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ON EULER'S FIRST TRANSFORMATION FORMULA FOR k-HYPERGEOMETRIC FUNCTION

SUNGTAE JUN AND INSUK KIM*

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GENERAL EDUCATION INSTITUTE, KONKUK UNIVERSITY, CHUNGJU 380-701, REPUBLIC OF KOREA.

DEPARTMENT OF MATHEMATICS EDUCATION, WONKWANG UNIVERSITY, IKSAN, 570-749, REPUBLIC OF KOREA.
sjun@kku.ac.kr
iki@wku.ac.kr

ABSTRACT. Mubeen *et al.* obtained Kummer's first transformation for the k-hypergeometric function. The aim of this note is to provide the Euler-type first transformation for the k-hypergeometric function. As a limiting case, we recover the results of Mubeen *et al.* In addition to this, an alternate and easy derivation of Kummer's first transformation for the k-hypergeometric function is also given.

Key words and phrases: Euler-type transformation; Kummer-type transformation; Hypergeometric function, k-hypergeometric function.

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1. INTRODUCTION

In 2007, Diaz and Pariguan [2] introduced the k-generalized gamma function Γ_k, k-beta function B_k and Pochhammer k-symbol $(x)_{j,k}$ as follows:

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma\left(\frac{x}{k}\right) = \int_0^\infty e^{-\frac{t^k}{k}}\,t^{x-1}dt \tag{1.1}$$

for Re$(x) > 0$ and $k > 0$.

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt \tag{1.2}$$

for $x > 0$ and $y > 0$.

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \tag{1.3}$$

and

$$(x)_{j,k} = x(x+k)(x+2k)\cdots(x+(j-1)k). \tag{1.4}$$

In the same paper, they have also defined the k-hypergeometric function ${}_mF_{m,k}$ as follows.

$$\begin{aligned} &{}_mF_{m,k}\begin{bmatrix}(a_1,k), & (a_2,k), & \cdots, & (a_m,k)\\ (b_1,k), & (b_2,k), & \cdots, & (b_m,k)\end{bmatrix};x\Bigg] \\ &= \sum_{n=0}^{\infty}\frac{(a_1)_{n,k}(a_2)_{n,k}\cdots,(a_m)_{n,k}}{(b_1)_{n,k}(b_2)_{n,k}\cdots,(b_m)_{n,k}}\frac{x^n}{n!}.\end{aligned} \tag{1.5}$$

Furthermore, it is easily seen that the k-hypergeometric function ${}_1F_{1,k}\left[\begin{matrix}(a,k)\\(b,k)\end{matrix};x\right]$ satisfies the differential equation,

$$kx\frac{d^2y}{dx^2} + (b-kx)\frac{dy}{dx} - ay = 0 \tag{1.6}$$

It is not out of place to mention here that for $k = 1$, (1.5) reduces at once to the ordinary hypergeometric function ${}_mF_m$.

Diaz *et al.* [1, 2, 3] also established a number of very interesting properties of this k-hypergeometric function. Later, Mansour [7], Kokologiannaki [4], Krasniqi [5, 6], Merorci [8] and Mubeen & Habibullah [10] developed this k-hypergeometric function by obtaining some more results.

In other paper, Mubeen and Habibullah [11] introduced the following integral representation of some confluent k-hypergeometric function ${}_mF_{m,k}$ and k-hypergeometric function ${}_{m+1}F_{m,k}$ as follows.

$$\begin{aligned} &{}_mF_{m,k}\left[\begin{matrix}(\frac{\beta}{m},k), & (\frac{\beta+1}{m},k), & \cdots, & (\frac{\beta+m-1}{m},k)\\ (\frac{\gamma}{m},k), & (\frac{\gamma+1}{m},k), & \cdots, & (\frac{\gamma+m-1}{m},k)\end{matrix};x\right] \\ &= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^m}dt\end{aligned} \tag{1.7}$$

and

$$(1.8) \qquad {}_{m+1}F_{m,k}\left[\begin{matrix}(\alpha,1), & (\frac{\beta}{m},k), & (\frac{\beta+1}{m},k), & \cdots, & (\frac{\beta+m-1}{m},k)\\ & (\frac{\gamma}{m},k), & (\frac{\gamma+1}{m},k), & \cdots, & (\frac{\gamma+m-1}{m},k)\end{matrix};x\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-xt^m)^{-\alpha}dt.$$

It is to be noted here that for $k=1$ and $m=1$, the results (1.7) and (1.8) reduce to the integral representation of the ordinary confluent hypergeometric function ${}_1F_1\left[\begin{matrix}\beta\\ \gamma\end{matrix};x\right]$ and ordinary hypergeometric function ${}_2F_1\left[\begin{matrix}\alpha, \ \beta\\ \gamma\end{matrix};x\right]$ respectively.

With the help of the result (1.8), Mubeen [9] obtained the following k-Gauss summation theorem as follows.

$$(1.9) \qquad {}_2F_{1,k}\left[\begin{matrix}(a,1), & (b,k)\\ & (c,k)\end{matrix};1\right] = \frac{\Gamma_k(c)\Gamma_k(c-b-ka)}{\Gamma_k(c-b)\Gamma_k(c-ka)}$$

and in particular, the following k-Vandermonde theorem as follows.

$$(1.10) \qquad {}_2F_{1,k}\left[\begin{matrix}(-n,1), & (b,k)\\ & (c,k)\end{matrix};1\right] = \frac{(c-b)_{n,k}}{(c)_{n,k}}.$$

Clearly, for $k=1$, the results (1.9) and (1.10) reduce to Gauss summation theorem and Vandermonde summation theorem respectively for the ordinary hypergeometric function.

Finally, with the aid of the formula (1.10), Mubeen [9] obtained the following interesting k-analogue of Kummer's first formula as follows:

$$(1.11) \qquad {}_1F_{1,k}\left[\begin{matrix}(b,k)\\ (c,k)\end{matrix};x\right] = e^x\,{}_1F_{1,k}\left[\begin{matrix}(c-b,k)\\ (c,k)\end{matrix};-x\right].$$

For $k=1$, the result (1.11) reduces to the following well known as useful Kummer's first transformation [12] viz.

$$(1.12) \qquad {}_1F_1\left[\begin{matrix}b\\ c\end{matrix};\ x\right] = e^x\,{}_1F_1\left[\begin{matrix}c-b\\ c\end{matrix};-x\right].$$

The aim of this note is to provide the Euler-type first transformation for the k-hypergeometric function. As a limiting case, we recover the results of Mubeen *et al.* In addition to this, an alternate and easy derivation of Kummer's first transformation for the k-hypergeometric function is also given.

2. AN EULER-TYPE TRANSFORMATION FOR k-HYPERGEOMETRIC FUNCTION

The Euler-type transformation for the k-hypergeometric function to be established in this note is given in the following theorem.

Theorem 2.1. *For $k>0$, Re$(x)>\frac{1}{2}$ and $|x|<1$, the following result holds true.*

$$(2.1) \qquad {}_2F_{1,k}\left[\begin{matrix}(a,1), & (b,k)\\ & (c,k)\end{matrix};x\right] = (1-x)^{-a}\,{}_2F_{1,k}\left[\begin{matrix}(a,1), & (c-b,k)\\ & (c,k)\end{matrix};-\frac{x}{1-x}\right].$$

Proof. In order to prove (2.1), we start with the right-hand side of (2.1), denoting it by S as

$$S = (1-x)^{-a} {}_2F_{1,k}\left[\begin{matrix}(a,1), & (c-b,k) \\ & (c,k)\end{matrix}; -\frac{x}{1-x}\right].$$

Expressing ${}_2F_{1,k}$ as a series, we have

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n (a)_{n,1} (c-b)_{n,k}\, x^n}{(c)_{n,k}\, n!} (1-x)^{-(a+n)}.$$

Using Binomial theorem, we have, after some simplification

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n (a)_{n,1} (a+n)_{m,1} (c-b)_{n,k}}{(c)_{n,k}\, m!\, n!}\, x^{m+n}.$$

Using the identity

$$(a+n)_{m,1} = \frac{(a)_{m+n,1}}{(a)_{n,1}}$$

we have

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n (c-b)_{n,k} (a)_{m+n,1}}{(c)_{n,k}\, m!\, n!}\, x^{m+n}.$$

Changing m to $m-n$ and using known result [12], we have

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(-1)^n (c-b)_{n,k}\, (a)_{m,1}}{(c)_{n,k}\, n!(m-n)!}\, x^{m+n}.$$

Using the identity

$$(m-n)! = \frac{(-1)^n\, m!}{(-m)_n},$$

we have after some algebra

$$S = \sum_{m=0}^{\infty} \frac{(a)_{m,1}}{m!}\, x^m \sum_{n=0}^{m} \frac{(-m)_{n,1}\, (c-b)_{n,k}}{(c)_{n,k}\, n!}.$$

Summing up the inner series, we have

$$S = \sum_{m=0}^{\infty} \frac{(a)_{m,1}}{m!}\, x^m\, {}_2F_{1,k}\left[\begin{matrix}(-m,1), & (c-b,k) \\ & (c,k)\end{matrix}; 1\right].$$

Now using the result (1.10), we have

$$S = \sum_{m=0}^{\infty} \frac{(a)_{m,1}\, (b)_{m,k}}{(c)_{m,k}}\, \frac{x^m}{m!}.$$

Finally summing up the series, we easily arrive at the left-hand side of (2.1). This completes the proof of our result (2.1). ∎

3. COROLLARIES

In this section, we shall mention a few known and unknown results of our main findings.

(1) In our main result (2.1), if we replace x by $\frac{x}{a}$ and taking $a \to \infty$, we get the known result (1.11) obtained by Mubeen [9].

(2) In our main result (2.1), if we take $x \to -1$, we get the following new results viz.

$$ {}_2F_{1,k}\left[\begin{matrix}(a,1), & (b,k) \\ & (c,k)\end{matrix}; -1\right] = 2^{-a}\, {}_2F_{1,k}\left[\begin{matrix}(a,1), & (c-b,k) \\ & (c,k)\end{matrix}; \frac{1}{2}\right]. \tag{3.1}$$

For $k = 1$, it reduces to the following well known and useful result recorded in [12] viz.

$$ {}_2F_1\left[\begin{matrix}a, & b \\ & c\end{matrix}; -1\right] = 2^{-a}\, {}_2F_1\left[\begin{matrix}a, & c-b \\ & c\end{matrix}; \frac{1}{2}\right]. \tag{3.2}$$

Similarly, other results can be obtained.

4. AN ALTERNATE DERIVATION OF (1.11)

In this section, we shall provide an alternate and easy derivation of the known result (1.11). For this, in (1.7), take $m = 1$, we have

$$ {}_1F_{1,k}\left[\begin{matrix}(\beta,k) \\ (\gamma,k)\end{matrix}; x\right] = \frac{\Gamma_k(\gamma)}{k\,\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1} e^{xt}\,dt. \tag{4.1}$$

Now let us make the substitution $1 - t = T$, then after some simplification, (4.1) takes the following form

$$ {}_1F_{1,k}\left[\begin{matrix}(\beta,k) \\ (\gamma,k)\end{matrix}; x\right] = e^x \frac{\Gamma_k(\gamma)}{k\,\Gamma_k(\gamma-\beta)\,\Gamma_k(\beta)} \int_0^1 T^{\frac{\gamma-\beta}{k}-1}(1-T)^{\frac{\beta}{k}-1} e^{-xT}\,dT, \tag{4.2}$$

which upon using the result (4.1) on the right-hand side, reduces to

$$ {}_1F_{1,k}\left[\begin{matrix}(\beta,k) \\ (\gamma,k)\end{matrix}; x\right] = e^x\, {}_1F_{1,k}\left[\begin{matrix}(\gamma-\beta,k) \\ (\gamma,k)\end{matrix}; -x\right] \tag{4.3}$$

which is (1.11).

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