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# SEMIVECTORIAL BILEVEL OPTIMIZATION ON AFFINE-FINSLER-METRIC MANIFOLDS

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ABSTRACT. A Finsler manifold is a differential manifold together with a Finsler metric, in this paper we construct a new class of Finsler metric affine mainfolds on bilevel semivectorial with optimization problems. The first steps for this purpose involve the study of bilevel optimization on affine manifolds. The bilevel programming problem can be viewed as a static version of the noncooperative, two-person game which was introduced in the context of unbalanced economic markets. Bilevel optimization is a special kind of optimization where one problem is embedded within another.

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## 1. COMPLETE AFFINE-FINSLER-METRIC MANIFOLDS

In some optimization problems [10, 11, 17, 18, 19, 20, 21, 22], one used an *affine-metric manifold* as a triple $(M, \Gamma, d)$, where $M$ is a smooth real $n$-dimensional manifold with metrizable topology induced by the metric $d$, $\Gamma$ is an affine symmetric connection on $M$, and $d$ is a distance function on $M$. The connection produces auto-parallel curves used for defining the convexity of subsets in $M$ and convexity of functions on $M$. The distance $d$ is used for introducing topological properties. Generally, one supposes that the topology of $M$ induced by the distance $d$ coincides with the manifold topology of $M$. Still not go on this way, since the paper [6] shows, in reasonable conditions on the metric $d$, that the manifold should be structured as a Finsler manifold.

In this paper we want to solve some optimization problems based on an affine-Finsler-metric structure on the basic manifold. An *affine-Finsler manifold* is a triple $(M, \Gamma, F)$, where $M$ is a smooth real $n$-dimensional manifold, $\Gamma$ is an affine symmetric connection on $M$, and $F$ is a Finsler function on the tangent bundle $TM$. An affine connection $\Gamma^h_{ij}$ in $M$ is a non-linear connection $N^h_j = y^i \Gamma^h_{ij}$ in $TM$. Such a non-linear connection is called *affine connection* (see [6], p. 211). Let $(M, \Gamma)$ be an affine manifold with the property that any two points $x, y \in M$ are joined by an auto-parallel curve. Then a Finsler structure $F$ induces two metrics: (i) the Finsler metric $d_F(x, y) = \inf_{\gamma \in C_{x,y}} \ell(\gamma)$, where $C_{x,y}$ the set of all curves which joins two points $x$, $y$, and (ii) the metric $d_{CAP}$ compatible to the connection $\Gamma$ given by $d_{CAP}(x, y) = \inf_{\gamma \in CAP} \ell(\gamma)$, where $CAP$ is the set of finite concatenations of auto-parallels (broken auto-parallels) which join the points $x, y \in M$.

We observe that

$$d_F(p, q) \leq d_{CAP}(p, q). \tag{1.1}$$

It follows that $d_{CAP}(p, q)$ is a distance on $M$ and

$$B_{CAP}(p; r) \subset B_F(p; r). \tag{1.2}$$

In the next Sections, we shall use the affine-Finsler-metric manifold

$$(M, \Gamma, F, d_{CAP}). \tag{1.3}$$

Concatenation of two curves: The curve $\gamma$ goes from $A$ to $B$, while the curve $\delta$ goes from $C$ to $D$. If one combines these curves by first going along $\gamma$ from $A$ to $B$ and then along $\delta$ from $C$ to $D$, the resulting curve from $A$ to $C$ is known as the concatenation of $\gamma$ and $\delta$ and is denoted by $\bar{\gamma} = \gamma \cup \delta$.

**Definition 1.1.** [**?**] An affine manifold $(M, \Gamma)$ is called auto-parallely complete if any auto-parallel $\gamma(t)$ starting at $p \in M$ is defined for all values of the parameter $t \in \mathbb{R}$.

**Theorem 1.1.** [1] *Let $M$ be a (Hausdorff, connected, smooth) compact $m$-manifold endowed with an affine connection $\Gamma$ and let $p \in M$. If the holonomy group Hol$_p(\Gamma)$ (regarded as a subgroup of the group $Gl(T_pM)$ of all the linear automorphisms of the tangent space $T_pM$) has compact closure, then $(M, \Gamma)$ is auto-parallely complete.*

Suppose that

$$\ell(\gamma) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} F(x(t), \dot{x}(t))dt \tag{1.4}$$

**Theorem 1.2.** *The solutions of the problem*

$$\inf_{\gamma \in CAP} \ell(\gamma) = \int_0^1 F(x(t), \dot{x}(t))dt \tag{1.5}$$

*are concatenations of auto-parallel curves whose tangent vectors satisfy*

$$\frac{\partial F}{\partial x^l} + \left(\lambda_i \frac{\partial \Gamma^i_{mn}}{\partial x^l} - 2\lambda_i \frac{\partial \Gamma^i_{ln}}{\partial x^m} + \left(\frac{\partial^2 F}{\partial \dot{x}^k \partial \dot{x}^l} + 2\lambda_i \Gamma^i_{lk}\right)\Gamma^k_{mn}\right)\dot{x}^m \dot{x}^n - \left(\frac{\partial^2 F}{\partial x^k \partial \dot{x}^l} + 2\dot{\lambda}_i \Gamma^i_{lk}\right)\dot{x}^k + \ddot{\lambda}_l = 0. \tag{1.6}$$

*Proof.* Without less the generality, suppose we have

The attached Lagrangian is

$$L = F(x(t), \dot{x}(t)) + \lambda_i(t)\left(\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)\right). \tag{1.7}$$

Consequently, the extremals must satisfy the ODE system

$$\frac{\partial L}{\partial x^l} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^l} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^l} = 0,\ \ \ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0.$$

On the other hand,

$$\frac{\partial L}{\partial x^l} = \frac{\partial F}{\partial x^l} + \lambda_i \frac{\partial \Gamma^i_{jk}}{\partial x^l}\dot{x}^j \dot{x}^k,\ \ \frac{\partial L}{\partial \dot{x}^l} = \frac{\partial F}{\partial \dot{x}^l} + 2\lambda_i \Gamma^i_{lk}\dot{x}^k,\ \ \frac{\partial L}{\partial \ddot{x}^l} = \lambda_l,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^l} = \frac{\partial^2 F}{\partial x^k \partial \dot{x}^l}\dot{x}^k + \frac{\partial^2 F}{\partial \dot{x}^k \partial \dot{x}^l}\ddot{x}^k + 2\dot{\lambda}_i \Gamma^i_{lk}\dot{x}^k + 2\lambda_i \frac{\partial \Gamma^i_{lk}}{\partial x^m}\dot{x}^m \dot{x}^k + 2\lambda_i \Gamma^i_{lk}\ddot{x}^k.$$

Writing the Euler-Lagrange ODE system and eliminating $\ddot{x}^k$, we find the condition on velocities, in the Theorem. ∎

Let us show that the completeness of the distance $d$ is equivalent to "any auto-parallel curve $\gamma(t)$ is defined for all values of the parameter $t$". This statement is similar to Hopf-Rinow Theorem in Riemannian manifolds theory (see, [5, p. 146]).

If there exists a minimizing auto-parallel $\gamma$ joining $p$ to $q$, then $d_{CAP}(p,q) = \ell(\gamma)$.

**Theorem 1.3.** *The topology induced by $d_{CAP}$ on $M$ coincides to the original topology on $M$.*

*Proof.* If $r$ is sufficiently small, the normal ball $B_r(p)$ coincides to the metric ball of radius $r$, centered at $p$. Hence, metric balls contain normal balls, and conversely. ∎

**Theorem 1.4.** *If $\exp_p$ is defined on all of $T_pM$, then for any $q \in M$, there exists an auto-parallel $\gamma$ joining $p$ to $q$ with $\ell(\gamma) = d_{CAP}(p,q)$.*

*Proof.* Let $d_{CAP}(p,q) = r$ and $B_\delta(p)$ a normal ball at $p$, with $S_\delta(p) = S$ the boundary of $B_\delta(p)$. We denote by $x_0$ a point where the continuous function $d_{CAP}(q,x),\ x \in S$, attains a minimum. Then $x_0 = exp_p\, \delta v$, where $v \in T_pM,\ |v| = 1$.

Let $\gamma(s) = exp_p\, sv$ be an auto-parallel curve. To show that $\gamma(r) = q$, we consider the equation

$$d_{CAP}(\gamma(s), q) = r - s, \tag{1.8}$$

and we introduce the set

$$A = \{s \in [0,r]\,|\,\text{with property that } 1.8 \text{ is valid}\}.$$

The set $A$ is nonempty, since the equation (1.8) is true for $s = 0$. Furthermore, the set $A \subset [0,r]$ is closed.

Let $s_0 \in A$. Let us show that if $s_0 < r$, then the equality 1.8 is true also for $s_0 + \delta'$, where $\delta'$ is sufficiently small. This implies $\sup A = r$. Since $A$ is closed, then $r \in A$, which shows that $\gamma(r) = q$.

∎

**Theorem 1.5.** *Let $(M, \Gamma, F, d)$ be a manifold, where $\Gamma$ is an affine connection, $F$ is a Finsler fundamental function and $d$ is a distance compatible to $\Gamma$ via $F$. Let $p \in M$. The following assertions are equivalent:*

*(i) any auto-parallel curve $\gamma(t)$, starting at $p$, is defined for all values of the parameter $t$;*
*(ii) $\exp_p$ is defined on all of $T_pM$;*
*(iii) the closed and bounded subsets of $M$ are compact.*
*(iv) the metric space $(M, d)$ is complete;*

*Proof.* $(iv) \Rightarrow (i)$. Suppose $M$ is not auto-parallely complete. Then some normalized (via $F$) auto-parallel $\gamma(s)$ is defined for $s < s_0$ and is not defined for $s_0$. Let $\{s_n\}$ , with $s_n < s_0$ and $\lim_{n\to\infty} s_n = s_0$. Given $\epsilon > 0$, there exists an index $n_0$ such that: if $n, m > n_0$, then $|s_n - s_m| < \epsilon$. It follows

$$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m| < \epsilon,$$

and hence $\gamma(s_n)$ is a Cauchy sequence. Then $\lim_{n\to\infty} \gamma(s_n) = p_0 \in M$, since $M$ is complete in the metric $d$.

Let $(U, \delta)$ be a totally normal neighborhood of $p_0$. Choose $n_1$ with the property: if $n, m > n_1$, then $|s_m - s_n| < \delta$ and $\gamma(s_n), \gamma(s_m) \in U$. Then there exists a unique auto-parallel $\alpha$ with the properties: its length is less than $\delta$, and joins the points $\gamma(s_n)$ and $\gamma(s_m)$. It is clear that $\alpha$ coincides to $\gamma$, wherever $\gamma$ is defined. Since $\exp_{\gamma(s_n)}$ is a diffeomorphism on $B_\delta(0)$ and $\exp_{\gamma(s_n)}(B_\delta(0)) \supset U$, the curve $\alpha$ extends $\gamma$ beyond $s_0$.

$(i) \Rightarrow (ii)$, obvious.

$(ii) \Rightarrow (iii)$. Let $A \subset M$ be closed and bounded. Since $A$ is bounded, $A \subset B$, where $B$ is a ball with center $p$ in the metric $d$. By previous Theorem, there exists a ball $\overline{B_r(0)} \subset T_pM$, such that $B \subset \exp_p \overline{B_r(0)}$. Being the continuous image of a compact set, $\exp_p \overline{B_r(0)}$ is compact. Hence, $A$ is a closed set contained in a compact set, and is therefore compact.

∎

1.1. **Finsler metric.** The function $y \to F(x, y)$ is called sub-homogeneous of degree $p$ if

$$F(x, \lambda y) \leq \lambda^p F(x, y).$$

The function $y \to F(x, y)$ is sub-homogeneous of degree $p$ iff it verifies the Euler inequality $y^i F_{y^i} \geq pF$.

Suppose the fundamental function $F(x, y)$ is non-negative, has the value zero only if $y = 0$, and is homogeneous of degree one in $y$. The homogeneity holds in particular for positive factors. Using Euler PDE, we have $y^i F_{y^i} = F$ (we abbreviate usual partial derivatives by subscripts). Repeated usual partial derivatives give

$$y^j F_{y^i y^j} = 0, \; y^k F_{y^i y^j y^k} = -F_{y^i y^j}, ...$$

Define $g_{ij}(x, y)$ using the usual partial derivatives,

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} = \frac{1}{2} (F^2)_{y^i y^j} = F F_{y^i y^j} + F_{y^i} F_{y^j}$$

and suppose $g_{ij}(x, y)$ is positive definite (the energy partial function $y \to F^2(x, y)$ is Euclidean strictly convex; Finsler metric). It follows

$$y^i g_{ij} = F F_{y^j}, \; y^i y^j g_{ij} = F^2, \; y^i \frac{\partial g_{ij}}{\partial y^k} = 0, \; y^k \frac{\partial g_{ij}}{\partial y^k} = 0.$$

The relations

$$g_{ij}(x, y) p^i p^j = F F_{y^i y^j} p^i p^j + \left(F_{y^i} p^i\right)^2$$
$$F^2 \left(F_{y^i} p^i\right)^2 = (g_{ij}(x, y) y^i p^j)^2 \leq F^2 g_{ij}(x, y) p^i p^j$$

implies that $F_{y^iy^j}(x,y)p^ip^j \geq 0$ (positive semidefinite). In other words, the partial function $y \to F(x,y)$ is Euclidean convex. Hence it satisfies

$$F(x,y) + F_{y^i}(x,y)(p-y)^i \leq F(x,p).$$

Adding Euler PDE, we obtain $F_{y^i}(x,y)p^i \leq F(x,p)$. Also, the convexity and the homogeneity of $y \to F(x,y)$ lead to triangle inequality

$$F(x,p+q) \leq F(x,p) + F(x,q),$$

with equality if and only if $p$ and $q$ are collinear. The last two inequalities permit to prove that Finsler geodesics minimize locally the distance.

Let $g_{ij} = g_{ij}(x,y)$ be the local components of Finsler metric $g(x,y)$. Denote

$$G^j = \frac{1}{2} g^{jh} \left( \frac{\partial^2 F}{\partial y^h \partial x^k} y^k - \frac{\partial F}{\partial x^h} \right),\ N_i^j = \frac{\partial G^j}{\partial y^j},\ \delta_i = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

The Finsler metric determines the Chern connection $\nabla$ of components

$$C^i_{jk} = \frac{1}{2} g^{ih} \left( \delta_k g_{jh} + \delta_j g_{kh} - \delta_h g_{jk} \right),\ i,j,k,h = 1,...,n.$$

The fundamental properties of this connection are: (i) is torsion-free, (ii) is almost compatible with the Finsler metric in the sense $g_{ij|k} = \frac{\partial g_{ij}}{\partial y^l} N^l_k$, (iii) the vector field $y^i \frac{\partial}{\partial x^i}$ is $h$-parallel, i.e., $y^i_{|k} = 0$.

Our basic manifold is in fact the *projectivized tangent bundle* $PTM$ (each tangent space to a manifold is taken to be a projective vector space). The fundamental function $F(x,y)$ defines the length of a $C^1$ curve $\gamma(t),\ t \in [a,b]$, namely

$$\ell(\gamma) = \int_a^b F(\gamma(t), \dot{\gamma}(t)) dt$$

and a functional whose $C^2$ extremals are called geodesics. In other words, a $C^2$ curve $\gamma : I \subset \mathbb{R} \to M$, with constant speed parametrization, is called geodesic if its tangent vector field $\dot{\gamma}$ is auto-parallel with respect to $\nabla$, i.e., $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. Let $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ be the lift of $\gamma$ to $PTM$. Then the equations of geodesics are

$$\frac{d^2 x^i}{dt^2} + \frac{dx^j}{dt}\frac{dx^k}{dt}\ C^i_{jk}|_{\tilde{\gamma}} = 0.$$

Since $\frac{\partial g}{\partial y}$ part disappears, it rests

$$\frac{d^2 x^i}{dt^2} + \frac{dx^j}{dt}\frac{dx^k}{dt}\ \frac{1}{2} g^{ih} \left( \frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right) |_{\tilde{\gamma}} = 0.$$

Let $\gamma(t)$ be a Finsler geodesic joining the points $\gamma(a) = p, \gamma(b) = q$. The Finsler metric $d_F$ is defined by $d_F(p,q) = \inf_\gamma \ell(\gamma)$. We do not use this metric.

## 2. COINCIDENCE BETWEEN AUTO-PARALLEL CURVES AND FINSLERIAN GEODESICS

The auto-parallel curves of $\Gamma(x)$ coincide to Finsler geodesics of $g_{ij}(x,y)$ if and only if

$$\frac{1}{2} g^{ih} \left( \frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right)(x,y) = \Gamma^i_{jk}(x)$$

or

$$\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} = 2 g_{ih} \Gamma^i_{jk}.$$

Changing $h$ with $j$, we find

$$\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^h} - \frac{\partial g_{hk}}{\partial x^j} = 2g_{ij}\Gamma^i_{hk}.$$

Adding the last two PDE relations, we get

$$\frac{\partial g_{jh}}{\partial x^k} = g_{ih}\Gamma^i_{jk} + g_{ij}\Gamma^i_{hk},$$

i.e., $g_{ij}(x,y)$ is parallel with respect to $\Gamma^i_{jk}(x)$.

2.1. **Affine-Hessian metric.** Suppose $F(x,y)$ is the fundamental function generating the Finsler structure on the manifold $(M,\Gamma)$.

The partial fundamental function $y \to F(x,y)$ and the connection $\Gamma(x)$ define the Hessian

$$(Hess_\Gamma F^2)_{ij} = \frac{\partial^2 F^2}{\partial y^i \partial y^j} - \Gamma^h_{ij}\frac{\partial F^2}{\partial y^h} = 2F(Hess_\Gamma F)_{ij} + 2\,\frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}.$$

Suppose the Hessian tensor

$$h_{ij}(x,y) = \frac{1}{2}\,(Hess_\Gamma F^2)_{ij}$$

is positive definite. By hypotheses we find

$$y^i h_{ij}(x,y) = F\,F_{y^h}\left(\delta^h_j - y^i\Gamma^h_{ij}\right),\ y^i y^j h_{ij}(x,y) = F^2 - FF_{y^h}\Gamma^h_{ij}y^i y^j > 0$$

It follows that the next relations are true:

$$h_{ij}(x,y)p^i p^j = F(Hess_\Gamma F)_{ij}p^i p^j + (F_{y^i}p^i)^2$$

$$F^2\left(F_{y^i}p^i - F_{y^h}\Gamma^h_{ij}y^i p^j\right)^2 = \left(h_{ij}(x,y)y^i p^j\right)^2 \le \left(F^2 - FF_{y^h}\Gamma^h_{ij}y^i y^j\right)h_{ij}(x,y)p^i p^j.$$

The equality (in the last relations) holds true only if the vectors $p$ and $y$ are collinear.

**Open problem** The partial function $y \to F(x,y)$ is affine convex, i.e., $(Hess_\Gamma F)_{ij}p^i p^j \ge 0$ or not?

## 3. SIGNIFICATIVE EXAMPLE

Let $\left(\mathbb{R}^2_+, \Gamma, F(x,y) = |y^1| + |y^2|\right)$ be an affine-Finsler manifold, with the tangent manifold $T\mathbb{R}^2_+ = \mathbb{R}^2_+ \times \mathbb{R}^2$. Suppose

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{2}\,\frac{\mu^1}{\mu^2 x^2},\ \Gamma^2_{12} = \Gamma^2_{21} = -\frac{1}{2}\,\frac{\mu^2}{\mu^1 x^1},\ \text{and otherwise } \Gamma^h_{ij} = 0.$$

Then the auto-parallel curves are of the form

$$x^i(t) = \lambda^i e^{\mu^i t},\ i = 1,2.$$

Also, the auto-parallel segment $\gamma : [0,1] \to \mathbb{R}^2_+$ joining the points $P = (a^1, a^2)$ and $Q = (b^1, b^2)$, i.e., $\gamma(0) = P$, $\gamma(1) = Q$ is given by

$$\gamma(t) = ((a^1)^{1-t}(b^1)^t, (a^2)^{1-t}(b^2)^t).$$

Let us define a distance $d$ on the Finsler space $\left(\mathbb{R}^2_+, \Gamma, F(x,y) = |y^1| + |y^2|\right)$, compatible to the affine structure $\Gamma$. For that we use the length of an auto-parallel curve between points $P$ and $Q$,

$$\ell(\gamma) = \int_0^1 F(\gamma(t), \dot\gamma(t)dt = \int_0^1 (|\dot x^1(t)| + |\dot x^2(t)|)dt = |b^1 - a^1| + |b^2 - a^2|.$$

Then, naturally, we define

$$d(P,Q) = |b^1 - a^1| + |b^2 - a^2|.$$

It follows easily the closed ball

$$B(P;R):\ |x^1-a^1|+|x^2-a^2|\leq R,$$

centered at the point $P=(a^1,a^2)\in\mathbb{R}^2_+$, of radius $R>0$.

In this way, it was created an affine-Finsler-metric manifold

$$\left(\mathbb{R}^2_+,\Gamma,F(x,y)=|y^1|+|y^2|,d\right).$$

**Proposition 3.1.** *Let* $F(x,y)=|y^1|+|y^2|$.

*(i) Any curve* $\gamma(t)=(x^1(t),x^2(t)),\ t\in[0,1]$, *joining the points* $(a^1,a^2),(b^1,b^2)$, *whose components* $x^1,x^2$ *are monotonic, is a Finsler geodesic.*

*(ii) The auto-parallels*

$$\gamma(t)=((a^1)^{1-t}(b^1)^t,(a^2)^{1-t}(b^2)^t)$$

*are Finsler geodesics.*

*Proof.* Indeed, for an arbitrary piecewise $C^1$ curve $\gamma(t)=(x^1(t),x^2(t)),\ t\in[0,1]$, joining the points $(a^1,a^2),(b^1,b^2)$, we have

$$\begin{aligned}\ell(\gamma)=\int_0^1|\dot{x}^1(t)|dt+\int_0^1|\dot{x}^2(t)|dt&=V_0^1x^1+V_0^1x^2\\&\geq|b^1-a^1|+|b^2-a^2|,\end{aligned}$$

where $V_0^1x$ means the variation of the function $x$ on the interval $[0,1]$. If $x$ is monotonic, then $V_0^1x=|x(1)-x(0)|$. ∎

**Example 3.1.** *Let us consider the function*

$$\varphi:M_1\times M_2\to\mathbb{R}^2,$$

$$\varphi(x,y)=(\varphi_1(x,y),\varphi_2(x,y))=(\ln^2(x^1)+y\sqrt{x^2},y\ln(x^1)-\ln x^2).$$

*The partial function* $x\to\varphi(x,y)$ *is affine convex. Indeed*

$$u(t)=\varphi_1(x(t),y)=((1-t)\ln a^1+t\ln a^2)^2+y(a^1)^{\frac{1-t}{2}}(a^2)^{\frac{t}{2}}$$

$$v(t)=\varphi_2(x(t),y)=y((1-t)\ln a^1+t\ln a^2)-((1-t)\ln b^1+t\ln b^2)$$

*verifies* $u''(t)>0$ *respectively* $v''(t)\geq0$.

Our paper is based also on some ideas in: [2] (convex mappings between Riemannian manifolds), [4] (geometric modeling in probability and statistics), [6] (arc length in metric and Finsler manifolds), [8] (applications of Hahn-Banach principle to moment and optimization problems), [14] (geodesic connectedness of semi-Riemannian manifolds), [25] (tangent and cotangent bundles), and see ([23], [24]).

## 4. THE SEMIVECTORIAL BILEVEL PROBLEM

Let $(M_1,{}^1\Gamma)$, the leader decision affine manifold, and $(M_2,{}^2\Gamma)$, the follower decision affine manifold, be two connected affine manifolds of dimension $m$ and $n$, respectively. Moreover, $(M_2,{}^2\Gamma,d)$ is supposed to be complete. Let also $f:M_1\times M_2\to\mathbb{R}$ be the leader objective function, and let $F=(F_1,...,F_r):M_1\times M_2\to\mathbb{R}^r$ be the follower multiobjective function.

Let $x\in M_1$, $y\in M_2$ be the generic points. The *weakly or properly Pareto solution set of the follower multiobjective optimization problem* is represented by the set-valued function

$$\psi:M_1\rightrightarrows M_2,\ \psi(x)=\sigma\text{-ARGMIN}^C_{y\in M_2}F(x,y).$$

We deal with two semivectorial bilevel problems:

(i) The *optimistic semivectorial bilevel problem*

$$\min_{x\in M_1} \inf_{y\in\psi(x)} f(x,y). \tag{4.1}$$

In this case, the follower cooperates with the leader; i.e., for each $x \in M_1$, the follower chooses among all its $\sigma$-Pareto solutions (his best responses) one which is the best for the leader (assuming that such a solution exists).

(ii) The *pessimistic semivectorial bilevel problem*

$$\min_{x\in M_1} \sup_{y\in\psi(x)} f(x,y). \tag{4.2}$$

In this case, there is no cooperation between the leader and the follower, and the leader expects the worst scenario; i.e., for each $x \in M_1$, the follower may choose among all its $\sigma$-Pareto solutions (his best responses) one which is unfavorable for the leader (in this case we prefer to use "sup" instead of "max").(see [3], [22], [27]).

**Example 4.1.** *Consider the bilevel programming problem*

$$\min_x \ [-y+x^2 : \ -0.5 \le x \le 0.5, \ y\in\psi(x)], \tag{4.3}$$

*where* $\psi(x) = Arg\min_y [yx^2 : -1 \leqslant y \leqslant 1]$. *Since* $\psi(x) = -1$ *for* $x \neq 0$ *and* $\psi(0) = [-1,1]$, *the unique optimistic optimal solution of the bilevel problem is* $(x,y) = (0,1)$. *The optimistic optimal function value is* $-1$.

*Now assume that the followers problem is perturbed*

$$\psi_\alpha(x) = Arg\min_y (yx^2 + \alpha y^2 : \ -1 \leqslant y \leqslant 1),$$

*for small* $\alpha > 0$. *Then,*

$$y_\alpha(x) = \begin{cases} -1 & \textit{if} \ \ x^2 > 2\alpha \\ -x^2/2\alpha & \textit{if} \ \ x^2 \le 2\alpha. \end{cases}$$

*Replacing this function into Leaders objective function gives*

$$F(x, y_\alpha(x)) = \begin{cases} x^2+1 & \textit{if} \ \ x^2 > 2\alpha \\ x^2 + x^2/2\alpha & \textit{if} \ \ x^2 \le 2\alpha. \end{cases}$$

*This function must be minimized on* $[-0.5, 0.5]$. *The unique optimal solution of this problem is* $x_\alpha = 0$, *for all* $\alpha > 0$, *with* $f(0, y_\alpha(0)) = 0$. *For* $\alpha \to 0$, *the Leaders objective function value tends to* $0$, *which is not the optimistic optimal objective function value in the original problem.*

**Example 4.2.** *Consider the bilevel programming problem*

$$\min_x \ [(x-y)^2 + x^2 : -20 \le x \le 20, \ y\in\psi(x)], \tag{4.4}$$

*where*

$$\psi(x) = Arg\min_y [xy : -x-1 \leqslant y \leqslant -x+1]$$

*or*

$$\psi(x) = \begin{cases} [-1,1] & \textit{if} \ \ x=0 \\ -x-1 & \textit{if} \ \ x>0 \\ -x+1 & \textit{if} \ \ x<0. \end{cases}$$

*Let* $F(x,y) = (x-y)^2 + x^2$. *Then the optimal solution of the Lower level problem into this function 4.4 where the solution is uniquely determined, we get*

$$F(x, y(x)) = \begin{cases} [0,1] & \textit{if} \ \ x=0 \\ (-2x-1)^2 + x^2 & \textit{if} \ \ x>0 \\ (-2x+1)^2 + x^2 & \textit{if} \ \ x<0. \end{cases}$$

*on the regions where the functions are defined. Taking infimum, for y tending to zero, we find* $\lim_{x\to 0} F(x, y(x)) = 1$.

*This can be used to confirm that* $(x^{\circ}, y^{\circ}) = (0, 0)$*, is the unique optimistic optimal solution of the problem in this example. Now, if the leader is not exactly enough in choosing his solution, then the real outcome of the problem has an objective function value above* $1$ *which is for away from the optimistic optimal value zero.*

## 5. OPTIMAL CONTROL PROBLEM

In this section we discuss an optimal control problem, for this let the function $F(.,.)$ be as defined in Section 1 of this paper, but with take $\dot{x}(.) = u(.)$, where $u(.)$ here is a piecewise continuous control function, thus with the performance index (or cost function) defined in equation 1.4, we have an optimal control problem, see [26]. Therefore by using the method of dynamic programming [26, Ch.IV], we can obtain an optimal control of our problem. This method deals with study the properties of the value function, this function of initial state defined as a minimum value of the performance index of the problem i.e., when the value function is differentiable and satisfies the partial differential equation of dynamic programming [26, Th.4.1], then we have an optimal control for the control problem.

## 6. CONCLUSION

Class of Finsler metric affine manifolds on bilevel semivectorial with optimization problems is constructed. Study the bilevel optimization on affine manifolds is the main purpose of this paper. So, we solve some optimization problems based on an affine-Finsler-metric structure on the basic manifold and the semivectorial Bilevel problem as well. Bile del optimization is special kind of optimization where one problem is embedded within another.

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