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OSCILLATORY BEHAVIOR OF SECOND-ORDER NON-CANONICAL RETARDED DIFFERENCE EQUATIONS

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ABSTRACT. Using monotonic properties of nonoscillatory solutions, we obtain new oscillatory criteria for the second-order non-canonical difference equation with retarded argument

$$\Delta(\mu(\ell)\Delta\eta(\ell)) + \phi(\ell)\eta(\sigma(\ell)) = 0.$$

Our oscillation results improve and extend the earlier ones. Examples illustrating the results are provided.

Key words and phrases: Second-order, Non-canonical, Difference equations, Oscillation.

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1. INTRODUCTION

This paper deals with the second-order non-canonical retarded difference equations of the form

$$\Delta(\mu(\ell)\Delta\eta(\ell)) + \phi(\ell)\eta(\sigma(\ell)) = 0,\ \ell \geq \ell_0 > 0, \tag{E}$$

where ℓ_0 is a positive integer and $\ell \in \mathbb{N}(\ell_0) = \{\ell_0, \ell_0 + 1, ...,\}$,

(H_1) $\{\mu(\ell)\}$ and $\{\phi(\ell)\}$ are positive real sequences with

$$\Omega(\ell_0) = \sum_{\ell=\ell_0}^{\infty} \frac{1}{\mu(\ell)} < \infty; \tag{1.1}$$

(H_2) $\{\sigma(\ell)\}$ is a monotone increasing sequence of integers with $\sigma(\ell) \leq \ell - 1$ for $\ell \geq \ell_0$.

By a solution of (E), we mean a real sequence $\{\eta(\ell)\}$ which is defined and satisfies (E) for all $\ell \geq \ell_0$. A nontrivial solution of (E) is said to be $oscillatory$ if it is neither eventually negative nor eventually positive, otherwise, it is $nonoscillatory$. Equation (E) is called oscillatory if all its solutions are oscillatory. We say that (E) is in non-canonical form if (1.1) holds.

Oscillatory and asymptotic behavior of solutions of various types of difference equations are discussed over the past few decades. A large amount of papers and monographs have been devoted to this problem, see for example [1, 2, 4, 6, 8, 9, 18, 16, 15, 10, 11, 5, 7, 3] and the references cited therein. There is a significant difference in the structure of nonoscillatory (say positive) solutions between canonical and non-canonical equations. It is well known that the first difference of any positive solution $\{\eta(\ell)\}$ of canonical equations is of one sign eventually where as non-canonical one both sign possibilities of the first difference of any positive solution have to be treated. These type of equations are studied in the literature (see [12, 13, 17]) by extending known results for canonical equations.

Very recently in [7], the authors extended the technique of Koplatadze [8] to half-linear second order delay differential equations. The objective of this paper is to obtain new difference inequalities that lead to new monotonicity properties of solutions, which are applied to obtain new oscillatory criteria for delay difference equations in non-canonical form. The obtained results would improve and complement to those obtained for linear equations reported in the literature.

2. BASIC LEMMAS

From a discrete lemma of Kiguradze [1], the set of positive solutions of (E) has the following structure.

Lemma 2.1. *Let $\{\eta(\ell)\}$ be an eventually positive solution of (E). Then $\{\eta(\ell)\}$ satisfies one of the following conditions:*

$(S_1):\ \mu(\ell)\Delta\eta(\ell) > 0,\ \Delta(\mu(\ell)\Delta\eta(\ell)) < 0,$
$(S_*):\ \mu(\ell)\Delta\eta(\ell) < 0,\ \Delta(\mu(\ell)\Delta\eta(\ell)) < 0,$

for all $\ell \geq \ell_1 \geq \ell_0$.

Lemma 2.2. *If*

$$\sum_{\ell=\ell_0}^{\infty} \Omega(\ell+1)\phi(\ell) = \infty, \tag{2.1}$$

then the positive solution $\{\eta(\ell)\}$ of (E) satisfies (S_*), and

(i) $\lim_{\ell\to\infty} \eta(\ell) = 0;$
(i) $\eta(\ell) + \mu(\ell)\Omega(\ell)\Delta\eta(\ell) \geq 0;$

(iii) $\{\frac{\eta(\ell)}{\Omega(\ell)}\}$ *is eventually increasing.*

Proof. Let $\{\eta(\ell)\}$ be an eventually positive solution of (E) satisfying condition (S_1) for $\ell \geq \ell_1 \geq \ell_0$. Summing up (E) from ℓ_1 to ∞, we obtain

$$\mu(\ell_1)\Delta\eta(\ell_1) \geq \sum_{\ell=\ell_1}^{\infty} \phi(\ell)\eta(\sigma(\ell)).$$

Since $\{\eta(\ell)\}$ is positive and increasing, there exists a positive constant M such that $\eta(\ell) \geq M$ and $\eta(\sigma(\ell)) \geq M$ eventually. Hence, we have

$$\mu(\ell_1)\Delta\eta(\ell_1) \geq M\sum_{\ell=\ell_1}^{\infty} \phi(\ell) \geq M\sum_{\ell=\ell_1}^{\infty} \Omega(\ell+1)\phi(\ell)$$

which contradicts (2.1), and we conclude that $\{\eta(\ell)\}$ satisfies (S_*). Therefore, there exists a finite limit $\lim_{\ell\to\infty} \eta(\ell) = K$.

We claim that $K = 0$. If not, then $\eta(\ell) \geq K > 0$. Summing up (E) from ℓ_1 to $\ell - 1$, we get

$$-\mu(\ell)\Delta\eta(\ell) \geq K\sum_{s=\ell_1}^{\ell-1} \phi(s).$$

Summing up again from ℓ_1 to ∞, we get

$$\eta(\ell_1) \geq K\sum_{\ell=\ell_1}^{\infty} \frac{1}{\mu(\ell)} \sum_{s=\ell_1}^{\ell-1} \phi(s) = K\sum_{\ell=\ell_1}^{\infty} \Omega(\ell+1)\phi(\ell) = \infty.$$

This contradicts $K > 0$, which proves part (i).

To prove part (ii) we proceed as follows: The monotonicity of $\mu(\ell)\Delta\eta(\ell)$ implies that

$$\begin{aligned}\eta(\ell) &\geq \sum_{s=\ell}^{\infty} \frac{-\mu(s)\Delta\eta(s)}{\mu(s)} \geq -\mu(\ell)\Delta\eta(\ell)\sum_{s=\ell}^{\infty} \frac{1}{\mu(s)} \\ &= -\Omega(\ell)\mu(\ell)\Delta\eta(\ell),\end{aligned}$$

which proves part (ii).

Next, we verify part (iii). Indeed, in view of part(ii), we have

$$\Delta\left(\frac{\eta(\ell)}{\Omega(\ell)}\right) = \frac{\Omega(\ell)\mu(\ell)\Delta\eta(\ell) + \eta(\ell)}{\Omega(\ell)\Omega(\ell+1)} \geq 0$$

which proves part (iii). The proof of the lemma is complete. ∎

Remark 2.1. In the above results, we do not assume that $\sigma(\ell)$ is retarded or advanced argument.

3. OSCILLATION RESULTS

In this section, first we will establish new monotonic properties for solutions of (E) from the class (S_*), and then we obtain new oscillation criteria for (E).

Lemma 3.1. *Let* (2.1) *holds. Assume that there exists a* $\delta > 0$ *such that*

$$\min_{\ell\geq\ell_0} \left\{\phi(\ell)\Omega^2(\ell+1)\mu(\ell), \phi(\ell)\Omega(\ell+1)\Omega(\ell+2)\mu(\ell+1)\right\} \geq \delta \tag{3.1}$$

eventually. If $\{\eta(\ell)\}$ *is a positive solution of* (E), *then*

(i) $\{\frac{\eta(\ell)}{\Omega^\delta(\ell)}\}$ *is decreasing;*

(ii) $\lim_{\ell\to\infty} \frac{\eta(\ell)}{\Omega^\delta(\ell)} = 0$;

(iii) $\{\frac{\eta(\ell)}{\Omega^{1-\delta}(\ell)}\}$ *is increasing.*

Proof. Let $\{\eta(\ell)\}$ be an eventually positive solution of (E). Then (2.1) implies that $\{\eta(\ell)\}$ satisfies (S_*) for $\ell \geq \ell_1 \geq \ell_0$.

Summing up (E) from ℓ_1 to $\ell - 1$, we get

$$\begin{aligned} -\mu(\ell)\Delta\eta(\ell) &= -\mu(\ell_1)\Delta\eta(\ell_1) + \sum_{s=\ell_1}^{\ell-1} \phi(s)\eta(\sigma(s)) \\ &\geq -\mu(\ell_1)\Delta\eta(\ell_1) + \eta(\ell)\sum_{s=\ell_1}^{\ell-1} \phi(s), \end{aligned}$$

which, in view of (3.1), leads to

$$\begin{aligned} -\mu(\ell)\Delta\eta(\ell) &\geq -\mu(\ell_1)\Delta\eta(\ell_1) + \delta\eta(\ell)\sum_{s=\ell_1}^{\ell-1} \frac{1}{\mu(s+1)\Omega(s+1)\Omega(s+2)} \\ &= -\mu(\ell_1)\Delta\eta(\ell_1) + \delta\eta(\ell)\sum_{s=\ell_1}^{\ell-1} \Delta\left(\frac{1}{\Omega(s+2)} - \frac{1}{\Omega(s+1)}\right) \\ &= -\mu(\ell_1)\Delta\eta(\ell_1) + \delta\eta(\ell)\left(\frac{1}{\Omega(\ell+1)} - \frac{1}{\Omega(\ell_1+1)}\right) \\ &\geq \frac{\delta\eta(\ell)}{\Omega(\ell+1)}, \end{aligned} \tag{3.2}$$

where we have used $\eta(\ell) \to 0$ as $\ell \to \infty$. Hence

$$\Delta\left(\frac{\eta(\ell)}{\Omega^\delta(\ell)}\right) = \frac{\Omega^\delta(\ell)\Delta\eta(\ell) - \eta(\ell)\Delta(\Omega^\delta(\ell))}{\Omega^\delta(\ell)\Omega^\delta(\ell+1)}. \tag{3.3}$$

By Mean-value Theorem, we have

$$\Delta\left(\Omega^\delta(\ell)\right) \geq \frac{-\delta}{\mu(\ell)}\frac{\Omega^\delta(\ell)}{\Omega(\ell+1)}. \tag{3.4}$$

Using (3.4) in (3.3) and, in view of (3.2), we get

$$\Delta\left(\frac{\eta(\ell)}{\Omega^\delta(\ell)}\right) \leq \frac{[\Omega(\ell+1)\mu(\ell)\Delta\eta(\ell) + \delta\eta(\ell)]}{\mu(\ell)\Omega^{\delta+1}(\ell+1)} \leq 0.$$

That is, $\frac{\eta(\ell)}{\Omega^\delta(\ell)}$ is decreasing, and therefore there exists $\lim_{\ell\to\infty} \frac{\eta(\ell)}{\Omega^\delta(\ell)} = k \geq 0$.

We claim that $k = 0$. Indeed, if $k > 0$, then $\eta(\ell) \geq k\Omega^\delta(\ell) > 0$ eventually. Now define the auxiliary sequence

$$z(\ell) = (\mu(\ell)\Omega(\ell)\Delta\eta(\ell) + \eta(\ell))\Omega^{-\delta}(\ell).$$

Based on Lemma 2.2, it is obvious that $z(\ell) \geq 0$ and

$$\begin{aligned} \Delta z(\ell) &= \Delta(\mu(\ell)\Delta\eta(\ell))\Omega^{1-\delta}(\ell+1) + \mu(\ell)\Delta\eta(\ell)\Delta(\Omega^{1-\delta}(\ell)) \\ &\quad +\Omega^{-\delta}(\ell+1)\Delta\eta(\ell) + \eta(\ell)\Delta(\Omega^{-\delta}(\ell)). \end{aligned} \tag{3.5}$$

By Mean-value Theorem

$$\Delta(\Omega^{1-\delta}(\ell)) \geq \frac{-(1-\delta)}{\mu(\ell)}\Omega^{-\delta}(\ell+1) \tag{3.6}$$

and

$$\Delta(\Omega^{-\delta}(\ell)) \leq \frac{\delta}{\mu(\ell)}\Omega^{-1-\delta}(\ell+1). \tag{3.7}$$

Using (3.6) and (3.7) in (3.5) yields

$$\begin{aligned}
\Delta z(\ell) &\leq -\phi(\ell)\eta(\sigma(\ell))\Omega^{1-\delta}(\ell+1) + \delta\Delta\eta(\ell)\Omega^{-\delta}(\ell+1) + \delta\frac{\eta(\ell)}{\mu(\ell)}\Omega^{-1-\delta}(\ell+1) \\
&\leq \frac{-\delta\eta(\ell)\Omega^{1-\delta}(\ell+1)}{\mu(\ell)\Omega^2(\ell+1)} + \delta\Delta\eta(\ell)\Omega^{-\delta}(\ell+1) + \delta\frac{\eta(\ell)}{\mu(\ell)}\Omega^{-1-\delta}(\ell+1) \\
&= \delta\Delta\eta(\ell)\Omega^{-\delta}(\ell+1). \qquad (3.8)
\end{aligned}$$

Since $\eta(\ell) \geq k\Omega^{\delta}(\ell) \geq k\Omega^{\delta}(\ell+1)$ and using (3.2), we get from (3.8) that

$$\Delta z(\ell) \leq \frac{-k\delta^2}{\Omega(\ell+1)\mu(\ell)} < 0.$$

Summing up the last inequality from ℓ_1 to $\ell-1$, we obtain

$$\begin{aligned}
z(\ell_1) &\geq k\delta^2 \sum_{s=\ell_1}^{\ell-1} \frac{1}{\mu(s)\Omega(s+1)} \geq k\delta^2 \sum_{s=\ell_1}^{\ell-1} \int_{\Omega(s+1)}^{\Omega(s)} \frac{dv}{v} \\
&= k\delta^2 \ln\frac{\Omega(\ell_1)}{\Omega(\ell)} \to \infty \;\; as \;\; \ell \to \infty
\end{aligned}$$

which is a contradiction. Thus

$$\lim_{\ell\to\infty} \frac{\eta(\ell)}{\Omega^{\delta}(\ell)} = 0.$$

Finally, we prove part (iii). Equation (E) can be rewritten in the equivalent form

$$\Delta(\Omega(\ell)\mu(\ell)\Delta\eta(\ell) + \eta(\ell)) + \Omega(\ell+1)\phi(\ell)\eta(\sigma(\ell)) = 0. \tag{3.9}$$

Summing up (3.9) from ℓ to ∞ and taking into account the fact that $\frac{\eta(\ell)}{\Omega(\ell)}$ is increasing, we get

$$\begin{aligned}
\Omega(\ell)\mu(\ell)\Delta\eta(\ell) + \eta(\ell) &\geq \sum_{s=\ell}^{\infty} \Omega(s+1)\phi(s)\eta(\sigma(s)) \\
&\geq \sum_{s=\ell}^{\infty} \Omega(s+1)\phi(s)\eta(s) \\
&\geq \frac{\eta(\ell)}{\Omega(\ell)} \sum_{s=\ell}^{\infty} \Omega(s)\Omega(s+1)\phi(s) \\
&\geq \frac{\eta(\ell)}{\Omega(\ell)} \sum_{s=\ell}^{\infty} \Omega^2(s+1)\phi(s) \\
&\geq \frac{\eta(\ell)}{\Omega(\ell)} \sum_{s=\ell}^{\infty} \frac{\delta}{\mu(s)} = \delta\frac{\eta(\ell)}{\Omega(\ell)} \sum_{s=n}^{\infty} \Delta(-\Omega(s)) \\
&= \delta\eta(s),
\end{aligned}$$

that is

$$\Omega(\ell)\mu(\ell)\Delta\eta(\ell) + (1-\delta)\eta(\ell) \;\; \geq \;\; 0. \tag{3.10}$$

Now

$$\Delta\left(\frac{\eta(\ell)}{\Omega^{1-\delta}(\ell)}\right)=\frac{\Omega^{1-\delta}(\ell)\Delta\eta(\ell)-\eta(\ell)\Delta(\Omega^{1-\delta}(\ell))}{\Omega^{1-\delta}(\ell)\Omega^{1-\delta}(\ell+1)}. \tag{3.11}$$

By Mean-value Theorem

$$-\Delta(\Omega^{1-\delta}(\ell))\geq\frac{(1-\delta)}{\mu(\ell)}\Omega^{-\delta}(\ell). \tag{3.12}$$

Using (3.12) in (3.11) and, in view of (3.10), we obtain

$$\Delta\left(\frac{\eta(\ell)}{\Omega^{1-\delta}(\ell)}\right)\geq\frac{\Omega(\ell)\mu(\ell)\Delta\eta(\ell)+(1-\delta)\eta(\ell)}{\Omega(\ell)\mu(\ell)\Omega^{1-\delta}(\ell+1)}\geq 0.$$

The proof of the lemma is complete. ∎

Remark 3.1. Based on the above lemma, it is obvious that

$$\left\{\frac{\eta(\ell)}{\Omega^{\delta}(\ell)}\right\}\ is\ decreasing\ and\ \left\{\frac{\eta(\ell)}{\Omega^{1-\delta}(\ell)}\right\}\ is\ increasing,$$

which immediately gives the following oscillatory criterion.

Theorem 3.2. *Assume that* (2.1) *and* (3.1) *hold. If*

$$\delta>\frac{1}{2}, \tag{3.13}$$

then (E) *is oscillatory.*

Proof. Assume, for the sake of contradiction, that $\{\eta(\ell)\}$ is an eventually positive solution of (E). Then $\{\eta(\ell)\}$ satisfies either (S_1) or (S_*). In view of condition (2.1), $\{\eta(\ell)\}$ satisfies condition (S_*). From Lemma 3.1, we see that (3.2) implies

$$-\Omega(\ell)\mu(\ell)\Delta\eta(\ell)\geq-\Omega(\ell+1)\mu(\ell)\Delta\eta(\ell)\geq\delta\eta(\ell)$$

and from (3.10), we have

$$(1-\delta)\eta(\ell)\geq\delta\eta(\ell),$$

i.e.,

$$\delta\leq\frac{1}{2},$$

which contradicts (3.13). The proof of the theorem is complete. ∎

If $\delta\leq\frac{1}{2}$, then one can improve the results given in Lemma 3.1. Since $\Omega(\ell)$ is decreasing, there exists a constant $\alpha\geq 1$ such that

$$\frac{\Omega(\sigma(\ell))}{\Omega(\ell)}\geq\alpha,$$

we introduce the constant $\delta_1>\delta$ as follows

$$\delta_1=\frac{\alpha^{\delta}\delta}{1-\delta}. \tag{3.14}$$

Lemma 3.3. *Assume that* (2.1) *and* (3.1) *hold. If* $\{\eta(\ell)\}$ *is a positive solution of* (E), *then*

$$\delta_1\eta(\ell)+\Omega(\ell)\mu(\ell)\Delta\eta(\ell)\leq 0. \tag{3.15}$$

Proof. Let $\{\eta(\ell)\}$ be an eventually positive solution of (E). In view of (2.1) of Lemma 2.2, it satisfies condition (S_*) for all $\ell \geq \ell_1 \geq \ell_0$.

Summing up (E) from ℓ_1 to $\ell - 1$ and using the fact that $\{\frac{\eta(\ell)}{\Omega^\delta(\ell)}\}$ is decreasing, we get

$$\begin{aligned}
-\mu(\ell)\Delta\eta(\ell) &\geq -\mu(\ell_1)\Delta\eta(\ell_1) + \sum_{s=\ell_1}^{\ell-1} \frac{\phi(s)\eta(s)\Omega^\delta(\sigma(s))}{\Omega^\delta(s)} \\
&\geq -\mu(\ell_1)\Delta\eta(\ell_1) + \frac{\eta(\ell)}{\Omega^\delta(\ell)} \sum_{s=\ell_1}^{\ell-1} \phi(s)\alpha^\delta \Omega^\delta(s+1) \\
&\geq -\mu(\ell_1)\Delta\eta(\ell_1) + \frac{\delta\alpha^\delta\eta(\ell)}{\Omega^\delta(\ell)} \sum_{s=\ell_1}^{\ell-1} \frac{\Omega^\delta(s+1)}{\mu(s)\Omega^2(s+1)} \\
&\geq -\mu(\ell_1)\Delta\eta(\ell_1) + \frac{\delta\alpha^\delta\eta(\ell)}{\Omega^\delta(\ell)} \sum_{s=\ell_1}^{\ell-1} \int_{\Omega(s+1)}^{\Omega(s)} \frac{dv}{v^{2-\delta}} \\
&= -\mu(\ell_1)\Delta\eta(\ell_1) - \frac{\delta_1\eta(\ell)}{\Omega^\delta(\ell)}\Omega^{\delta-1}(\ell) + \delta_1\frac{\eta(\ell)}{\Omega(\ell)}.
\end{aligned}$$

Since $\frac{\eta(\ell)}{\Omega^\delta(\ell)} \to 0$ as $\ell \to \infty$, we obtain

$$-\Omega(\ell)\mu(\ell)\Delta\eta(\ell) \geq \delta_1\eta(\ell).$$

The proof of the lemma is complete. ∎

Now we are ready to present the main results of this section.

Theorem 3.4. *Assume that* (2.1), (3.1) *and* (3.14) *hold. If*

$$\liminf_{\ell\to\infty} \sum_{s=\sigma(\ell)}^{\ell-1} \phi(s)\Omega(s+1) > \frac{1-\delta_1}{e}, \tag{3.16}$$

then (E) *is oscillatory.*

Proof. Assume, for the sake of contradiction, that (E) has an eventually positive solution $\{\eta(\ell)\}$. Condition (2.1) implies that $\{\eta(\ell)\}$ satisfies condition (S_*). Consider the auxiliary sequence

$$w(\ell) = \Omega(\ell)\mu(\ell)\Delta\eta(\ell) + \eta(\ell).$$

It follows from Lemma 2.2 (ii) that $w(\ell) > 0$, and furthermore

$$\Delta w(\ell) = \Delta(\mu(\ell)\Delta\eta(\ell))\Omega(\ell+1) = -\phi(\ell)\Omega(\ell+1)\eta(\sigma(\ell)). \tag{3.17}$$

On the other hand, since $\{\frac{\eta(\ell)}{\Omega^\delta(\ell)}\}$ is decreasing, then from (3.15) we have

$$\Omega(\ell)\mu(\ell)\Delta\eta(\ell) + \delta_1\eta(\ell) \leq 0.$$

Thus

$$w(\ell) \leq (1-\delta_1)\eta(\ell).$$

Using the last inequality into (3.17), we see that $\{w(\ell)\}$ is a positive solution of

$$\Delta w(\ell) + \frac{\phi(\ell)\Omega(\ell+1)}{1-\delta_1} w(\sigma(\ell)) \leq 0. \tag{3.18}$$

This is a contradiction since by Theorem 2.1 of [14], condition (3.16) implies that (3.18) has no positive solution. The proof of the theorem is complete. ∎

Corollary 3.5. *Assume that* (2.1), (3.1) *and* (3.14) *hold. If* $\sigma(\ell) = \ell - \tau$, *where* $\tau \geq 1$ *is an integer such that*

$$\liminf_{\ell \to \infty} \sum_{s=\ell-\tau}^{\ell-1} \phi(s)\Omega(s+1) > (1-\delta_1)\left(\frac{\tau}{\tau+1}\right)^{\tau+1},$$

then (E) *is oscillatory.*

Proof. The proof follows by applying Theorem 6.1.1 of [2] instead of Theorem 2.1 of [14]. The proof of the corollary is complete. ∎

Theorem 3.6. *If*

$$\limsup_{\ell \to \infty} \left\{ \Omega(\sigma(\ell)) \sum_{s=\ell_0}^{\sigma(\ell)-1} \phi(s) + \sum_{s=\sigma(s)}^{\ell-1} \Omega(s+1)\phi(s) \right.$$
$$\left. + \frac{1}{\Omega(\sigma(\ell))} \sum_{s=\ell}^{\infty} \Omega(s+1)\phi(s)\Omega(\sigma(s)) \right\} > 1, \tag{3.19}$$

then (E) *is oscillatory.*

Proof. Assume, for the sake of contradiction, that $\{\eta(\ell)\}$ is an eventually positive solution of (E). It follows from (3.19), that there exists a constant $M > 0$ such that

$$\limsup_{\ell \to \infty} \left\{ \Omega(\sigma(\ell)) \sum_{s=\ell_0}^{\sigma(\ell)-1} \phi(s) + \sum_{s=\sigma(s)}^{\ell-1} \Omega(s+1)\phi(s) \right.$$
$$\left. + \frac{1}{\Omega(\sigma(\ell))} \sum_{s=\ell}^{\infty} \Omega(s+1)\phi(s)\Omega(\sigma(s)) \right\} \geq M. \tag{3.20}$$

We claim that (3.20) implies (2.1). Indeed, if not, then $\sum_{\ell=\ell_0}^{\infty} \Omega(\ell+1)\phi(\ell) < \infty$, which means that there exists an integer $\ell_* \geq \ell_1 \geq \ell_0$ such that

$$\sum_{\ell=\ell_*}^{\infty} \Omega(\ell+1)\phi(\ell) < \frac{M}{6}. \tag{3.21}$$

That is, for $\ell \geq \ell_0$

$$\begin{aligned}
\Omega(\sigma(\ell)) \sum_{s=\ell_1}^{\sigma(\ell)-1} \phi(s) &= \Omega(\sigma(\ell)) \sum_{s=\ell_1}^{\ell_*-1} \phi(s) + \Omega(\sigma(\ell)) \sum_{s=\ell_*}^{\sigma(\ell)-1} \phi(s) \\
&\leq \Omega(\sigma(\ell)) \sum_{s=\ell_*}^{\ell_*-1} \phi(s) + \sum_{s=\ell_*}^{\sigma(\ell)-1} \Omega(s)\phi(s) \\
&\leq \Omega(\sigma(\ell)) \sum_{s=\ell_1}^{\ell_*-1} \phi(s) + \frac{M}{6}.
\end{aligned}$$

Hence, for $\ell \geq \ell_*$

$$\sum_{s=\sigma(\ell)}^{\ell-1} \Omega(s+1)\phi(s) \leq \sum_{s=\sigma(\ell)}^{\ell-1} \Omega(s)\phi(s) \leq \frac{M}{6}.$$

On the other hand, for $\ell \geq \ell_*$

$$\frac{1}{\Omega(\sigma(\ell))}\sum_{s=\ell}^{\infty}\Omega(s+1)\phi(s)\Omega(\sigma(s)) \leq \sum_{s=\ell}^{\infty}\Omega(s+1)\phi(s) < \frac{M}{6}.$$

Considering the above inequalities, we see that

$$\limsup_{\ell\to\infty}\left\{\Omega(\sigma(\ell))\sum_{s=\ell_1}^{\sigma(\ell)-1}\phi(s) + \sum_{s=\sigma(\ell)}^{\ell-1}\Omega(s+1)\phi(s)\right.$$
$$\left. +\frac{1}{\Omega(\sigma(\ell))}\sum_{s=\ell}^{\infty}\Omega(s+1)\phi(s)\Omega(\sigma(s))\right\} \leq \frac{M}{2},$$

which contradicts (3.20), and therefore (2.1) holds. Thus $\{\eta(\ell)\}$ satisfies the conclusions of Lemma 2.2. Simple calculation shows that (E) can be rewritten as follows

$$\Delta(\Omega(\ell)\mu(\ell)\Delta\eta(\ell) + \eta(\ell)) + \Omega(\ell+1)\phi(\ell)\eta(\sigma(\ell)) = 0.$$

Summing up the last equation from ℓ to ∞, we have

$$\Omega(\ell)\mu(\ell)\Delta\eta(\ell) + \eta(\ell) \geq \sum_{s=\ell}^{\infty}\Omega(s+1)\phi(s)\eta(\sigma(s)). \tag{3.22}$$

On the other hand, summing up of (E) from ℓ_1 to $\ell - 1$, we get

$$-\mu(\ell)\Delta\eta(\ell) \geq \sum_{s=\ell_1}^{\ell-1}\phi(s)\eta(\sigma(s)). \tag{3.23}$$

Using (3.23) in (3.22), we obtain

$$\eta(\ell) \geq \Omega(\ell)\sum_{s=\ell_1}^{\ell-1}\phi(s)\eta(\sigma(s)) + \sum_{s=\ell}^{\infty}\Omega(s+1)\phi(s)\eta(\sigma(s)).$$

Hence

$$\begin{aligned}\eta(\sigma(\ell)) \quad &\geq \quad \Omega(\sigma(\ell))\sum_{s=\ell_1}^{\sigma(\ell)-1}\phi(s)\eta(\sigma(s)) + \sum_{s=\sigma(\ell)}^{\ell-1}\Omega(s+1)\phi(s)\eta(\sigma(s)) \\ &\quad + \sum_{s=\ell}^{\infty}\Omega(s+1)\phi(s)\eta(\sigma(s)). \qquad (3.24)\end{aligned}$$

Using that $\{\eta(\ell)\}$ is decreasing and $\{\eta(\ell)/\Omega(\ell)\}$ is increasing, we obtain

$$1 = \frac{\eta(\sigma(\ell))}{\eta(\sigma(\ell))} \quad \geq \quad \left\{\Omega(\sigma(\ell))\sum_{s=\ell_1}^{\sigma(\ell)-1}\phi(s) + \sum_{s=\sigma(\ell)}^{\ell-1}\Omega(s+1)\phi(s)\right.$$
$$\left. +\frac{1}{\phi(\sigma(\ell))}\sum_{s=\ell}^{\infty}\Omega(s+1)\Omega(\sigma(s))\phi(s)\right\}.$$

Taking $\limsup$ as $n \to \infty$ on both sides of the last inequality we get a contradiction. The proof of the theorem is complete. ∎

4. EXAMPLES

In this section, we illustrate the importance of the obtained results via some examples.

Example 4.1. *Consider the second-order retarded difference equation*

$$\Delta(\ell(\ell+1)\Delta\eta(\ell)) + \lambda(\ell+1)\eta(\ell-2) = 0,\ \ell \geq 1, \tag{4.1}$$

where $\lambda > 0$.

Here,$\sigma(\ell) = \ell - 2$, $\Omega(\ell) = \frac{1}{\ell}$, $\delta = \lambda$, and $\alpha = 1$.
So $\delta_1 = \frac{\lambda}{1-\lambda}$ and the condition (3.17) becomes

$$\lim_{\ell\to\infty}\inf \sum_{s=\ell-2}^{\ell-1} \lambda(s+1)\frac{1}{(s+1)} = 2\lambda > \left(\frac{1-2\lambda}{1-\lambda}\right)\left(\frac{8}{27}\right),$$

which will be satisfied if $\lambda \geq \frac{1}{7}$. Hence (4.1) is oscillatory if $\lambda > \frac{1}{2}$ by Theorem 3.2 and by Corollary 3.5 if $\lambda \geq \frac{1}{7}$.

Example 4.2. *Consider the second-order retarded difference equation*

$$\Delta(\ell(\ell+1)\Delta\eta(\ell)) + \lambda\eta([\ell/2]) = 0,\ \ell \geq 2, \tag{4.2}$$

where $[\ell/2]$ *is a greatest integer function and* $\lambda > 0$.

Here, $\sigma(\ell) = [\ell/2]$, $\Omega(\ell) = \frac{1}{\ell}$, $\delta = \frac{2}{3}\lambda$, and $\alpha = 2$.
So $\delta_1 = 2^{2/3\lambda}\frac{(2/3\lambda)}{(1-2/3\lambda)}$ and the condition (3.16) becomes

$$\begin{aligned}\lim_{\ell\to\infty}\inf \sum_{s=[\ell/2]}^{\ell-1} \lambda\left(\frac{1}{(s+1)}\right) \quad &\geq \quad \lim_{\ell\to\infty}\inf \sum_{s=[\ell/2]}^{\ell-1} \lambda = \lim_{\ell\to\infty}\frac{\lambda}{\ell}\left[\ell - [\ell/2]\right] \\ &\geq \quad \frac{\lambda}{2} > \left[1 - \frac{\frac{2}{3}\lambda 2^{\frac{2}{3}\lambda}}{(1-\frac{2}{3}\lambda)}\right]\frac{1}{e}\end{aligned}$$

and this is true for $\lambda \geq \frac{1}{2}$. Therefore by Theorem 3.4 the equation (4.2) is oscillatory if $\lambda \geq \frac{1}{2}$ and oscillatory by Theorem 3.2 if $\lambda \geq \frac{3}{4}$.

Remark 4.1. In this paper by establishing new monotonic properties of nonoscillatory solution of (E), we present new oscillation criteria for (E). Employing the results in [12, 13], we see that the solutions of (4.1) and (4.2) are either oscillatory or tend to zero asymptotically. Further both the examples are non-canonical type, the results obtained in [4, 9, 5, 7, 17, 3] cannot be applicable. So our results improve and complement to that reported in [4, 9, 12, 13, 5, 7, 17, 3].

REFERENCES

[1] R.P. AGARWAL, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.

[2] R.P. AGARWAL, M. BOHNER, S.R. GRACE and D.O'REGAN, *Discrete Oscillation Theory*, Hindawi Publ. Corp., New York, 2005.

[3] G.E. CHATZARAKIS, S.R. GRACE and I. JADLOVSKA, Oscillation theorems for certain second-order nonlinear retarded difference equations, *Math.Slovaca*, **71** (2021), pp. 871–880.

[4] S.S. CHENG, T.CH. YAN and H.J. LI, Oscillation criteria for second order difference equations, *Funk. Equvac.*, **34** (1999), pp. 233–239.

[5] H.A. EL-MORSHEDY and S.R. GRACE, Comparison theorems for second order nonlinear difference equations, *J. Math. Anal. Appl.*, **306** (2005), pp. 100–121.

[6] S.R. GRACE and B.S. LALLI, Oscillation theorems for second order delay and neutral difference equations, *Utilitas Math.*, **45** (1994), pp. 197–211.

[7] R. KANAGASABAPATHI, S. SELVARANGAM, J.R. GRAEF and E. THANDAPANI, Oscillation results using linearization of quasi-linear second order delay difference equations, *Medit J. Math.*, (2021), (to appear).

[8] R. KOPLATADZE, Oscillation of linear difference equations with deviating arguments, *Comput. Math. Appl.*, **42** (2001), pp. 477–486.

[9] Z.R. LIU, W.D. CHEN and Y.H. YU, Oscillation criteria for second order nonlinear difference equations, *Kyungpook Math. J.*, **39** (1999), pp. 127–132.

[10] P. MOHANKUMAR and A. RAMESH, An oscillatory behavior for second order delay diffference equations, *J. Comput. Math. Soc.*, **5** (2014), pp. 123–127.

[11] P. REHAK, Oscillatory criteria for second order half-linear difference equations, *J. Differ. Equations Appl.*, **7** (2001), pp. 483–505.

[12] S.H. SAKER, Oscillation of second order nonlinear delay difference equations, *Bull. Korean Math. Soc.*, **40** (2003), pp. 489–501.

[13] S.H. SAKER, Oscillation theorems for second order nonlinear delay difference equations, *Period. Math. Hunger.*, **47** (2003), pp. 201–213.

[14] Y. SHOUKAKU, On the oscillation of solutions of first-order difference equations with delay, *Comm.Math. Analysis*, **20** (2017), pp. 62–67.

[15] E. THANDAPANI and K. RAVI, Oscillation of second-order half-linear difference equations, *Appl. Math. Lett.*, **13** (2000), pp. 43–49.

[16] E. THANDAPANI, K. RAVI and J.R. GRAEF, Oscillation and comparison theorems for half-linear second-order difference equations, *Comput. Math. Appl.*, **42** (2001), pp. 953–960.

[17] Z. ZHANG and J. ZHANG, Oscillation criteria for second-order functional difference equations with “summation small” coefficient, *Comput. Math. Appl.*, **38** (1999), pp. 25–31.

[18] Z. ZHANG and BI PING, Linear oscillation of second-order nonlinear difference equations with damped term, *Comput. Math. Appl.*, **41** (2001), pp. 659–667.