $\eta_t^{(1)}$, and $\eta_{rr}^{(2)}$ into the determining equation 4.17 results in:

$$\begin{aligned} & g_t + f_t V + [f - \xi_t^1]V_t - \xi_t^2 V_r + \frac{1}{2}A^2 \left\{ g_{rr} + f_{rr}V + [2f_r - \xi_{rr}^2]V_r \right\} \\ & + \frac{1}{2}A^2 \left\{ -\xi_{rr}^1 V_t + [f - 2\xi_r^2](-\frac{2}{A^2}[V_t + DV_r - CV]) - 2\xi_r^1 V_{tr} \right\} \\ & + D[g_r + f_r V + [f - \xi_r^2]V_r - \xi_r^1 V_t] - CfV - Cg = 0 \end{aligned} \tag{4.23}$$

Afterwards, this single equation result in a system of equations (overdetermining equations) below, which is obtained by setting the coefficients of the derivative of V to zero.

$$V_{tr} : \xi_r^2 = 0 \tag{4.24}$$

$$V_t : -\xi_t^1 + 2\xi_r^2 = 0 \tag{4.25}$$

$$V_r : -\xi_t^2 + A^2 g_{rr} + D\xi_r^2 - \frac{1}{2}A^2\xi_{rr}^2 = 0 \tag{4.26}$$

$$V^0 : g_t + \frac{1}{2}A^2 g_{rr} + Dg_r - Cg = 0 \tag{4.27}$$

$$V : f_t + \frac{1}{2}A^2 f_{rr} + Df_r - 2C\xi_r^2 = 0. \tag{4.28}$$

Differentiating equation (4.24) with respect to r yields

$$\xi_{rr}^2 = 0. \tag{4.29}$$

Integrating equation (4.29) twice with respect to r yields

$$\xi^2 = ar + b. \tag{4.30}$$

Applying Manale's formula we obtain

$$\xi^2 = \frac{a \sin\left(\frac{\omega r}{i}\right) + b\phi \cos\left(\frac{\omega r}{i}\right)}{-i\omega}$$

where $\phi = \sin\left(\frac{\omega}{i}\right)$ and a and b are arbitrary functions of t. Therefore,

$$\begin{aligned} \xi_r^2 &= \frac{1}{-i\omega}\left[\frac{\omega a}{i}\cos\left(\frac{\omega r}{i}\right) - b\phi\frac{\omega}{i}\sin\left(\frac{\omega r}{i}\right)\right] \\ &= a\cos\left(\frac{\omega r}{i}\right) - b\phi\sin\left(\frac{\omega r}{i}\right), \end{aligned} \tag{4.31}$$

$$\xi_{rr}^2 = -\frac{\omega a}{i}\sin\left(\frac{\omega r}{i}\right) - b\phi\frac{\omega}{i}\cos\left(\frac{\omega r}{i}\right), \tag{4.32}$$

and

$$\xi_t^2 = \frac{\dot{a}\sin\left(\frac{\omega r}{i}\right) + \dot{b}\phi\cos\left(\frac{\omega r}{i}\right)}{-i\omega}. \tag{4.33}$$

Now, substituting equation (4.32) into equation (4.25) gives

$$\begin{aligned} \xi_t^1 &= 2\xi_r^2 \\ &= 2a\cos\left(\frac{\omega r}{i}\right) - b\phi\sin\left(\frac{\omega r}{i}\right), \end{aligned} \tag{4.34}$$

which means that

$$\xi^1 = 2at\cos\left(\frac{\omega r}{i}\right) - 2bt\phi\sin\left(\frac{\omega r}{i}\right) + C. \tag{4.35}$$

The expression for f_r is found by substituting in equations (4.31), (4.32), and (4.33) into equation (4.26) to get

$$f_r = \cos\left(\frac{\omega r}{i}\right)\left\{-\frac{\omega b\phi}{2i} - \frac{\dot{b}\phi}{A^2 i\omega} - \frac{Da}{A^2}\right\} + \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{\omega a}{2i} - \frac{\dot{a}}{A^2 i\omega} + \frac{Db\phi}{A^2}\right\}. \tag{4.36}$$

Equation (4.36) can easily be integrated to obtain

$$f = \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{b\phi}{2} - \frac{\dot{b}\phi}{A^2\omega^2} - \frac{Dai}{A^2\omega}\right\} + \cos\left(\frac{\omega r}{i}\right)\left\{\frac{a}{2} + \frac{\dot{a}}{A^2\omega^2} - \frac{Dib\phi}{A^2\omega}\right\} + k(t). \tag{4.37}$$

The expressions for f_{rr} and f_t are given by

$$f_{rr} = \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{\omega^2 b\phi}{2} - \frac{\dot{b}\phi}{A^2\omega^2} + \frac{Da\omega}{A^2 i}\right\} + \cos\left(\frac{\omega r}{i}\right)\left\{\frac{\omega^2 a}{2} + \frac{\dot{a}}{A^2} + \frac{D\omega b\phi}{A^2 i}\right\} \tag{4.38}$$

and

$$f_t = \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{\dot{b}\phi}{2} - \frac{\ddot{b}\phi}{A^2\omega^2} - \frac{D\dot{a}i}{A^2\omega}\right\} + \cos\left(\frac{\omega r}{i}\right)\left\{\frac{\dot{a}}{2} + \frac{\ddot{a}}{A^2\omega^2} - \frac{Di\dot{b}\phi}{A^2\omega}\right\} + k'(t). \tag{4.39}$$

Now, being equipped with expressions for ξ_r^2, f_r, f_{rr}, and f_t, these can be substituted into the last determining equation (4.28) to obtain

$$\begin{aligned}
&\sin\left(\frac{\omega r}{i}\right)\left\{-\frac{\dot{b}\phi}{2} - \frac{\ddot{b}\phi}{A^2\omega^2} - \frac{Di\dot{a}}{A^2\omega} + 2Cb\phi\right\} + \cos\left(\frac{\omega r}{i}\right)\left\{\frac{\dot{a}}{2} + \frac{\ddot{a}}{A^2\omega^2} - \frac{Di\dot{b}\phi}{A^2\omega} - 2Ca\right\}\\
&+k'(t) + \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{bA^2\omega^2\phi}{4} - \frac{\dot{b}\phi}{2} + \frac{Da\omega}{2i}\right\} + \cos\left(\frac{\omega r}{i}\right)\left\{\frac{aA^2\omega^2}{4} + \frac{\dot{a}}{2} + \frac{Db\omega\phi}{2i}\right\}\\
&+\cos\left(\frac{\omega r}{i}\right)\left\{-\frac{Db\omega\phi}{2i} - \frac{D\dot{b}\phi}{A^2 i\omega} - \frac{D^2 a}{A^2}\right\} + \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{Da\omega}{2i} - \frac{D\dot{a}}{2i} - \frac{D\dot{a}}{A^2 i\omega} + \frac{D^2 b\phi}{A^2}\right\} = 0.
\end{aligned} \tag{4.40}$$

Grouping together all the coefficients of Sine alone and those of Cosine alone and equating them to zero gives

$$\begin{aligned}
\sin\left(\frac{\omega r}{i}\right):\quad & -\frac{\dot{b}\phi}{2} - \frac{\ddot{b}\phi}{A^2\omega^2} - \frac{Di\dot{a}}{A^2\omega} - \frac{bA^2\omega^2\phi}{4} - \frac{\dot{b}\phi}{2} - \frac{Da\omega}{2i}\\
& + \frac{Da\omega}{2i} - \frac{D\dot{a}}{A^2\omega i} - \frac{D^2 b\phi}{A^2} + 2Cb\phi = 0
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
\cos\left(\frac{\omega r}{i}\right):\quad & \frac{\dot{a}}{2} + \frac{\ddot{a}\phi}{A^2\omega^2} - \frac{Di\dot{b}\phi}{A^2\omega} + \frac{aA^2\omega^2\phi}{4} + \frac{\dot{a}}{2} + \frac{Db\omega\phi}{2i} - \frac{Db\omega\phi}{2i}\\
& - \frac{D\dot{b}\phi}{A^2\omega i} - \frac{D^2 a}{A^2} + 2aC = 0.
\end{aligned} \tag{4.42}$$

The equation for Sine simplifies to

$$\ddot{b} + \dot{b}A^2\omega^2 + \frac{bA^4\omega^4}{4} - \frac{D^2 b}{A^2} + 2CA^2\omega^2 b = 0. \tag{4.43}$$

To obtain b, the following change of variables is considered

$$\beta = \frac{A^2\omega^2}{2} \tag{4.44}$$

$$k = -\frac{D^2}{A^2} - 2C \tag{4.45}$$

$$\alpha_1 = b\beta^2 - k. \tag{4.46}$$

Equation (4.46) implies that

$$\dot{\alpha_1} = \dot{b}\beta^2 \tag{4.47}$$

$$\ddot{\alpha_1} = \ddot{b}\beta^2. \tag{4.48}$$

Therefore, equation (4.43) becomes

$$\ddot{\alpha_1} + 2\dot{\alpha_1} + \alpha_1\beta^2 = 0. \tag{4.49}$$

Now, setting

$$\alpha_1 = c(t)z(t), \tag{4.50}$$

so that

$$\alpha_1' = c'z + cz', \tag{4.51}$$

$$\alpha_1'' = c''z + 2c'z' + cz''. \tag{4.52}$$

Hence,

$$cz'' + (2c + 2c')z + (c'' + 2c' + \beta^2 c)z = 0. \tag{4.53}$$

The function c is chosen so that

$$2c + 2c' = 0 \tag{4.54}$$

making

$$c = e^{-t} \tag{4.55}$$

which simplifies (4.53) to

$$z'' + (\beta^2 - 1)z = 0. \tag{4.56}$$

Defining

$$\hat{\omega} = \sqrt{\beta^2 - 1}. \tag{4.57}$$

The solution of equation (4.49) becomes

$$\alpha_1 = e^{-t}\left(C_1\frac{\sin\hat{\omega}\cos\hat{\omega}t}{\hat{\omega}}\right) + C_2 e^{-t}\frac{\sin\hat{\omega}t}{\hat{\omega}}. \tag{4.58}$$

If $\hat{\omega} \to 0$ then $\beta = \pm 1$. Solving for b in equation (4.46) and invoking equation (4.58) gives

$$b = \frac{e^{-t}}{\beta^2}\left\{\left(C_1\frac{\sin\hat{\omega}\cos\hat{\omega}t}{\hat{\omega}} + C_2 e^{-t}\frac{\sin\hat{\omega}t}{\hat{\omega}}\right)\right\} + \frac{D^2}{\beta^2 A^2} + \frac{4C}{\beta^2}. \tag{4.59}$$

In a similar way, one can solve for a from the Cosine equation (4.42), which simplifies to a second-order linear differential equation

$$\ddot{a} + \dot{a}A^2\omega^2 + \frac{aA^4\omega^4}{4} - \frac{aD^2}{A^2} - 2aA^2\omega^2 C = 0 \tag{4.60}$$

and gives

$$a = \frac{e^{-t}}{\beta^2}\left\{\left(C_3\frac{\sin\hat{\omega}\cos\hat{\omega}t}{\hat{\omega}}\right) + C_4\frac{\sin\hat{\omega}t}{\omega}\right\} + \frac{D^2}{\beta^2 A^2} - \frac{4C}{\beta^2}. \tag{4.61}$$

Hence, the expressions for $\dot{a}$ and $\dot{b}$ are:

$$\dot{a} = -\frac{e^{-t}}{\beta^2}\left(C_3\frac{\sin\hat{\omega}\cos\hat{\omega}t}{\hat{\omega}} + C_4\frac{\sin\hat{\omega t}}{\hat{\omega}}\right) + \frac{e^{-t}}{\beta^2}\left(-C_3\sin\hat{\omega}\sin\hat{\omega}t + C_4\cos\hat{\omega}t\right) \tag{4.62}$$

and

$$\dot{b} = -\frac{e^{-t}}{\beta^2}\left(C_1\frac{\sin\hat{\omega}\cos\hat{\omega}t}{\hat{\omega}} + C_2\frac{\sin\hat{\omega}t}{\hat{\omega}}\right) + \frac{e^{-t}}{\beta^2}\left(-C_1\sin\hat{\omega}\sin\hat{\omega}t + C_2\cos\hat{\omega}t\right). \tag{4.63}$$

Now, from equation (4.45) the following is obtained

$$k'(t) = 0 \tag{4.64}$$

$$k(t) = C_5 \tag{4.65}$$

Therefore ξ^1, ξ^2, and f become

$$\begin{aligned} \xi^1 &= \cos\left(\frac{\omega r}{i}\right)\left\{\frac{2te^{-t}}{\beta^2\hat{\omega}}\left\{(C_3\sin\hat{\omega}\cos\hat{\omega}t) + C_4\sin\hat{\omega}\right\} + \frac{2tD^2}{\beta^2 A^2}\right\} \\ &- \sin\left(\frac{\omega r}{i}\right)\left\{\frac{2t\phi e^{-t}}{\beta^2\hat{\omega}}\left\{(C_1\sin\hat{\omega}\cos\hat{\omega}t) + C_2\sin\hat{\omega}t\right\} + \frac{2t\phi D^2}{\beta^2 A^2}\right\} \\ &- \frac{8Ct}{\beta}\cos\left(\frac{\omega r}{i}\right) + \frac{8Ct\phi}{\beta}\cos\left(\frac{\omega r}{i}\right) + C_5, \end{aligned} \tag{4.66}$$

$$\begin{aligned} \xi^2 &= \sin\left(\frac{\omega r}{i}\right)\left\{\frac{e^{-t}}{-\beta^2\omega\hat{\omega}i}\left(C_3\sin\hat{\omega}\sin\hat{\omega}t + C_4\sin\hat{\omega}\right) - \frac{D^2}{i\omega\beta^2 A^2}\right\} \\ &+ \cos\left(\frac{\omega r}{i}\right)\left\{\frac{e^{-t}}{-\beta^2\omega\hat{\omega}i}\left(C_1\sin\hat{\omega}\cos\hat{\omega}t - C_2\sin\hat{\omega}\right) - \frac{D^2\phi}{i\omega\beta^2 A^2}\right\} \\ &+ \frac{4Ci}{\omega\beta}\sin\left(\frac{\omega r}{i}\right) + \frac{4Ci\phi}{\omega\beta}\cos\left(\frac{\omega r}{i}\right), \end{aligned} \tag{4.67}$$

and

(4.68)

$$f = \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{C_1\phi e^{-t}\sin\hat{\omega}\cos\hat{\omega t}}{2\beta^2\hat{\omega}} - \frac{C_2\phi e^{-t}\sin\hat{\omega}t}{2\beta^2\hat{\omega}} - \frac{D^2\phi}{2\beta^3A^2} - \frac{4C\phi}{\beta}\right\}$$
$$+ \sin\left(\frac{\omega r}{i}\right)\left\{\frac{C_1\phi e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3\hat{\omega}} + \frac{C_2\phi e^{-t}\sin\hat{\omega}t}{2\beta^3\hat{\omega}} + \frac{C_1\phi e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3}\right\}$$
$$+ \sin\left(\frac{\omega r}{i}\right)\left\{-\frac{C_2\phi e^{-t}\cos\hat{\omega}t}{2\beta^3} - \frac{C_3Di\omega e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3} - \frac{C_4Di\omega e^{-t}\sin\hat{\omega}t}{2\beta^3} - \frac{D^3i\omega}{2\beta^3A^2}\right\}$$
$$+ \cos\left(\frac{\omega r}{i}\right)\left\{\frac{C_3e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^2\hat{\omega}} + \frac{C_4e^{-t}\sin\hat{\omega}t}{2\beta^3\hat{\omega}} + \frac{D^2}{2\beta^3A^2} - \frac{4C\phi}{\beta}\right\}$$
$$+ \cos\left(\frac{\omega r}{i}\right)\left\{-\frac{C_3e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3\hat{\omega}} - \frac{C_4e^{-t}\sin\hat{\omega}t}{2\beta^3\hat{\omega}} - \frac{C_3e^{-t}\sin\hat{\omega}\sin\hat{\omega}t}{2\beta^3}\right\}$$
$$+ \cos\left(\frac{\omega r}{i}\right)\left\{-\frac{C_4e^{-t}\cos\hat{\omega}t}{2\beta^3} - \frac{C_1\phi Di\omega e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3}\right\}$$
$$+ \cos\left(\frac{\omega r}{i}\right)\left\{-\frac{C_2\phi Di\omega e^{-t}\sin\hat{\omega}t}{2\beta^3} - \frac{D^3\phi i\omega}{2\beta^3A^2}\right\} + C_5.$$

Hence, the equation (4.1) admits the following eight symmetries [7]

$$\begin{aligned} X_1 = &\left(-\frac{2te^{-t}\phi}{\beta^2\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right)\right)\partial_t + \left(\frac{e^{-t}i\phi}{\beta^2\hat{\omega}\omega}\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\partial_r \\ &+ \{\frac{-e^{-t}\phi}{2\beta^2\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) + \frac{e^{-t}\phi}{2\beta^3\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) \\ &+ \frac{e^{-t}\phi}{2\beta^3}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) - \frac{Di\phi e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3}\cos\left(\frac{\omega r}{i}\right)\}V\partial_V \end{aligned} \tag{4.69}$$

$$\begin{aligned} X_2 = &\left(-\frac{2te^{-t}\phi}{\beta^2\hat{\omega}}\sin\hat{\omega}t\sin\left(\frac{\omega r}{i}\right)\right)\partial_t - \left(\frac{e^{-t}i\phi}{\beta^2\hat{\omega}\omega}\sin\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\partial_r \\ &+ \{-\frac{e^{-t}\phi}{2\beta^2\hat{\omega}}\sin\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) + \frac{e^{-t}\phi}{2\beta^3\hat{\omega}}\sin\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) \\ &- \frac{e^{-t}\phi}{2\beta^3}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) - \frac{Di\phi\omega e^{-t}\sin\hat{\omega}t}{2\beta^3}\cos\left(\frac{\omega r}{i}\right)\}V\partial_V \end{aligned} \tag{4.70}$$

$$\begin{aligned} X_3 = &\left(-\frac{-2te^{-t}\phi}{\beta^2\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\partial_t + \left(\frac{e^{-t}i\phi}{\beta^2\hat{\omega}\omega}\sin\omega\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\partial_r \\ &+ \{\frac{e^{-t}\phi}{2\beta^3}\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right) + \frac{e^{-t}\phi}{2\beta^3\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) \\ &- \frac{e^{-t}\phi}{2\beta^3}\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right) - \frac{Di\omega e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3}\sin\left(\frac{\omega r}{i}\right)\}V\partial_V \end{aligned} \tag{4.71}$$

$$X_4 = \left(\frac{2te^{-t}\phi}{\beta^2\hat{\omega}}\sin\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\partial_t + \left(\frac{e^{-t}i\phi}{\beta^2\hat{\omega}\omega}\sin\hat{\omega}t\sin\left(\frac{\omega r}{i}\right)\right)\partial_r$$
$$+\{\frac{e^{-t}\phi}{2\beta^2\hat{\omega}}\sin\hat{\omega}t\cos\left(\frac{\omega r}{i}\right) - \frac{e^{-t}\phi}{2\beta^3\hat{\omega}}\sin\hat{\omega}t\cos\left(\frac{\omega r}{i}\right) \tag{4.72}$$
$$+\frac{e^{-t}\phi}{2\beta^3}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right) - \frac{Di\omega e^{-t}\sin\hat{\omega}t}{2\beta^3}\sin\left(\frac{\omega r}{i}\right)\}V\partial_V$$

$$X_5 = \frac{2tD^2}{\beta^2\sigma^2}\left(\cos\left(\frac{\omega r}{i}\right) - \phi\sin\left(\frac{\omega r}{i}\right)\right)\partial_t$$
$$+\frac{D^2}{\omega\beta^2\sigma^2}\left(\phi\cos\left(\frac{\omega r}{i}\right) + \sin\left(\frac{\omega r}{i}\right)\right)\partial_r - \frac{D^2 i\omega}{2\beta^3\sigma^2}\left(\sin\left(\frac{\omega r}{i}\right) + \phi\cos\{\frac{\omega r}{i}\right) \tag{4.73}$$
$$+D\sin\left(\frac{\omega r}{i}\right) + D\phi\cos\left(\frac{\omega r}{i}\right)\}V\partial_V$$

$$X_6 = V\partial_V \tag{4.74}$$

$$X_7 = \partial_t \tag{4.75}$$

$$X_8 = -\frac{8rt}{\beta}\left(\cos\left(\frac{\omega r}{i}\right) + \phi\sin\left(\frac{\omega r}{i}\right)\right)\partial_t + \frac{4rt}{\beta\omega}\left(\phi\cos\left(\frac{\omega r}{i}\right) + \sin\left(\frac{\omega r}{i}\right)\right)\partial_r$$
$$+\frac{4\sigma}{\beta}\left(\phi\sin\left(\frac{\omega r}{i}\right) - \cos\left(\frac{\omega r}{i}\right)V\partial_V\right) \tag{4.76}$$

where

$$\beta = \frac{\sigma^2\omega^2}{2}, \text{ and} \tag{4.77}$$

$$k_1 = -\frac{D^2}{\sigma^2} - 2r. \tag{4.78}$$

5. INVARIANT SOLUTION FOR X_1

Masebe and Manale in[7] presented an invariant solution through the symmetry X_3. In this section, the invariant solution through the symmetry X_1 is constructed, and its evolution against time is represented graphically. The invariants are determined from solving the equation

$$X_1 I = \left(-\frac{2te^{-t}\phi}{\beta^2\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right)\right)\frac{\partial I}{\partial t} + \left(\frac{e^{-t}i\phi}{\beta^2\hat{\omega}\omega}\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\frac{\partial I}{\partial r}$$
$$+\left\{\frac{-e^{-t}\phi}{2\beta^2\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) + \frac{e^{-t}\phi}{2\beta^3\hat{\omega}}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right)\right. \tag{5.1}$$
$$\left.+\frac{e^{-t}\phi}{2\beta^3}\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) - \frac{Di\phi e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3}\cos\left(\frac{\omega r}{i}\right)\right\}V\frac{\partial I}{\partial V}$$

The operator X_1 has characteristic system

$$\frac{dt}{\left(\frac{-2te^{-t}\phi\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right)}{\beta^2\hat{\omega}}\right)} = \frac{dr}{\left(\frac{e^{-t}i\phi\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)}{\beta^2\hat{\omega}}\right)} = \frac{dV}{VK}. \tag{5.2}$$

where

$$K = \left\{ \frac{-e^{-t}\phi}{2\beta^2\hat{\omega}} \sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) + \frac{e^{-t}\phi}{2\beta^3\hat{\omega}} \sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) + \frac{e^{-t}\phi}{2\beta^3} \sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right) - \frac{Di\phi e^{-t}\sin\hat{\omega}\cos\hat{\omega}t}{2\beta^3}\cos\left(\frac{\omega r}{i}\right) \right\}. \tag{5.3}$$

There are two equations that can be formed from the above characteristic system. The first one is

$$\frac{dt}{\left(\frac{-2te^{-t}\phi\sin\hat{\omega}\cos\hat{\omega}t\sin\left(\frac{\omega r}{i}\right)}{\beta^2\hat{\omega}}\right)} = \frac{dr}{\left(\frac{e^{-t}i\phi\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)}{\beta^2\hat{\omega}}\right)}. \tag{5.4}$$

Equation (5.4) simplifies to

$$\frac{dt}{t} = -\frac{2\omega}{i}\tan\left(\frac{\omega r}{i}\right) dr. \tag{5.5}$$

Integrating both sides gives

$$t = K_1\cos^2\left(\frac{\omega r}{i}\right). \tag{5.6}$$

K_1 is the constant of integration. Therefore one of the invariants is

$$\psi_1 = \frac{\cos^2\left(\frac{\omega r}{i}\right)}{t}. \tag{5.7}$$

The second equation that can also be formed from equation (5.2) is

$$\frac{dr}{\left(\frac{e^{-t}i\phi}{\beta^2\hat{\omega}\omega}\sin\hat{\omega}\cos\hat{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)} = \frac{dV}{VK}. \tag{5.8}$$

Simplifying leads to

$$-\frac{\omega}{2i}\tan\left(\frac{\omega r}{i}\right) + \frac{\omega}{2\beta i}\tan\left(\frac{\omega r}{i}\right) + \frac{\hat{\omega}\omega}{2\beta i}\tan\left(\frac{\omega r}{i}\right) - \frac{\hat{\omega}\omega^2}{2\beta}dr = \frac{dV}{V} \tag{5.9}$$

Integrating equation (5.9) with respect to r on the left-hand side, and with respect to V on the right-hand side, gives

$$\frac{1}{2}\ln|\cos\left(\frac{\omega r}{i}\right)| - \frac{1}{2\beta}\ln|\cos\left(\frac{\omega r}{i}\right)| - \frac{\hat{\omega}}{2\beta}\ln|\cos\left(\frac{\omega r}{i}\right)| - \frac{D\hat{\omega}\omega^2 r}{2\beta} = \ln V + C. \tag{5.10}$$

For very small values of ω the first three terms of equation (5.10) disappear, so that

$$-\frac{D\hat{\omega}\omega^2 r}{2\beta} = \ln V + C \tag{5.11}$$

β is either -1 or 1 [8] and $\hat{\omega}$ is also constant; therefore, without any loss of generality, $\frac{\hat{\omega}}{2\beta}$ can be absorbed into the single constant D; which means that

$$-\frac{D\hat{\omega}\omega^2 r}{2\beta} \approx D\omega^2 r = \ln V + C \tag{5.12}$$

Therefore,

$$V = K_2 e^{D\omega^2 r} \tag{5.13}$$

K_2 is the constant of integration, so another invariant is

$$\psi_2 = \frac{V}{e^{D\omega^2 r}} \tag{5.14}$$

Now, designating one of the invariants as a function of the other gives

$$\psi_2 = \varphi(\psi_1), \tag{5.15}$$

which implies that

$$V = e^{D\omega^2 r}\varphi(\psi_1) \tag{5.16}$$

Differentiating equation (5.16) with respect to t and r using chain rule gives

$$\begin{aligned} V_t &= e^{D\omega^2 r}\frac{\partial\varphi}{\partial\psi_1}\frac{\partial\psi_1}{\partial t} \\ &= -\frac{e^{D\omega^2 t}}{t^2}\cos^2\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) \end{aligned} \tag{5.17}$$

$$V_r = \frac{\partial}{\partial r}\left(e^{D\omega^2 r}\right)\varphi(\psi_1) + e^{D\omega^2 r}\frac{\partial}{\partial r}\left(\varphi(\psi_1)\right) \tag{5.18}$$

$$= D\omega^2 e^{D\omega^2 r}\varphi(\psi_1) - \frac{2\omega r}{i}e^{D\omega^2 r}\frac{\sin\left(\frac{\omega r}{i}\right)\cos\left(\frac{\omega r}{i}\right)}{t}\varphi^{'}(\psi_1) \tag{5.19}$$

and

(5.20)

$$\begin{aligned} V_{rr} = &D\omega^2\frac{\partial}{\partial r}\left(e^{D\omega^2 r}\right)\varphi(\psi_1) + D\omega^2 e^{D\omega^2 r}\frac{\partial}{\partial r}\left(\varphi(\psi_1)\right) \\ &- \frac{2\omega}{it}\frac{\partial}{\partial r}\left(e^{D\omega^2 r}\right)\sin\left(\frac{\omega r}{i}\right)\cos\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) - \frac{2\omega}{it}e^{D\omega^2 r}\frac{\partial}{\partial r}\left(\sin\left(\frac{\omega r}{i}\right)\right)\cos\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) \\ &- \frac{2\omega}{it}e^{D\omega^2 r}\frac{\partial}{\partial r}\left(\cos\left(\frac{\omega r}{i}\right)\right)\sin\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) - \frac{2\omega}{it}e^{D\omega^2 r}\sin\left(\frac{\omega r}{i}\right)\cos\left(\frac{\omega r}{i}\right)\frac{\partial}{\partial r}\varphi^{'}(\psi_1). \end{aligned}$$

Which gives

$$\begin{aligned} V_{rr} = &D^2\omega^4 e^{D\omega^2 r}\varphi(\psi_1) - \frac{2D}{it}e^{D\omega^2 r}\sin\left(\frac{\omega r}{i}\right)\cos\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) \\ &- \frac{2D\omega^3}{it}e^{D\omega^2 r}\sin\left(\frac{\omega r}{i}\right)\cos\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) + \frac{2\omega^2}{t}e^{D\omega^2 r}\cos^2\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) \\ &- \frac{2\omega^2}{t}\sin^2\left(\frac{\omega r}{i}\right)\varphi^{'}(\psi_1) - \frac{4\omega^2}{t^2}e^{D\omega^2 r}\cos^2\left(\frac{\omega r}{i}\right)\sin^2\left(\frac{\omega r}{i}\right)\varphi^{''}(\psi_1). \end{aligned} \tag{5.21}$$

(5.22)

The substitution of equations (5.17), (5.18), and (5.21) into (4.1) gives

(5.23)

$$\begin{aligned} 0 = &\varphi^{''}(\psi_1)\left\{-\frac{2\omega^2\sigma^2}{t^2}e^{D\omega^2 r}\sin^2\left(\frac{\omega r}{i}\right)\cos^2\left(\frac{\omega r}{i}\right)\right\} \\ &+ \varphi^{'}(\psi_1)\left\{-\frac{e^{D\omega^2 r}}{t^2}\cos^2\left(\frac{\omega r}{i}\right)\right\} - \varphi^{'}(\psi_1)\left\{\frac{D\omega^3\sigma^2}{it}e^{\omega^2 r}\sin\left(\frac{\omega r}{i}\right)\cos\left(\frac{\omega r}{i}\right)\right\} \\ &+ \varphi^{'}(\psi_1)\left\{\frac{\omega^2\sigma^2}{t}e^{D\omega^2 r}\cos^2\left(\frac{\omega r}{i}\right) - \frac{\omega^2\sigma^2}{t}e^{D\omega^2 r}\sin^2\left(\frac{\omega r}{i}\right)\right\} \\ &- \varphi^{'}(\psi_1)\left\{\frac{2D\omega}{it}e^{D\omega^2 r}\sin\left(\frac{\omega r}{i}\right)\cos\left(\frac{\omega r}{i}\right)\right\} + \varphi(\psi_1)\left\{\frac{D^2\omega^4\sigma^2}{2}e^{D\omega^2 r} + D^2\omega^2 e^{D\omega^2 r} - Ce^{D\omega^2 r}\right\}. \end{aligned}$$

At this stage, one can take the advantage of the fact that $\omega \to 0$, which reduces the partial differential equation (5.23) to

$$-\frac{1}{t^2}\varphi'(\psi_1) - C\varphi(\psi_1) = 0 \tag{5.24}$$

To solve equation (5.24) the following change of variables are applied

$$d\lambda = -t^2 d\psi_1 \tag{5.25}$$

$$\varphi(\psi_1) = h \tag{5.26}$$

and then equation (5.25) implies

$$\frac{d\varphi}{d\psi_1} = -t^2\frac{d\varphi}{d\lambda}. \tag{5.27}$$

Effecting these change of variables into equation 5.24 leads to

$$h_\lambda - hC = 0. \tag{5.28}$$

Solving by separation of variables gives

$$h = C_2 e^{C\lambda}. \tag{5.29}$$

To put the solution in terms of the original variable t, put equation (5.25) in integration form as follows:

$$\lambda = -t^2 \int d\psi_1. \tag{5.30}$$

Applying equation (5.7) for ψ_1 gives

$$\begin{aligned} \lambda &= -t^2 \int d\frac{\cos^2\left(\frac{\omega r}{i}\right)}{t} \\ &= -\cos^2\left(\frac{\omega r}{i}\right)\int \frac{d\frac{1}{t}}{\frac{1}{t^2}} \\ &= t\cos^2\left(\frac{\omega r}{i}\right) + C_3. \end{aligned} \tag{5.31}$$

Taking $\omega \to 0$ gives

$$\lambda = t + C_3. \tag{5.32}$$

Combining equation (5.32) with equation (5.29) and setting back $h = \varphi$, the following solution is obtained in terms of t

$$\varphi(\psi_1) = C_4 e^{Ct}. \tag{5.33}$$

Substitute equation (5.33) in 5.16 gives

$$V = C_4 e^{D\omega^2 r} e^{Ct}. \tag{5.34}$$

Taking $\omega \to 0$ give

$$V = C_4 e^{Ct}. \tag{5.35}$$

If this is the solution for the transformed Black-Scholes equation (4.13), it is expected that when derivatives V_t, V_r, and V_{rr} are substituted into equation (4.13), the result should be zero. Checking if this is the case

$$\begin{aligned} V_t &= CC_4 e^{Ct} \\ V_r &= 0 \\ V_{rr} &= 0, \end{aligned}$$

which means that from (4.13) we have

$$CC_4e^{Ct} - CC_4e^{Ct} = 0.$$

6. NUMERICAL SOLUTIONS

The Option's value against time is plotted in Figure 1 below.

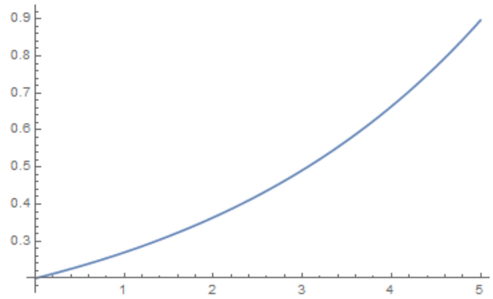

Figure 1: Plot for solution 5.35

The constants C_4 and C were chosen to be 0.2 and 0.3, respectively. This was to ensure that we obtain positive solutions as the price of an option is always a positive function of time. This solution is consistent with one of the solutions obtained by Ibragimov and Gazizov in [4].

7. CONCLUSION

We applied Lie Symmetry technique to a Mathematical Model which describes an option pricing. The model was formulated by Fischer Black and Myron Scholes in 1979. Their model forms the corner-stone for modern financial theory, it is not quite often that one can talk about the modern finance without ever mentioning the revolutionary Black-Scholes (BS) model. Furthermore, we found new symmetries by using Manale's formula as presented by Masebe and Manale in [7]. Lie group analysis is indeed the most powerful tool to find the exact solution of partial differential equations. We constructed an invariant solution through the obtained symmetries and represented graphically its evolution against time.

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