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EXISTENCE OF COMPOSITIONAL SQUARE ROOTS OF CIRCLE MAPS

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ABSTRACT. In this paper, we discuss the existence of compositional square roots of circle maps. If f and g are two maps such that $g \circ g = f$, we say that g is a compositional square root of f.

Key words and phrases: Compositional square roots; Circle maps; Periodic points; Lift map; Conjugacy; Sharkovskii's theorem.

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1. SUMMARY OF RESULTS

For interval maps, we have the following theorems:

(1) No interval map with a periodic point with period $n \geq 2$ admits a square root.
(2) There are uncountably many interval maps (up to conjugacy) with square root and uncountably many interval maps (up to conjugacy) without square root whenever the only periodic points are fixed points.

For circle maps, we have the following theorems:

(1) No circle map with a periodic point with period $n \geq 2$ admits a square root.
(2) There are uncountably many circle maps (up to conjugacy) with square root and uncountably many circle maps (up to conjugacy) without square root whenever the only periodic points are fixed points.
(3) There are uncountably many circle maps (up to conjugacy) with square root and uncountably many circle maps (up to conjugacy) without square root whenever there are no periodic points.

2. INTRODUCTION

A compositional square root (simply square root) of a map $f : X \to X$ is a map $g : X \to X$ such that $g \circ g = f$. An interval map is a continuous self map of any closed interval. If n is a positive integer, f^n always denotes the composition of f with itself n-times. A compositional n^{th} root of a map $f : X \to X$ is a map $g : X \to X$ such that $g^n = f$. An element $x \in X$ is called a fixed point of f if $f(x) = x$, and a periodic point if there is a positive integer n such that $f^n(x) = x$. The set of all fixed points of a function f is denoted by $Fix(f)$, the set of all periodic points of period 2 (i.e., $\{x \in X : f^2(x) = x,\ f(x) \neq x\}$) of a function f is denoted by $P_2(f)$ and the set of all periodic points of a function f is denoted by $P(f)$. A subset E of the domain is said to be f-invariant if $f(E) \subset E$. In the recent paper [2], the author proved that (i) every increasing interval map admits a square root, (ii) no decreasing interval map admits a square root, (iii) every interval map f can be extended to another map g on a bigger interval such that $g \circ g = f$ on the smaller interval, (iv) every piecewise linear map from $[0, 1]$ to $[0, 1]$ that interchanges 0 and 1 and the interior interval is invariant, fails to admit a square root. In this paper, we consider continuous maps on the circle and obtained some more results in the case of interval maps. We also discuss many criteria about the existence of a square root.

3. SQUARE ROOTS OF CIRCLE MAPS

There are several ways of defining the circle. The usual Euclidean circle is given by $\{(x, y) \in \mathbb{R} : x^2 + y^2 = r^2\}$, where r is the radius. Another common definition of the circle about 0 in $\mathbb{C}$ with radius r is $\{z \in \mathbb{C} : z = re^{2\pi i\theta}, \theta \in \mathbb{R}\}$. Another one is $S^1 = \mathbb{R}/\mathbb{Z}$, the real numbers modulo the integers. However, the first definition is very useful for visualization and the second one simplifies some computations and the third one is less intuitive but computationally simpler. This space can be imagine as the unit interval $[0, 1]$ identified at its endpoints. If we have two points a and b in the circle, we denote $[a, b]$ for the interval (i.e., the arc) wrapping forward from a to b. We will call a map $f : S^1 \to S^1$ orientation-preserving if, for any two points $a,\ b$ in the circle, every point in $[a, b]$ is mapped into $[f(a), f(b)]$, and we call f orientation-reversing if $[a, b]$ is mapped into $[f(b), f(a)]$. Define $\pi : \mathbb{R} \to S^1$ by $\pi(x) = x(mod\mathbb{Z})$. The map π is continuous, orientation-preserving, surjective, injective on any half-open interval of length 1, and satisfies $\pi(x + n) = \pi(x)$ for all real number x and integer n. Let $f : S^1 \to S^1$ be continuous. We denote $\mathcal{F}$ for a countable family of closed intervals in $\mathbb{R}$ such that for any interval I in the family, $f \circ \pi(I)$ is a closed, proper subinterval of S^1 and the union of all elements from the family is $\mathbb{R}$. It is possible to divide $\mathbb{R}$ into such a family $\mathcal{F}$. A continuous map $F : \mathbb{R} \to \mathbb{R}$ is called a lift of f to $\mathbb{R}$ if $\pi \circ F = f \circ \pi$. That is., lift F is a map which is semi-conjugate to f. If $F : \mathbb{R} \to \mathbb{R}$ is a lift of f to $\mathbb{R}$ then by adding any constant integer m to F, the new function $\tilde{F}$ satisfies $\pi \circ \tilde{F} = f \circ \pi$., and thus $\tilde{F}$ is a lift. Also, since $\pi(x) = \pi(y)$ if and only if x and y differ by an integer, any two lifts must differ from each other by a constant integer. Therefore, all lifts of f are equal to $F(x) + m$ for some integer m. Hence, orientation-preserving maps on the circle corresponds to increasing for maps of $\mathbb{R}$, and orientation-reversing corresponds to decreasing.

Proposition 3.1. *$F : \mathbb{R} \to \mathbb{R}$ is a lift of a continuous map $f : S^1 \to S^1$ if and only if F is continuous and there exists $d \in \mathbb{Z}$ such that, for all $x \in \mathbb{R}$, $F(x + 1) = F(x) + d$. Inductively, $F(x + m) = F(x) + dm$, for any natural number (or integer, if f is a homeomorphism) m.*

Proof. Any lift F must satisfy $F(x+1) = F(x)+d$ for any $x \in \mathbb{R}$ and for some integer d since $\pi \circ F = f \circ \pi$. Furthermore, d must be the same integer for all points since $F(x+1)$ is continuous. For the other direction, we can define $f : S^1 \to S^1$ by considering $\pi \circ F$ on $[0, 1) = \pi([0, 1))$. Since $F(0)$ and $F(1)$ differ by an integer, f will be continuous at $0 = 1 \in S^1$. ∎

The following theorem provides a relationship between the existence of square roots of circle maps and the interval maps.

Theorem 3.2. *Let $f : S^1 \to S^1$ be continuous. A lift $F : \mathbb{R} \to \mathbb{R}$ of f has a square root if and only if f has a square root.*

Proof. The map $\pi : \mathbb{R} \to S^1$ restricted to any half open interval of length 1 in $\mathbb{R}$, is a homeomorphism from that interval to S^1. Thus, if J is a closed interval which is a proper subset of S^1, then $\pi^{-1}(J)$ is a family of disjoint closed intervals in $\mathbb{R}$ of length less than 1. Hence, for each $j \in J$, if we choose a point $p \in \pi^{-1}(\{j\})$, p belongs to exactly one interval L in $\pi^{-1}(J)$ which is homeomorphic to J through the restriction of π to L. Let $f : S^1 \to S^1$ be continuous and $I_0 \in \mathcal{F}$. Suppose $0 \in I_0 \in \mathcal{F}$ and choose $p_0 \in \pi^{-1}(f(0))$. Next set $F(0) = p_0$ and let $L_0 \in \mathcal{F}$ be such that $p_0 \in L_0$. Then $\pi|_{L_0}$, the restriction of π to L_0, is a homeomorphism from L_0 to $f(\pi(I_0))$. On I_0, we set $F = \pi^{-1}|_{L_0} \circ f \circ \pi$. By continuing this process, inductively we can define F on $\mathbb{R}$. Let $\mathcal{U} = \{\mathbb{R} \setminus I : I \in \mathcal{F}\}$. Then $\mathcal{C} = \{\pi(U) : U \in \mathcal{U}\}$ is an open cover for S^1. Since S^1 is compact, $\mathcal{C}$ has a finite sub cover. Hence finitely many interval in $\mathcal{F}$ will determine the existence of square root of a continuous map $f : S^1 \to S^1$ and hence by Proposition 3.1, the theorem follows. ∎

Proposition 3.3. *Let $f : X \to X$ be continuous. Suppose there is an f-invariant set $E \subset X$ such that $f(a) = b$ and $f(b) = a$ for some $a, b \in E$ with $a \neq b$. If f has no periodic of period 2 in $E \setminus \{a, b\} \neq \emptyset$ then the equation $\phi^2 = f$ has no solution.*

Proof. Let $f : X \to X$ be continuous. Suppose there is an f-invariant set $E \subset X$ such that $f(a) = b$ and $f(b) = a$ for some $a, b \in E$ with $a \neq b$, and f has no periodic of period 2 in the non-empty set $E \setminus \{a, b\}$. Contrary to assume that $\phi : X \to X$ is a solution of $\phi^2 = f$. Then $\phi^2(E) \subset E$. Put $c := \phi(a)$. Then $c \in E$ and $f^2(c) = c$. Which implies either $c = a$ or $c = b$ or $f(c) = c$, and hence it follows that $a = b$. It is a contradiction. Hence the equation $\phi^2 = f$ has no solution. ∎

The following total order on $\mathbb{N}$ is called the Sharkovskii's ordering:
$3 \succ 5 \succ 7 \succ 9 \succ ... \succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ...$
$\succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ...$
$...2^n \succ .... \succ 2^2 \succ 2 \succ 1.$

Theorem 3.4. [3] *(Sharkovskii's Theorem) Let $m \succ n$ in the Sharkovskii's ordering. For every continuous self map of $\mathbb{R}$, if there is an m-cycle, then there is an n-cycle.*

Corollary 3.5. *No interval map with a periodic point of period $n \geq 2$ admits a square root.*

Proof. Let $f : I \to I$ be an interval map with a periodic point of period $n \geq 2$. Then by Theorem 3.4, f has a periodic point x with period 2. Consider $E = (P(f) \setminus P_2(f)) \cup \{x, f(x)\}$. Then E is f-invariant. Hence by Proposition 3.3, the proof follows. ∎

Remark 3.1. Any orientation preserving homeomorphism on S^1 has a square root, and for any orientation reversing homeomorphism on S^1 has no square root. This is because, a map $f : S^1 \to S^1$ is an orientation preserving (reversing) homeomorphism if and only if the lift $F : \mathbb{R} \to \mathbb{R}$ is an increasing (decreasing) bijection. Hence by Theorem 3.2, the remark follows.

The family $\mathcal{PER}(S^1) := \{Per(f) : f : S^1 \to S^1$ is continuous$\}$ has been completely described by Block and Coppel as follows:

Theorem 3.6. [1] *The following are equivalent for a subset S of $\mathbb{N}$:*

(1) $1 \in S \in \mathcal{PER}(S^1)$

(2) *If $n \in S$ for some $n > 1$, (at least) one of the following should hold:*

(i) *Every integer greater than n belongs to S.*

(ii) *Every integer that comes later than n in the Sharkovskii's ordering, belongs to S.*

Theorem 3.7. *Let $f : S^1 \to S^1$ be a continuous map such that $Fix(f) \neq \emptyset$. If $P(f) \neq Fix(f)$ then f has no square root.*

Proof. Let $f : S^1 \to S^1$ be continuous with $Fix(f) \neq \emptyset$. Suppose that $P(f) \neq Fix(f)$. By Theorem 3.6, if $1 \in Per(f) = S$ and $n \in S$ with $n > 1$ then either every integer that comes later than n in the Sharkovskii's ordering, belong to S or every integer greater than n belongs to S. If $n \in S$ and every integer that comes later than n in the Sharkovskii's ordering, belong to S, then f has a periodic point x of period 2. In this case, consider $\tilde{P}(f) = (P(f) \setminus P_2(f)) \cup \{x, f(x)\}$. By Proposition 3.3, the map f has no square root since $\tilde{P}(f)$ is an f-invariant set with no periodic point of period 2 other than x. If every integer greater than n belongs to S then consider $g = f^{2m}$ for some natural number m with $2m \geq n$. Let x be a periodic point of f with period $2m$. By a similar argument involved above, g has no square root. Contrary to assume that f has a square root. Which implies g has a square root. Which is a contradiction. Hence the proof. ∎

Theorem 3.8. *Let $\mathcal{F}$ be the set of all interval maps $f : I \to I$ such that $P(f) = Fix(f)$. There are uncountably many elements (up to conjugacy) in $\mathcal{F}$ with square root and uncountably many elements (up to conjugacy) in $\mathcal{F}$ without square root.*

Proof. Without loss of generality assume that $I = \mathbb{R} \cup \{-\infty, \infty\}$. Let $\{I_n := [a_n, b_n] : a_n, b_n \in \mathbb{R} \cup \{-\infty, \infty\}\}$ be a countable family of closed intervals in $\mathbb{R} \cup \{-\infty, \infty\}$ with disjoint interiors. For each $n \in \mathbb{N} \setminus \{1\}$, consider an increasing continuous bijection $f_n : I_n \to I_n$ such that $f(a_n) = a_n$, $f(b_n) = b_n$ and f_n has n-number of fixed points. For each $A \subset \mathbb{N} \setminus \{1\}$, let $f_A : I \to I$ be an increasing bijection such that the restriction of f_A to I_n is f_n for each $n \in A$ and $Fix(f) = \cup_{n \in A} Fix(f_n)$. Then f_A has a square root and $P(f_A) = Fix(f_A)$ for each $A \subset \mathbb{N} \setminus \{1\}$. Also f_A is not conjugate to f_B for distinct subsets A, B of $\mathbb{N} \setminus \{1\}$. Hence there are uncountably many elements (up to conjugacy) in $\mathcal{F}$ with a square root. Next, let $\{J_n := [a_n, b_n] : a_n, b_n \in [0, \infty]\}$ be a countable family of closed intervals in $[0, \infty]$ with disjoint interiors. For each $A \subset \mathbb{N} \setminus \{1\}$, let f_A be an increasing bijection on $[0, \infty]$ such that the restriction of f_A to J_n is f_n for each $n \in A$ and $Fix(f) = \cup_{n \in A} Fix(f_n)$. Consider a decreasing bijection $g_A : I \to I$ such that $g_A^2 = f_A$ on $[0, \infty]$. Then g_A has a no square root and

$P(g_A) = Fix(f_A)$ for each $A \subset \mathbb{N} \setminus \{1\}$. Also g_A is not conjugate to g_B for distinct subsets A, B of $\mathbb{N} \setminus \{1\}$. Hence there are uncountably many elements (up to conjugacy) in $\mathcal{F}$ without square root. ∎

Corollary 3.9. *Let $\mathcal{G}$ be the set of all continuous maps $f : S^1 \to S^1$ such that $P(f) = Fix(f) \neq \emptyset$. There are uncountably many elements (up to conjugacy) in $\mathcal{G}$ with square root and uncountably many elements (up to conjugacy) in $\mathcal{G}$ without square root.*

Proof. Let $a, b \in S^1$ and $I = [a, b]$ be a closed arc in S^1. Suppose that $f : I \to I$ is a continuous such that $f(a) = a$ and $f(b) = b$. If $g : S^1 \to S^1$ is a continuous map such that $g := f$ on I and $g(x) = x$ on $S^1 \setminus I$ then $P(g) = P(f)$. Hence by Theorem 3.8, the results follows. ∎

Theorem 3.10. *Let $\mathcal{H}$ be the set of all continuous maps $f : S^1 \to S^1$ such that $Fix(f) = \emptyset$. There are uncountably many elements (up to conjugacy) in $\mathcal{H}$ with square root and uncountably many elements (up to conjugacy) in $\mathcal{H}$ without square root.*

Proof. Every orientation preserving irrational rotation has a square root and no orientation reversing irrational rotation has a square root. Hence the proof follows by Remark 3.1. ∎

4. CONCLUSION

Studying compositional square roots of a map $f : X \to X$ and finding a criteria for the existence are very much interested in mathematics. This paper gives a criteria for continuous circle maps and it is listed in the Section 1. The same question can be asked in the case of compositional n^{th} roots and we shall address this question in sequel to this paper.

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