COEFFICIENT ESTIMATES OF SAKAGUCHI KIND FUNCTIONS USING LUCAS POLYNOMIALS

H. PRIYA AND B. SRUTHA KEERTHI

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DEPARTMENT OF MATHEMATICS, SCHOOL OF ADVANCED SCIENCES, VIT CHENNAI CAMPUS, CHENNAI - 600 048, INDIA.

priyahririkrishnan1@gmail.com

DEPARTMENT OF MATHEMATICS, SCHOOL OF ADVANCED SCIENCES, VIT CHENNAI CAMPUS, CHENNAI - 600 048, INDIA.

sruthilaya06@yahoo.co.in

ABSTRACT. By means of \((p, q)\) Lucas polynomials, we estimate coefficient bounds and Fekete-Szego inequalities for functions belonging to this class. Several corollaries and consequences of the main results are also obtained.

Key words and phrases: Lucas polynomial; Analytic functions; Univalent functions; Bi-univalent functions.

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1. Introduction

Let $\mathcal{A}$ indicate an analytic functions family, which is normalized under the condition $f(0) = f'(0) - 1 = 0$ in $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

\begin{equation}
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. With a view to recalling the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\mathbb{U}$. Then we say that the function $f$ is subordinate to $g$ if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$ with

$$\omega(0) = 0, \ |\omega(z)| < 1, (z \in \mathbb{U})$$

such that $f(z) = g(\omega(z))$

We denote this subordination by,

$$f < g \ (or) \ f(z) < g(z)$$

In particular, if the function $g$ is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0), f(\mathbb{U}) \subset g(\mathbb{U})$

The Koebe-One Quarter theorem [11] asserts that image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disc of radius $\frac{1}{4}$, thus every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w, (|\omega| < r_0(f), r_0(f) > \frac{1}{4})$

\begin{equation}
 f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \ldots
\end{equation}

A function $f \in \mathcal{A}$ is said to be bi-univalent functions in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if there exist a function $g \in \mathcal{S}$ such that $g(z)$ is an univalent extension of $f^{-1}$ to $\mathbb{U}$. Let $\Lambda$ denote the class of bi-univalent functions in $\mathbb{U}$. The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log(1+2z)$ are in the class $\Lambda$ (see details in [20]). However, the familiar Koebe function is not bi-univalent. Lewin [17] investigated the class of bi-univalent functions $\Lambda$ and obtained a bound $|a_2| \leq 1.51$. Motivated by the work of Lewin [17], Brannan and Clunie [9] conjectured that $|a_2| \leq \sqrt{2}$. The coefficient estimate problem for $|a_n| (n \in \mathbb{N}, n \geq 3)$ is still open [20]. Brannan and Taha [10] also worked on certain subclasses of the bi-univalent function class $\Lambda$ and obtained estimates for their initial co-efficients.

Various classes of bi-univalent functions were introduced and studied in recent times, the study of bi-univalent functions gained momentum mainly due to the work of Srivastava et al [20]. Motivates by this, many researchers [11, 13, 15, 20, 21] and [27, 29], also the references cited there in recently investigated several interesting subclasses of the class $\Lambda$ and found non-sharp estimates on the first two Taylor-Maclaurin co-efficients. Recently, many researchers have been exploring bi-univalent functions, few to mention Fibonacci polynomials, Lucas polynomials, Chebyshev polynomials, Pell polynomials, Lucas-Lehmer polynomials, Orthogonal polynomials and the other special polynomials and their generalizations are of great importance in a variety of branches such as Physics, Engineering, Architecture, Nature, Art, Number theory, Combinatorics and Numerical analysis. These polynomials have been studied in several papers from a theoretical point view (see for example [23, 30] also see references therein)

We recall the following results relevant for our study as stated in [3]. Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p, q)$-Lucas polynomials $L_{p,q,n}(x)$ are defined by the
recurrence relation.
\[ L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x) \quad (n \geq 2) \]

From which the first few Lucas polynomials can be found as
\[
\begin{align*}
L_{p,q,0}(x) &= 2 \\
L_{p,q,1}(x) &= p(x) \\
L_{p,q,2}(x) &= p^2(x) + 2q(x) \\
L_{p,q,3}(x) &= p^3(x) + 3p(x)q(x), \ldots
\end{align*}
\]

(1.3)

For the special cases of \( p(x) \) and \( q(x) \), we can get the polynomials given \( L_{x,1,n}(x) \equiv L_n(x) \) Lucas polynomials, \( L_{2x,1,n}(x) \equiv D_n(x) \) Pell-Lucas polynomials, \( L_{1,2x,n}(x) \equiv j_n(x) \) Jacobsthal-Lucas polynomials, \( L_{3x,-2,n}(x) \equiv F_n(x) \) Fermate-Lucas polynomials, \( L_{2x,-1,n}(x) \equiv T_n(x) \) Chebyshev polynomials first kind.

**Lemma 1.1.** [16] Let \( G_{(L(x))}(z) \) be the generating function of the \((p,q)\)-Lucas polynomial sequence \( L_{p,q,n}(x) \) Then,

\[
G_{(L(x))}(z) = \sum_{n=0}^{\infty} L_{p,q,n}(x)z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}
\]

and

\[
G_{(L(x))}(z) = G_{(L(x))}(z) - 1 = 1 + \sum_{n=1}^{\infty} L_{p,q,n}(x)z^n = \frac{1 + q(x)z^2}{1 - p(x)z - q(x)z^2}
\]

**Definition 1.1.** A function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is said to be in the class \( C(t, \lambda) \) if it satisfies the subordination conditions which are as follows

\[
(1 - t) \frac{[\lambda z^3 f'''(z) + (1 + 2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z^2 [f''(z) - tf''(tz)] + z [f'(z) - tf'(tz)]} < G_{(L(x))}(z) \quad (z \in \mathbb{U})
\]

and

\[
(1 - t) \frac{[\lambda \omega^3 g'''(\omega) + (1 + 2\lambda)\omega^2 g''(\omega) + \omega g'(\omega)]}{\lambda \omega^2 [g''(\omega) - tg''(t\omega)] + \omega [g'(\omega) - tg'(t\omega)]} < G_{(L(x))}(\omega) \quad (\omega \in \mathbb{U})
\]

where \( G_{(L(p,q,n))}(z) \in \phi \) and the function \( g \) is described as \( g(\omega) = f^{-1}(\omega) \)

**Definition 1.2.** For the case \( t = 0 \) the class \( C(t, \lambda) \) reduces to the class \( C(0, \lambda) \) satisfying the following conditions,

\[
[\lambda z^3 f'''(z) + (1 + 2\lambda)z^2 f''(z) + zf'(z)] < G_{(L(x))}(z)
\]

and

\[
[\lambda \omega^3 g'''(\omega) + (1 + 2\lambda)\omega^2 g''(\omega) + \omega g'(\omega)] < G_{(L(x))}(\omega)
\]

**Definition 1.3.** For the case \( \lambda = 0 \) the class \( C(t, \lambda) \) reduces to the class \( C(t, 0) \) satisfying the following conditions,

\[
\frac{(1 - t) \ [z^2 f''(z) + zf'(z)]}{z [f'(z) - tf'(tz)]} < G_{(L(x))}(z)
\]
\[
\frac{(1-t) [\omega^2 g''(\omega) + \omega g'(\omega)]}{\omega [g'(\omega) - t g'(t \omega)]} < G_{(L(x))}(\omega)
\]

2. COEFFICIENT BOUNDS FOR THE FUNCTION OF THE CLASS C(t, \lambda)

**Theorem 2.1.** Let the function \( f(z) \) given by (1.1) be in the class \( C(t, \lambda) \). Then

\[
|a_2| \leq \frac{|p(x)|}{\sqrt{2} \sqrt{4(T_2 - 2) T_2 p'(x) - (p'(x))(T_2 - 2)(1 + \lambda) - 2q(x)(T_2 - 2)(1 + \lambda) - 3(T_3 - 3)p^2(x)}}
\]

\[
|a_3| \leq \frac{|p^1(x)|}{3(1 + 2\lambda)(3 - T_3)} + \frac{|p^2(x)|}{4(1 + \lambda)^2(2 - T_2)^2}
\]

**Proof.** Let \( f \in C(t, \lambda) \) there exists two analytic functions \( u, v : \mathbb{U} \to \mathbb{U} \) with \( u(0) = v(0) \) such that \( |u(z)| < 1, |v(\omega)| < 1 \), we can write from (1.4) and (1.5), we have

(2.1) \[
\frac{(1-t) [\lambda z^3 f'''(z) + (1 + 2\lambda)z^2 f''(z) + z f'(z)]}{\lambda z^2 [f''(z) - t f''(t z)] + z [f'(z) - t f'(t z)]} = G_{(L(x))}(z) \quad (z \in \mathbb{U})
\]

and

(2.2) \[
\frac{(1-t) [\lambda \omega^2 g'''(\omega) + (1 + 2\lambda)\omega^2 g''(\omega) + \omega g'(\omega)]}{\lambda \omega^2 [g''(\omega) - t g''(t \omega)] + \omega [g'(\omega) - t g'(t \omega)]} = G_{(L(x))}(\omega) \quad (\omega \in \mathbb{U})
\]

It is fairly well known that if

| \( u(z) \) | = | \( u_1 z + u_2 z^2 + \cdots \) | < 1  
| \( v(\omega) \) | = | \( v_1 \omega + v_2 \omega^2 + \cdots \) | < 1  
then \( |a_k| \leq 1 \) and \( |v_k| \leq 1 \) \( (k \in \mathbb{N}) \)

It follows that,

\[
G_{(L(x))}(u(z)) = 1 + L_{p,q,1}(x) u(z) + L_{p,q,2}(x) u^2(z) + \cdots
\]

(2.3)

\[
G_{(L(x))}(v(\omega)) = 1 + L_{p,q,1}(x) v(\omega) + L_{p,q,2}(x) v^2(\omega) + \cdots
\]

(2.4)

From the equalities (2.1) and (2.2), we obtain that

(2.5) \[
\frac{(1-t) [\lambda z^3 f'''(z) + (1 + 2\lambda)z^2 f''(z) + z f'(z)]}{\lambda z^2 [f''(z) - t f''(t z)] + z [f'(z) - t f'(t z)]} = 1 + L_{p,q,1}(x) u_1(z)
\]

(2.6)

It is follows that, from (2.5) and (2.6), we obtain,

(2.7) \[
2a_2(1 + \lambda)(2 - T_2) = L_{p,q,1}(x) u_1
\]

(2.8) \[
3a_3(1 + 2\lambda)(3 - T_3) + 4a_2^2 T_2(1 + \lambda)^2(T_2 - 2) = L_{p,q,1}(x) u_2 + L_{p,q,2}(x) u_1^2
\]

and

(2.9) \[
-2a_2(1 + \lambda)(2 - T_2) = L_{p,q,1}(x) v_1
\]

From (2.7) and (2.9), we get,

\[ u_1 = -v_1 \]

and

\[ 2 \left[ 4(1 + \lambda)^2 (2 - T_2)^2 \right] a_2^2 = \mathcal{L}_{p,q,1}^2(x)(u_1^2 + v_1^2) \]

By adding (2.8) to (2.10), we get,

\[ 2 \left[ 3(3 - T_3) + 4T_2(T_2 - 2) \right] a_2^2 = \mathcal{L}_{p,q,1}^2(x)(u_2 + v_2) + \mathcal{L}_{p,q,2}(x)(u_1^2 + v_1^2) \]

By using (2.12) in (2.13) we have,

\[ a_2^2 = \frac{\mathcal{L}_{p,q,1}^3(x)(u_2 + v_2)}{2 \left\{ 4(T_2 - 2) \{ T_2 \mathcal{L}_{p,q,1}^2(x) - \mathcal{L}_{p,q,2}(x)(T_2 - 2)(1 + \lambda) \} - 3(T_3 - 3)\mathcal{L}_{p,q,1}^2(x) \} \}

Thus From (1.3) and (2.14) we get,

\[ |a_2| \leq \sqrt{2 \left\{ 4(T_2 - 2) \{ T_2p^2(x) - (p^2(x))(T_2 - 2)(1 + \lambda) - 2q(x)(T_2 - 2)(1 + \lambda) \} - 3(T_3 - 3)p^2(x) \} \] \]

Next, in order to find the bound on \( |a_3| \), by subtracting (2.10) from (2.8), we obtain

\[ 6(1 + 2\lambda)(3 - T_3)(a_3 - a_2^2) = \mathcal{L}_{p,q,1}^2(x)(u_2 - v_2) \]

\[ a_3 = \frac{\mathcal{L}_{p,q,1}^2(x)(u_2 - v_2)}{6(1 + 2\lambda)(3 - T_3)} + a_2^2 \]

\[ a_3 = \frac{\mathcal{L}_{p,q,1}^2(x)(u_2 - v_2)}{6(1 + 2\lambda)(3 - T_3)} + \frac{\mathcal{L}_{p,q,1}^2(x)(u_1^2 + v_1^2)}{8(1 + \lambda)^2(2 - T_2)^2} \]

\[ |a_3| \leq \frac{|p^1(x)|}{3(1 + 2\lambda)(3 - T_3)} + \frac{|p^2(x)|}{4(1 + \lambda)^2(2 - T_2)^2} \]

This completes the proof  

If we take \( \lambda = 0 \) and \( t = 0 \) in theorem (2.1) we obtain the following corollaries respectively,

**Corollary 2.2.** A function \( f \in \mathcal{A} \) of the form (1.1) is in the class \( C(t, \lambda) \) where \( \lambda = 0 \), we obtain,

\[ |a_2| \leq \frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{2 \left\{ 4(T_2 - 2) \{ T_2p^2(x) - (p^2(x))(T_2 - 2)(1 + \lambda) \} - 3(T_3 - 3)p^2(x) \} \] \]

\[ |a_3| \leq \frac{|p^1(x)|}{3(3 - T_3)} + \frac{|p^2(x)|}{4(2 - T_2)^2} \]

**Corollary 2.3.** A function \( f \in \mathcal{A} \) of the form (1.1) is in the class \( C(t, \lambda) \) where \( t = 0 \), we obtain

\[ |a_2| \leq \frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{2 \left\{ -4 \{ T_2p^2(x) + p^2(x)(1 + \lambda) + 2q(x)(1 + \lambda) \} + 3p^2(x) \} \] \]

\[ |a_3| \leq \frac{|p^1(x)|}{6(1 + 2\lambda)} + \frac{|p^2(x)|}{4(1 + \lambda)^2} \]
3. Fekete-Szegö Inequality For The Class \( C(t, \lambda) \)

Fekete-Szegö inequality is one of the famous problems related to coefficient of univalent analytic functions. It was first given by [12], the classical Fekete-Szegö inequality for the coefficients of \( f \in S \) is

\[
|a_3 - \mu a_2^2| \leq 1 + 2e^{\frac{2\mu}{1 - \mu}} \quad \text{for } \mu \in [0, 1)
\]

As \( \mu \to 1^- \) we have the elementary inequality \( |a_3 - a_2^2| \leq 1 \). Moreover, the coefficient functional \( S_\mu(f) = a_3 - \mu a_2^2 \) on the normalized analytic functions \( f \) in the unit disk \( U \) plays an important role in function theory. The problem of maximizing the absolute value of the functional \( S_\mu(f) \) is called the Fekete-Szegö problem. In this section, we are ready to find the sharp bounds of Fekete-Szegö functional \( S_\mu(f) \) for \( f \in C(t, \lambda) \) given by (1.1)

Theorem 3.1. Let \( f \) given by (1.1) be in the class \( C(t, \lambda) \) and \( \mu \in \mathbb{R} \). Then,

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
|p(x)| / (3(1 + 2\lambda)(3 - T_3)), & 0 \leq |h(\mu)| \leq \frac{1}{6(1 + 2\lambda)(3 - T_3)} \\
2|p(x)||h(\mu)|, & |h(\mu)| \geq \frac{1}{6(1 + 2\lambda)(3 - T_3)} 
\end{cases}
\]

where

\[
h(\mu) = \frac{(1 - \mu)\mathcal{L}_{p,q,1}^2(x)}{2 \{ [3(1 + 2\lambda)(3 - T_3) - 2T_2(1 + \lambda)^2(2 - T_2)] \mathcal{L}_{p,q,1}^2(x) - 4\mathcal{L}_{p,q,2}(x)(1 + \lambda)^2(2 - T_2)^2 \}}
\]

Proof. From (2.14) & (2.15) we conclude that

\[
a_3 - \mu a_2^2 = \left[ \frac{(1 - \mu)\mathcal{L}_{p,q,1}^3(x)(u_2 + v_2)}{2 \{ [3(1 + 2\lambda)(3 - T_3) - 2T_2(1 + \lambda)^2(2 - T_2)] \mathcal{L}_{p,q,1}^2(x) - 4\mathcal{L}_{p,q,2}(x)(1 + \lambda)^2(2 - T_2)^2 \}} \right] + \left[ \mathcal{L}_{p,q,1}(x)(u_2 - v_2) \right] / (6(1 + 2\lambda)(3 - T_3))
\]

\[
a_3 - \mu a_2^2 = \mathcal{L}_{p,q,1}(x) \left[ \left( h(\mu) + \frac{1}{6(1 + 2\lambda)(3 - T_3)} \right) u_2 + \left( h(\mu) - \frac{1}{6(1 + 2\lambda)(3 - T_3)} \right) v_2 \right]
\]

where,

\[
h(\mu) = \frac{(1 - \mu)\mathcal{L}_{p,q,1}^2(x)}{2 \{ [3(1 + 2\lambda)(3 - T_3) - 2T_2(1 + \lambda)^2(2 - T_2)] \mathcal{L}_{p,q,1}^2(x) - 4\mathcal{L}_{p,q,2}(x)(1 + \lambda)^2(2 - T_2)^2 \}}
\]

Then, in view of (1.3), we obtain

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
|p(x)| / (3(1 + 2\lambda)(3 - T_3)), & 0 \leq |h(\mu)| \leq \frac{1}{6(1 + 2\lambda)(3 - T_3)} \\
2|p(x)||h(\mu)|, & |h(\mu)| \geq \frac{1}{6(1 + 2\lambda)(3 - T_3)} 
\end{cases}
\]

we end this section with some corollaries.

Taking \( \mu = 1 \) in theorem (3.1) we get the following corollary.
Corollary 3.2. If \( f \in C(t, \lambda) \) then,
\[
|a_3 - a_2^2| \leq \frac{|p(x)|}{3(1 + 2\lambda)(3 - T_3)}
\]

For \( \lambda = 0 \) and \( t = 0 \) in theorem (3.1), the following Fekete-Szego inequality is obtained:

Corollary 3.3. Let \( f \) given by (1.1) be in the class \( C(t, \lambda) \), then

For \( t = 0 \)
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|p(x)|}{6(1 + 2\lambda)}, & 0 \leq |h(\mu)| \leq \frac{1}{12(1 + 2\lambda)} \\
2|p(x)||h(\mu)|, & |h(\mu)| \geq \frac{1}{12(1 + 2\lambda)}
\end{cases}
\]

For \( \lambda = 0 \)
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|p(x)|}{3(3 - T_3)}, & 0 \leq |h(\mu)| \leq \frac{1}{6(3 - T_3)} \\
2|p(x)||h(\mu)|, & |h(\mu)| \geq \frac{1}{6(3 - T_3)}
\end{cases}
\]

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