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AN INTEGRATION TECHNIQUE FOR EVALUATING QUADRATIC HARMONIC SUMS

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ABSTRACT. The modified Abel lemma on summation by parts has been applied in many ways recently to determine closed-form evaluations for infinite series involving generalized harmonic numbers with an upper parameter of two. We build upon such results using an integration technique that we apply to "convert" a given evaluation for such a series into an evaluation for a corresponding series involving squared harmonic numbers.

Key words and phrases: Harmonic number; Infinite series; Beta integral; Closed form; Riemann zeta function.

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1. INTRODUCTION

Series with summands that involve consecutive entries in the famous sequence $\left(H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right)$ of harmonic numbers are used in many disciplines within mathematics; in particular, research endeavors based on the development of new identities concerning such summations are of great significance within the fields of classical analysis and number theory, and in the theory of special functions. Sums involving variants and analogues of the classical harmonic numbers are also very much of importance within these disciplines, making note, in particular, of the sequence of *alternating harmonic numbers* given by expressions of the form $\overline{H}_n = 1 - \frac{1}{2} + \cdots + \frac{(-1)^{n+1}}{n}$, along with *generalized harmonic numbers* of the form $H_n^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ and *odd harmonic numbers* $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$. In this article, we apply integral identities for harmonic-type numbers to improve upon recent results on infinite series derived from an Abel-type summation lemma.

The classical result known as *Abel's lemma on summation by parts*, as formulated in 1826 by Niels Henrik Abel [1], is a widely used tool in classical analysis [10, 34]. The *modified Abel lemma on summation by parts* (cf. [8, 10, 34, 36]) is such that

$$\sum_{n=1}^{\infty} B_n \nabla A_n = \left(\lim_{m\to\infty} A_m B_{m+1}\right) - A_0 B_1 + \sum_{n=1}^{\infty} A_n \Delta B_n \tag{1.1}$$

if this limit exists and if one of the two infinite series given above converges, letting the operators ∇ and Δ be such that $\nabla\tau_n = \tau_n - \tau_{n-1}$ and $\Delta\tau_n = \tau_n - \tau_{n+1}$ for a mapping $\tau\colon \mathbb{N}_0 \to \mathbb{C}$. Many remarkable identities for infinite series involving harmonic-type numbers are proved in [8, 10, 11, 34, 36] through direct applications of this lemma. The main goal for our article is to devise systematic ways of generalizing results from [8, 10, 34, 36] using integral identities for harmonic-type numbers, inspired by identities of this form recently considered in [6, 7].

The results from [8, 34] mainly concern series involving expressions as in $H_n^{(2)}$ or $O_n^{(2)} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots + \frac{1}{(2n-1)^2}$, making note of the identity

$$O_n^{(2)} = H_{2n}^{(2)} - \frac{H_n^{(2)}}{4}. \tag{1.2}$$

However, sums involving "quadratic" harmonic-type numbers as in H_n^2 or O_n^2 are not evaluated in closed form in [8, 34]. So, keeping (1.2) in mind, we are prompted to consider the following question: Given a series evaluation involving $H_{pn}^{(2)}$ derived from the modified Abel lemma as in [8, 34], how can we use such a result to find a symbolic form for the corresponding series obtained by replacing $H_{pn}^{(2)}$ with H_{pn}^2? The main purpose of this article is to apply an integration method, as given in Section 2, to answer this question. The integration results that we use often rely on non-trivial algorithms for determining antiderivative evaluations involving the polylogarithm function $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$.

1.1. Preliminaries.

The *beta integral* refers to the integral $\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ for $\Re(x) > 0$ and for $\Re(y) > 0$. It was shown in [3] how beta-type integrals may be used to construct remarkable Ramanujan-like rational series for $\frac{1}{\pi}$ involving harmonic numbers [35]. Through ingenious applications of coefficient-extraction methods, Wang and Chu in [35] managed to successfully "split" a number of Campbell's series for $\frac{1}{\pi}$ involving factors of the form $\binom{2n}{n}^2\left(H_n^2 + H_n^{(2)}\right)$, providing evaluations for the series obtained by expanding the summands according to such factors, so as to obtain two separate series involving H_n^2 and $H_n^{(2)}$ as summand factors. In this article, we intend to make use of a somewhat similar "splitting" strategy (cf. [4]). In this regard, the following well-known identity (cf. Section 5 below) is to play a prominent role:

$$\int_0^1 kx^{k-1}\ln^2(1-x)\,dx = H_k^2 + H_k^{(2)}. \tag{1.3}$$

In particular, for a parameter p, if we have an evaluation for a series as in [8, 34] involving $H_{pn}^{(2)}$ as a summand factor, it turns out that the moment formula in (1.3) may often be used to "convert" this evaluation into an evaluation for the series obtained by replacing $H_{pn}^{(2)}$ with H_{pn}^2, as we illustrate and clarify in Section 3.

The integral identity in (1.3) is a "beta-like" identity, making note of the following generalization of the integral in (1.3) given in [38], in which a variety of results on series involving expressions as in $H_n^2 + H_n^{(2)}$ are offered:

$$\int_0^1 x^{\alpha-1}\ln^m x\ln^k(1-x)\,dx = \left.\frac{\partial^{m+k}B(\alpha,\beta)}{\partial\alpha^m\partial\beta^k}\right|_{\beta=1}. \tag{1.4}$$

It turns out that evaluations for many series of the forms

$$\sum_{n=0}^{\infty}\frac{H_n^{(2)}}{Q(n)} \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{H_{2n}^{(2)}}{Q(n)} \tag{1.5}$$

for a polynomial $Q(n)$ may be "converted" into evaluations for the corresponding series

$$\sum_{n=0}^{\infty}\frac{H_n^2}{Q(n)} \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{H_{2n}^2}{Q(n)} \tag{1.6}$$

through the "splitting" method outlined above, using (1.3). Some series as in (1.5) had been proved using the modified Abel lemma in [8, 34], but, on the other hand, it turns out that by making use of the identity whereby

$$\int_0^1 \frac{x^k\ln(x)}{1-x}\,dx = H_k^{(2)} - \frac{\pi^2}{6}, \tag{1.7}$$

we may construct alternate and simplified proofs of many such results. In this regard, alternate integral formulas for the *polygamma function* may often be used, noting that we have that $H_z^{(s)}$ must equal $\zeta(s) + \frac{(-1)^{s-1}}{(s-1)!}\psi^{(s-1)}(z+1)$ for $z \in \mathbb{C}\setminus\{-1,-2,\ldots\}$, letting ψ denote the polygamma function, i.e., so that $\psi^m(z) := \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z)$, and letting $\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}$ denote the Riemann zeta function. However, our "splitting" method, as given by using (1.3) to obtain a series evaluation involving H_{pn}^2 from a corresponding sum containing $H_{pn}^{(2)}$, often relies on non-trivial integral evaluations; furthermore, it is not, in general, clear as to how to make use of Abel-type summation lemmas in order to go about with this replacement process.

We record the following integral identities for harmonic-type numbers:

$$\begin{aligned}
H_n^{(m+1)} &= \sum_{k=1}^{n}\frac{1}{k^{m+1}} &&= \frac{(-1)^m}{m!}\int_0^1\frac{1-x^n}{1-x}\ln^m x\,dx,\\
O_n^{(m+1)} &= \sum_{k=1}^{n}\frac{1}{(2k-1)^{m+1}} &&= \frac{(-1)^m}{m!}\int_0^1\frac{1-x^{2n}}{1-x^2}\ln^m x\,dx,\\
\overline{H}_n^{(m+1)} &= \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k^{m+1}} &&= \frac{(-1)^m}{m!}\int_0^1\frac{1-(-1)^n x^n}{1+x}\ln^m x\,dx,\\
\overline{O}_n^{(m+1)} &= \sum_{k=1}^{n}\frac{(-1)^{k+1}}{(2k-1)^{m+1}} &&= \frac{(-1)^m}{m!}\int_0^1\frac{1-(-1)^n x^{2n}}{1+x^2}\ln^m x\,dx.
\end{aligned}$$

Recalling (1.2), let us record the identity whereby

$$\overline{H}_n^{(m)}\left(\frac{1}{2}\right) = 2^m \overline{O}_n^{(m)}, \tag{1.8}$$

letting $H_n^{(m)}(x) = \sum_{i=0}^{n-1} \frac{1}{(x+i)^m}$ denote the generalized harmonic number function. With regard to (1.2) and to (1.8), these kinds of identities often allow us to "convert" our results involving $H_{pn}^{(2)}$ or H_{pn}^2 into corresponding results involving expressions as in $O_{pn}^{(2)}$, O_{pn}^2, $\overline{O}_{pn}^2$, etc. For example, we record the elegant evaluation (cf. Corollary 7 in [36])

$$\sum_{n=1}^{\infty} \frac{\overline{O}_n^{(2)}}{(n+1)(n+2)} = \frac{1}{9} + \frac{2G}{3} - \frac{\pi}{18} - \frac{\ln 2}{9}$$

that we may prove through a direct application of the integration technique given in Section 2 below, letting $G := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$ denote the famous *Catalan constant*.

Motivated by the famous Basel problem, Euler investigated the problem of evaluating series of the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(m)}}{n^q}, \tag{1.9}$$

which are referred to as *Euler sums*, making note of Euler's famous recurrence relation whereby $2\sum_{n=1}^{\infty} \frac{H_n}{n^m}$ equals $(m+2)\zeta(m+1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j)$. The phrase *Euler-type sum* is often used quite broadly, e.g., in reference to series given by replacing the denominator in (1.9) with a rational function in n, or in reference to alternating variants of (1.9), etc. Non-alternating Euler-type sums are to be mainly considered in this article, and we refer the reader to [9] for identities on extended Euler sums as in $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}^{(p)}}{n^q}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \overline{H}_{kn}^{(p)}}{n^q}$.

2. A "SPLITTING" METHOD FOR EVALUATING SERIES INVOLVING H_{pn}^2

The integration technique as formulated in the following proposition gives us a systematic and powerful way of constructing proofs for new closed-form evaluations for families of series involving quadratic/squared harmonic numbers or generalized harmonic numbers with an upper parameter of 2. Using Abel-type summation lemmas to try to find identities for series involving H_{pn}^2 seems to be more "ad hoc" compared to how we may systematically apply the following result.

Proposition 2.1. *Let α and p be fixed parameters and let $r(n)$ be a complex-valued sequence such that the following properties hold: It is possible to reverse the order of summation and integration in both*

$$\sum_{n=0}^{\infty} \alpha^n \, r(n) \int_0^1 \left(\frac{x^{pn} \ln(x)}{1-x} \right) dx$$

and

$$\sum_{n=0}^{\infty} \alpha^n \, r(n) \int_0^1 \left(p \, n \, x^{pn-1} \ln^2(1-x) \right) dx,$$

and both of these infinite series are convergent. It then follows that the identity whereby

$$\sum_{n=0}^{\infty} \alpha^n \, r(n) \, H_{pn}^{(2)} = \int_0^1 \left(\sum_{n=0}^{\infty} \alpha^n \, r(n) \frac{x^{pn} \ln(x)}{1-x} \right) dx + \frac{\pi^2}{6} \sum_{n=0}^{\infty} \alpha^n \, r(n) \tag{2.1}$$

must hold, and we also must have that:

(2.2)
$$\sum_{n=0}^{\infty} \alpha^n \, r(n) \, H_{pn}^2 = \int_0^1 \left(\sum_{n=0}^{\infty} \alpha^n \, r(n) \, \left(p \, n \, x^{pn-1} \, \ln^2(1-x) \right) \right) dx - \int_0^1 \left(\sum_{n=0}^{\infty} \alpha^n \, r(n) \, \frac{x^{pn} \ln(x)}{1-x} \right) dx - \frac{\pi^2}{6} \sum_{n=0}^{\infty} \alpha^n \, r(n).$$

We see that from the commutativity assumptions given above, the desired identities follow from Abel's continuity theorem together with the integral formulas in (1.3) and (1.7), noting that the right-hand side of the last equality in the above Proposition is $\sum_{n=0}^{\infty} \alpha^n \, r(n) \left(H_{pn}^{(2)} + H_{pn}^2 \right)$ minus the initial series in the penultimate equation. It is remarkable how the splitting method, as formulated above as Proposition 2.1, is so versatile and useful, in conjunction with the Wolfram *Mathematica* `Integrate` function, in the evaluation of intractable series involving H_{pn}^2. To rigorously prove evaluations for such series, we determine antiderivatives for the last two integrands in the above Proposition, and determining the appropriate limits of the symbolic forms for such antiderivatives is usually a straightforward computational exercise; typically, if such antiderivative evaluations, as provided by *Mathematica*, are correct, then this is easily verified, but actually *determining* these evaluations, in the first place, requires, for the most part, very non-trivial algorithms concerning the polylogarithm function. On this last point, we note that *Maple 2020* is not, in general, able to evaluate the required series or antiderivatives involved in our computations. In particular, *Maple 2020* is not able to evaluate the antiderivatives for any of the following expressions that we later apply in this article: (2.5), (3.6), (4.6), (4.8), (4.9), (4.14), (4.15), (4.16), (4.18), and (4.19).

We note that: In 2015, [32], Vălean and Furdui evaluated a well-known series due to Au-Yeung using the modified Abel summation lemma together with (1.3), so much of our present work may be regarded as a non-trivial extension of [32], as we apply (1.3) to obtain many very difficult antiderivatives based on developments in the application of the modified Abel lemma subsequent to [32]; with regard to [32], we also refer the reader to Furdui and Vălean's closely related work in [13]. Apart from [32], the idea of using the moments of $\ln^2(1-x)$ for $x \in [0,1)$ in order to evaluate a sum involving $H_k^2 + H_k^{(2)}$ and to then "split" this evaluation using a separate evaluation for a corresponding series with H_k^2 or $H_k^{(2)}$ as a factor has also been considered in [31]; we also refer the reader to the very recent article [28], in which the generating function (g.f.) for the sequence of squared harmonic numbers is used to calculate some infinite series involving H_{2n}^2.

2.1. **On the non-trivial nature of our applications of the splitting method.** Inspired by the applications of the modified Abel lemma on summation by parts from [8, 10, 34, 36], we are, for the purposes of this article, mainly interested in the evaluation of sums as in

(2.3)
$$\sum_{n=1}^{\infty} \frac{\Delta(n)}{Q(n)}$$

where $\Delta(n)$ is a harmonic-type number as in $H_{pn}^{(2)}$ or H_{pn}^2 and $Q(n)$ is a product of two distinct linear polynomials, as opposed to series as in Au–Yeung's famous formula

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n} \right)^2 = \frac{17\pi^4}{360}$$

and its recent generalizations [5, 28]. On the other hand, if we set $\Delta(n)$ to be a product of harmonic numbers with distinct indices, then such series may often be easily evaluated in a straightforward way using previously known g.f. identities. For example, it is not difficult to determine a closed form for the sum of all expressions as in $H_n H_{2n}$ quotiented by $(2n+1)(2n+3)$ for all natural numbers n; in this regard, we may begin by simply applying an appropriate operator to the g.f. for the sequence of even-indexed harmonic numbers so as to provide a simple closed form for $\left(\frac{H_{2n}}{2n+1} : n \in \mathbb{N}_0\right)$, which, in turn, easily gives us an elementary closed form for

$$\left(\frac{H_{2n}}{(2n+1)(2n+3)} : n \in \mathbb{N}_0\right);$$

differentiating this resultant identity, we may easily apply the usual moment formula for the harmonic sequence to evaluate the series with which we had started. Such elementary g.f. manipulations cannot, in general, be applied to harmonic sums that we have successfully evaluated using Proposition 2.1. In particular, let us consider the difficult series

$$\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)(2n+3)} \tag{2.4}$$

that is highlighted, as below, in Section 3.2. If we evaluate

$$\sum_{n=0}^{\infty} \frac{-n\, x^{n-1} \ln(1-x) H_n}{(2n+1)(2n+3)} \tag{2.5}$$

symbolically, staring from the g.f. for the classical sequence of harmonic numbers, we obtain a very complicated expression involving terms such as

$$\frac{\mathrm{Li}_2\left(\frac{1}{2} - \frac{\sqrt{x}}{2}\right) \ln(1-x)}{x^{5/2}},$$

and state-of-the-art computer algebra system (CAS) software cannot evaluate the required antiderivative for the polylogarithmic evaluation for (2.5). So, this is very much indicative of the remarkable nature about the elegant closed-form evaluation highlighted as (3.8) below.

As had been suggested, as above, our article is organized in such a way so as to be focused on applications of Proposition 2.1 based on the kinds of series derived in [8, 10, 34, 36] from the modified Abel lemma, with a particular regard to (2.3). However, Proposition 2.1 may be applied much more generally to obtain new and non-trivial identities, and we encourage the further application of this result, motivatived by the elegant equation

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \binom{2n}{n} \frac{H_n^2}{n+1} = \pi^2 + 4\ln^2(2)$$

that we may prove in a direct way using Proposition 2.1.

3. APPLICATIONS OF RECENT RESULTS FROM CHEN AND CHEN AND FROM WANG

Let us consider some known results on harmonic sums derived from the modified Abel lemma to illustrate how such results may often be improved upon so as to produce new results via the identity in (1.3), and to determine simplified proofs of results from [8, 10, 34, 36].

3.1. **Series inspired by the work of Chen and Chen.** An explicit identity for

$$\sum_{n=1}^{\infty} \frac{O_n^{(s)}}{(n+a)(n+a+1)} \tag{3.1}$$

for $a \in \mathbb{N}_0$ and $s \in \mathbb{N}$ is introduced in [8], and proved via the modified Abel lemma that we have frequently referred to. Let us use this identity to provide an illustration as to how Proposition 2.1 may be applied to evaluate series containing squares of doubly indexed harmonic numbers (cf. [28]).

From Chen and Chen's identity for (3.1) from [8], we see that

$$\sum_{n=1}^{\infty} \frac{O_n^{(2)}}{(n+1)(n+2)} = \frac{3\pi^2 - 8\ln(2) - 4}{36}, \tag{3.2}$$

and from the identity displayed in (1.2), it is easily seen that

$$\sum_{n=1}^{\infty} \frac{H_{2n}^{(2)}}{(n+1)(n+2)} = \frac{\pi^2}{8} - \frac{2\ln(2)}{9} - \frac{13}{36}. \tag{3.3}$$

This may also be easily verified through a direct application of the identity in (1.7), and the splitting technique from Section 2 can be shown to give us that:

$$\sum_{n=1}^{\infty} \frac{H_{2n}^{2}}{(n+1)(n+2)} = \frac{\pi^2}{8} + \frac{32\ln(2)}{9} - \frac{2\ln^2(2)}{3} + \frac{25}{36}. \tag{3.4}$$

This can also be shown by starting with the g.f. for the sequence of even-indexed harmonic numbers, so as to evaluate, as an elementary function, the g.f. for the sequence of expressions of the form $\frac{H_{2n}}{n+1}$, and from the identity whereby the $(2n+3)^{\text{th}}$ moment of $\ln(1-x)$ for $x \in [0,1)$ equals $-\frac{H_{2n+4}}{2n+4}$, we may obtain the desired result, by rewriting this expression as

$$-\frac{H_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} + \frac{1}{2n+3} + \frac{1}{2n+4}}{2(n+2)}.$$

This is a much less direct approach compared to our application of the splitting method to prove (3.4). Furthermore, this alternate approach relies on a moment formula for expressions of the form $\frac{H_{mx+b}}{mx+b}$, and it is not clear as to how to apply similar methods to evaluate series as in (2.3) in the case whereby the leading coefficient of the index of the numerator is not equal to the leading coefficients for the denominator factors, as in (2.4). Also, by evaluating the g.f. for the sequence of squared harmonic numbers (cf. [28]) as

$$\frac{2\text{Li}_2\left(\frac{x}{x-1}\right) - \ln^2(1-x)}{2(x-1)},$$

we are able to symbolically evaluate the g.f. for the sequence of expressions of the form $\frac{H_n^2}{2n+1}$, but this symbolic form is so extremely complicated that it is not possible for current CAS software to evaluate the required antiderivative, in this case, to evaluate (2.4). This illustrates the truly remarkable nature of our evaluation for (2.4).

3.2. **An alternate proof for and an extension of an infinite series due to Wang.** We offer a demonstration of an application of our splitting technique to the series

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)(2n+3)} = \frac{\pi^2}{12} + 2\ln(2) - 2 \tag{3.5}$$

discovered by Wang in 2018, in [34], and highlighted as a Corollary in [34] to an identity introduced and proved in [34] using the modified Abel summation lemma. Let us produce an alternate proof of (3.5), again with the use of the polygamma identity from (1.7).

Alternate proof of Wang's series evaluation in (3.5)*:* We replace the numerator of the summand in (3.5) with $\frac{x^n \ln(x)}{1-x}$, and we make use of the Maclaurin series identity that is such that:

$$\sum_{n=1}^{\infty} \frac{\left(\frac{x^n \ln(x)}{1-x}\right)}{(2n+1)(2n+3)} = \frac{\left(\sqrt{x}(2x-3) - 3(x-1)\tanh^{-1}\left(\sqrt{x}\right)\right)\ln(x)}{6(x-1)x^{3/2}}. \tag{3.6}$$

Using *Mathematica*, we may easily verify that the antiderivative of the right-hand side of the above equality is

$$\frac{1}{6}\Big(\mathrm{Li}_2(1-x) + \frac{3}{\sqrt{x}}\Big(\sqrt{x}\,\mathrm{Li}_2(x) + \sqrt{x}(\ln(1-x)(\ln(x)+2) - 2\ln(x)) + 2(\ln(x)+2)\tanh^{-1}\left(\sqrt{x}\right)\Big)\Big),$$

which, in turn, gives us the desired result, taking $x \to 0$ and $x \to 1$. ∎

If we are able to evaluate

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)} + H_n^2}{(2n+1)(2n+3)}, \tag{3.7}$$

then, of course, such an evaluation, together with Wang's formula in (3.5) would allow us to "split" an analytic evaluation for (3.7), i.e., according to the terms of the above numerator. Following this approach, we express (3.7) as the integral of

$$\sum_{n=1}^{\infty} \frac{n x^{n-1} \ln^2(1-x)}{(2n+1)(2n+3)}$$

for $x \in (0,1)$, i.e., the integral of

$$-\frac{\ln^2(1-x)\left(3\sqrt{x} + x\tanh^{-1}\left(\sqrt{x}\right) - 3\tanh^{-1}\left(\sqrt{x}\right)\right)}{4x^{5/2}}$$

over $(0,1)$. We may easily evaluate

$$\int -\frac{\left(3\sqrt{x} - 3\tanh^{-1}\left(\sqrt{x}\right) + x\tanh^{-1}\left(\sqrt{x}\right)\right)\ln^2(1-x)}{4x^{5/2}}\,dx$$

as an elementary function; following the splitting method, this gives us the following elegant formula:

$$\boxed{\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)(2n+3)} = 2 - \frac{\pi^2}{12} - 2\ln(2) + 2\ln^2(2).} \tag{3.8}$$

4. FURTHER APPLICATIONS INSPIRED BY THE MODIFIED ABEL LEMMA

The integral identities as given in Proposition 2.1 turn out to be extremely powerful in terms of the main goal of this paper, i.e., to improve upon the results from [8, 10, 34, 36] using integral identities as in Section 1.1.

4.1. **Generalizations of Wang's series.** Adopting notation from [34], we let *generalized harmonic numbers* be defined so that

$$h_0^{(m)}(a,b) = 0 \quad \text{and} \quad h_n^{(m)}(a,b) = \sum_{k=1}^{n} \frac{1}{(ak-a+b)^m}.$$

The modified Abel lemma was used directly in [34] to prove that

$$\sum_{k=1}^{\infty} \frac{a h_k^{(2)}(a,b)}{(ak+b)(ak-a+b)} = \sum_{k=0}^{\infty} \frac{1}{(ak+b)^3}, \tag{4.1}$$

but it would not, in general, be clear as to how to use Abel-type summation lemmas if we were to sum over products of $h_k^{(2)}(a,b)$ with rational functions apart from that displayed in the initial summand in (4.1). As an immediate consequence of the above identity, we have that

$$\sum_{n=1}^{\infty} \frac{O_n^{(2)}}{(2n-1)(2n+1)} = \frac{7\zeta(3)}{16}, \tag{4.2}$$

an evaluation that is also proved in [34] in a direct way via the Abel-type lemma in (1.1); recalling the identity in (1.2), we rewrite this as

$$\sum_{n=1}^{\infty} \frac{H_{2n}^{(2)} - \frac{H_n^{(2)}}{4}}{(2n-1)(2n+1)} = \frac{7\zeta(3)}{16}. \tag{4.3}$$

Applying reindexing to (3.5), we may "split" the above evaluation according to the numerator of the above summand.

Example 4.1. *We claim that both*

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+z_1)(2n+z_2)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n^2}{(2n+z_1)(2n+z_2)}$$

admit closed-form evaluations that may be determined with our master integral identity, for distinct odd integers z_1 and z_2. For $z_1 = 1$ and $z_2 = 3$, we recall Wang's series in (3.5)*, along with our splitting technique applied to this series, giving us* (3.8)*. As a natural generalization of these results, let us consider the case whereby $|z_1 - z_2| > 2$. We restrict our attention to the case whereby $z_1, z_2 \in \mathbb{N}$, as reindexing arguments may be easily applied to our results for $z_1, z_2 > 0$. Through a direct application of our splitting method, we can show that*

$$\sum_{n=0}^{\infty} \frac{H_n^{(2)}}{(2n+1)(2n+5)} = \frac{\pi^2}{18} + \frac{10\ln(2)}{9} - \frac{31}{27}$$

and that

$$\sum_{n=0}^{\infty} \frac{H_n^2}{(2n+1)(2n+5)} = \frac{4\ln^2(2)}{3} - \frac{16\ln(2)}{9} - \frac{\pi^2}{18} + \frac{49}{27}.$$

Example 4.2. *The identity whereby*

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)(2n+2m+1)} =$$

$$\frac{\pi^2}{12m} O_m + \frac{2\ln 2}{m} O_m^{(2)} - \frac{2}{m} \sum_{k=1}^{m} \frac{O_k}{(2k-1)^2} \tag{4.4}$$

for $m \in \mathbb{N}$ may be proved using Theorem 5.1 from [8]. *Thanks to our splitting technique, this allows us to evaluate the series given by replacing $H_n^{(2)}$ with H_n^2 in the above summand:*

$$\sum_{n=0}^{\infty} \frac{H_n^2}{(2n+1)(2n+7)} = \frac{46\ln^2(2)}{45} - \frac{1058\ln(2)}{675} - \frac{23\pi^2}{540} + \frac{16564}{10125},$$

$$\sum_{n=0}^{\infty} \frac{H_n^2}{(2n+1)(2n+9)} = \frac{88\ln^2(2)}{105} - \frac{15488\ln(2)}{11025} - \frac{11\pi^2}{315} + \frac{1729753}{1157625},$$

$$\sum_{n=0}^{\infty} \frac{H_n^2}{(2n+1)(2n+11)} = \frac{1126\ln^2(2)}{1575} - \frac{633938\ln(2)}{496125} - \frac{563\pi^2}{18900} + \frac{215844634}{156279375}.$$

Through a direct application of our splitting integration method as applied to the series in (4.4) for an arbitrary parameter $m \in \mathbb{N}$, we have that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)(2n+2m+1)} \tag{4.5}$$

must be equal to the following, letting Φ denote the Hurwitz–Lerch transcendent:

$$\int_0^1 \frac{\left((2m+1)\sqrt{x}\,\Phi\left(x,1,m+\frac{1}{2}\right) - 2\tanh^{-1}\left(\sqrt{x}\right)\right)\ln^2(1-x)}{8mx^{3/2}}\,dx - \frac{\pi^2 O_m}{12m} - \frac{2\ln(2)\,O_m^{(2)}}{m} + \frac{2}{m}\sum_{k=1}^{m}\frac{O_k}{(2k-1)^2}.$$

This identity is powerful enough, on its own, to be applied directly to evaluate series as in (4.5) for $m \in \mathbb{N}$.

Example 4.3. *Through a direct application of the above identity for* (4.5)*, by setting $m = 6$, we find that*

$$\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)(2n+13)}$$

evaluates as

$$\frac{6508\ln^2(2)}{10395} - \frac{42354064\ln(2)}{36018675} - \frac{1627\pi^2}{62370} + \frac{160848964597}{124804708875}.$$

Again with regard to Wang's series as given in (4.2) and (4.3), let us now consider generalizing this series by setting the index parameter p in Proposition 2.1 to be equal to 2.

Example 4.4. *We highlight the evaluation whereby*

$$\sum_{n=0}^{\infty} \frac{H_{2n}^{(2)}}{(2n+1)(2n+3)} = \frac{5\pi^2}{96} + \frac{\ln(2)}{2} - \frac{5}{8},$$

which, according to our splitting method, comes from indefinitely integrating the following, setting $\alpha = 1$, and again letting $p = 2$, and writing $r(n)$ in place of the rational function $\frac{1}{(2n+1)(2n+3)}$:

$$\sum_{n=0}^{\infty} \alpha^n \, r(n) \left(\frac{x^{pn} \ln(x)}{1-x} \right) = \tag{4.6}$$
$$-\frac{\ln(x)\left(x^2 \tanh^{-1}(x) + x - \tanh^{-1}(x)\right)}{2(x-1)x^3}.$$

As for our evaluation whereby

$$\sum_{n=0}^{\infty} \frac{H_{2n}^2}{(2n+1)(2n+3)} = \frac{5\pi^2}{96} + \frac{\ln^2(2)}{2} - \ln(2) + \frac{9}{8},$$

this requires, following our splitting technique, the determination of an antiderivative for the following, which is non-trivial:

$$-\frac{\ln^2(1-x)\left(x^2 \tanh^{-1}(x) + 3x - 3\tanh^{-1}(x)\right)}{2x^4}.$$

Example 4.5. *Recalling Example 4.1, we also find that both*

$$\sum_{n=1}^{\infty} \frac{H_{2n}^{(2)}}{(2n+z_1)(2n+z_2)} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{H_{2n}^2}{(2n+z_1)(2n+z_2)} \tag{4.7}$$

admit closed-form evaluations according to our splitting method, for distinct odd elements z_1 and z_2 of $\mathbb{Z}$. For the sake of brevity, we record the following two evaluations:

$$\sum_{n=0}^{\infty} \frac{H_{2n}^{(2)}}{(2n+1)(2n+5)} = \frac{43\pi^2}{1152} + \frac{5\ln(2)}{18} - \frac{10}{27},$$
$$\sum_{n=0}^{\infty} \frac{H_{2n}^2}{(2n+1)(2n+5)} = \frac{43\pi^2}{1152} + \frac{\ln^2(2)}{3} - \frac{31\ln(2)}{36} + \frac{481}{432}.$$

4.2. **A series involving $H_{4n}^{(2)}$.** Our integration methods have led us to discover a new family of series for Catalan's constant $G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ that naturally extend Wang's series from (4.3). To begin with, we highlight the following evaluation as a Corollary to Proposition 2.1, as this evaluation is of interest in its own right: Harmonic-type numbers of the form $H_{pn}^{(m)}$ for $p > 2$ are not considered in [8, 10, 34, 36], and the simple summand and series evaluation given below are in stark contrast to the very non-trivial and complicated antiderivative required in our proof. We also make note of the below evaluation relating the following fundamentally important mathematical constants in an elegant way, again with regard to the simplicity of the below summand: Catalan's constant, the Basel constant $\zeta(2) = \frac{\pi^2}{6}$, Archimedes' constant π, and the natural logarithm of 2.

Corollary 4.1. *The symbolic evaluation*

$$\sum_{n=0}^{\infty} \frac{H_{4n}^{(2)}}{(2n+1)(2n+3)} = -\frac{G}{15} + \frac{89\pi^2}{1920} + \frac{2\pi}{225} + \frac{157\ln(2)}{1800} - \frac{1669}{7200}$$

holds true.

Proof. Using our notation from Proposition 2.1, we set $\alpha = 1$ and $p = 4$, and we let the rational function $r(n)$ be equal to $\frac{1}{(2n+1)(2n+3)}$. With this set-up, we have that:

$$\sum_{n=0}^{\infty} \alpha^n \, r(n) \left(\frac{x^{pn} \ln(x)}{1-x} \right) = - \frac{\ln(x) \left(x^2 - \tanh^{-1}\left(x^2\right) + x^4 \tanh^{-1}\left(x^2\right) \right)}{2(x-1)x^6}. \tag{4.8}$$

We may verify the following antiderivative evaluation for the right-hand side of the above equality, i.e., by differentiating the following *Mathematica* output and simplifying.

```
(-288/x^3 - 450/x^2 - 800/x + 512*ArcTan[x] + (288*ArcTanh[x^2])/x^5 +
(450*ArcTanh[x^2])/x^4 + (800*ArcTanh[x^2])/x^3 + (1800*ArcTanh[x^2])/x^2 +
544*Log[1 - x] - 3600*Log[x] - (1440*Log[x])/x^3 - (1800*Log[x])/x^2 -
(2400*Log[x])/x + 960*ArcTan[x]*Log[x] + (1440*ArcTanh[x^2]*Log[x])/x^5 +
(1800*ArcTanh[x^2]*Log[x])/x^4 + (2400*ArcTanh[x^2]*Log[x])/x^3 + (3600*
ArcTanh[x^2]*Log[x])/x^2 + 2820*Log[1 - x]*Log[x] - 544*Log[1 + x] - 1020*
Log[x]*Log[1 + x] + 225*Log[1 - x^2] - 225*Log[1 + x^2] - 900*Log[x]*
Log[1 + x^2] + 900*Log[1 - x^4] + 1800*Log[x]*Log[1 - x^4] + 7200*
PolyLog[2, 1 - x] - 1020*PolyLog[2, -x] - (480*I)*PolyLog[2, (-I)*x] +
(480*I)*PolyLog[2, I*x] + 2820*PolyLog[2, x] - 450*PolyLog[2, -x^2] + 450*
PolyLog[2, x^4])/14400
```

Computing the required limits for $x \to 0$ and for $x \to 1$, we obtain the desired result, according to Proposition 2.1. ∎

We note that our splitting method cannot be applied directly with α, p, and $r(n)$ as in the above proof, at least with current versions of *Mathematica*, since this would require the evaluation of the intractable integral

$$- \int \frac{\ln^2(1-x) \left(3x^2 - 3\tanh^{-1}\left(x^2\right) + x^4 \tanh^{-1}\left(x^2\right) \right)}{x^7} \, dx, \tag{4.9}$$

which state-of-the-art CAS software cannot manage to compute. Furthermore, although we may mimic the above proof to show that

$$\sum_{n=0}^{\infty} \frac{H_{4n}^{(2)}}{(2n+1)(2n+5)} = -\frac{13G}{315} + \frac{5473\pi^2}{161280} + \frac{491\pi}{99225} + \frac{36881\ln(2)}{793800} - \frac{42058}{297675},$$

if we replace the denominator factor $(2n+5)$ with $(2n+7)$, we again encounter an indefinite integral that *Mathematica* cannot evaluate, which makes Corollary 4.1 all the more remarkable.

4.3. **A bisection argument.** We return to the problem of evaluating the series obtained by replacing $H_{4n}^{(2)}$ with H_{4n}^2 in Corollary 4.1. Since it is far from clear as to how it may be possible to compute the integral in (4.9), we intend to devise an alternative approach to applying our splitting method in order to evaluate the companion

$$\sum_{n=0}^{\infty} \frac{H_{4n}^2}{(2n+1)(2n+3)} \tag{4.10}$$

to the series in the aforementioned Corollary. In particular, if we are able to evaluate

$$\sum_{n=0}^{\infty} \frac{H_{2n}^2}{(n+1)(n+3)} \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \frac{H_{2n}^2}{(n+1)(n+3)} \tag{4.11}$$

separately, then this immediately gives us an evaluation for (4.10), i.e., simply by adding the two series in (4.11) and writing this sum as a single series, and then bisecting this resultant series. Considering the "elusive" nature of the series in (4.10), as given by how recalcitrant the corresponding integral in (4.9) is, we highlight the evaluation below as a Corollary.

Corollary 4.2. *The symbolic evaluation whereby*

$$\sum_{n=0}^{\infty} \frac{H_{4n}^2}{(2n+1)(2n+3)}$$

equals

$$-\frac{G}{15} - \frac{\pi^2}{640} + \frac{247\pi}{900} + \frac{\pi\ln(2)}{60} - \frac{3\ln^2(2)}{40} + \frac{107\ln(2)}{450} + \frac{5929}{7200}$$

must hold.

Proof. Setting $\alpha = 1$ and $p = 2$ and $r(n) = \frac{1}{(n+1)(n+3)}$ in Proposition 2.1, we find that

$$\sum_{n=0}^{\infty} \alpha^n \, r(n) \left(\frac{x^{pn}\ln(x)}{1-x} \right)$$

is equal to

$$\frac{\ln(x)\left(-x^4 - 2x^2 - 2\ln\left(1-x^2\right) + 2x^4\ln\left(1-x^2\right)\right)}{4(x-1)x^6},$$

and, applying $\int \cdot \, dx$ to the right-hand side, we may verify the following verbatim *Mathematica* output.

```
-(144*x^2 + 225*x^3 + 472*x^4 - 1669*x^5*Log[1 - x] + 720*x^2*Log[x] + 900*
x^3*Log[x] + 1560*x^4*Log[x] + 2250*x^5*Log[x] - 581*x^5*Log[1 + x] - 780*
x^5*Log[x]*Log[1 + x] + 144*Log[1 - x^2] + 225*x*Log[1 - x^2] + 400*x^2*
Log[1 - x^2] + 900*x^3*Log[1 - x^2] + 720*Log[x]*Log[1 - x^2] + 900*x*
Log[x]*Log[1 - x^2] + 1200*x^2*Log[x]*Log[1 - x^2] + 1800*x^3*Log[x]*
Log[1 - x^2] - 780*x^5*PolyLog[2, 1 - x] - 780*x^5*PolyLog[2, -x])/(7200*
x^5)
```

According to our splitting method, this can be shown to give us that:

$$\sum_{n=0}^{\infty} \frac{H_{2n}^{(2)}}{(n+1)(n+3)} = \frac{47\pi^2}{480} - \frac{34\ln(2)}{225} - \frac{3707}{14400}.$$

We leave it to the reader to verify the remaining computations required to compute our series in (4.11) according to our splitting technique, which gives us that

$$\sum_{n=0}^{\infty} \frac{H_{2n}^2}{(n+1)(n+3)} = \frac{47\pi^2}{480} - \frac{8\ln^2(2)}{15} + \frac{679\ln(2)}{225} + \frac{12587}{14400}.$$

The first part of Proposition 2.1 can be shown to give us that

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}^{(2)}}{(n+1)(n+3)} = -\frac{2G}{15} - \frac{\pi^2}{192} + \frac{4\pi}{225} + \frac{293\ln(2)}{900} - \frac{2969}{14400},$$

and, again by our splitting method, the above evaluation can be used to give us that:

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}^2}{(n+1)(n+3)}$$

equals

$$-\frac{2G}{15}-\frac{97\pi^2}{960}+\frac{247\pi}{450}+\frac{\pi\ln(2)}{30}+\frac{23\ln^2(2)}{60}-\frac{572\ln(2)}{225}+\frac{11129}{14400}.$$

This gives us the desired result, according to the bisection approach indicated above. ∎

We claim that all series of the forms

$$\sum_{n=0}^{\infty}\frac{H_{4n}^{(2)}}{(2n+z_1)(2n+z_2)} \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{H_{4n}^{2}}{(2n+z_1)(2n+z_2)}$$

must admit closed-forms involving Catalan's constant, for odd integers z_1 and z_2. From the bisection scheme employed in our proof of Corollary 4.2, this motivates the study of sums as in

$$\sum_{n=0}^{\infty}\frac{H_{pn}^{(2)}}{(n+z_1)(n+z_2)} \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{(-1)^n H_{pn}^{(2)}}{(n+z_1)(n+z_2)}$$

and as in

$$\sum_{n=0}^{\infty}\frac{H_{pn}^{2}}{(n+z_1)(n+z_2)} \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{(-1)^n H_{pn}^{2}}{(n+z_1)(n+z_2)} \tag{4.12}$$

for $z_1, z_2 \in \mathbb{N}$. However, even for the seemingly manageable case whereby $z_1 = 1$, $z_2 = 2$, and $p = 2$, the above alternating sum in (4.12) proves to be difficult, with *Mathematica* unable to evaluate the required antiderivative, in this case.

4.4. **Polynomial denominators of arbitrary degree.** Let us consider summations of the following form:

$$A(k) := \sum_{n=0}^{\infty}\frac{H_n^{(2)}}{(2n+1)(2n+3)\cdots(2n+2k-1)},$$

where $k \geq 2$ is an integer. We consider two different ways of investigating series of this form: We may either use an integral formula for the numerator, or we may apply partial fraction decomposition to the rational function factor in the above summand. This former approach gives us that

$$A(k) = \frac{k-\frac{1}{2}}{(k-1)(2k-1)!!}\frac{\pi^2}{6}+\frac{1}{(2k-1)!!}\int_0^1\frac{\ln x}{1-x}\,{}_2F_1\!\left[\begin{matrix}1,\frac{1}{2}\\ k+\frac{1}{2}\end{matrix}\,\middle|\,x\right]dx,$$

where ${}_2F_1$ is the ordinary hypergeometric function and $(2k-1)!! = 1\cdot 3\cdot 5\cdots(2k-1)$. Alternatively, the latter approach considered above necessarily gives us that:

$$\begin{aligned} A(k) &= \frac{\pi^2}{12(k-1)(2k-3)!!}+\frac{2\ln 2 O_{k-1}}{(k-1)(2k-3)!!} \\ &\quad -\frac{1}{2^k(k-1)!}\sum_{r=0}^{k-2}\binom{k-2}{r}(-1)^r O_{r+1}. \end{aligned}$$

We find that our master integral identity is strong enough to be able to directly evaluate series as in

$$\sum_{n=0}^{\infty}\frac{H_{pn}^{(2)}}{(n+1)(n+2)\cdots(n+k)}$$

and

$$\sum_{n=0}^{\infty}\frac{H_{pn}^{2}}{(n+1)(n+2)\cdots(n+k)},$$

and many similar expressions.

Example 4.6. *Through a direct application of our splitting method, we find that the equalities whereby*

$$\sum_{n=0}^{\infty} \frac{H_{2n}^{(2)}}{(n+1)(n+2)(n+3)} = \frac{13\pi^2}{480} - \frac{16\ln(2)}{225} - \frac{1493}{14400}$$

and

$$\sum_{n=0}^{\infty} \frac{H_{2n}^{2}}{(n+1)(n+2)(n+3)} = \frac{13\pi^2}{480} + \frac{121\ln(2)}{225} - \frac{2\ln^2(2)}{15} - \frac{2587}{14400}$$

both hold.

4.5. **Denominator factors with distinct leading coefficients.** Let $\overline{O}_n = 1 - \frac{1}{3} + \cdots + \frac{(-1)^{n+1}}{2n-1}$ denote the *alternating odd harmonic number* of a given order $n \in \mathbb{N}$. Quite recently, Sofo and Nimbran, in [26], made use of the moments of expressions as in $\ln^q x \ln^p(1+\delta x)$ for $x \in (0,1]$ to obtain some new Euler-like sum identities, including:

$$\sum_{n=1}^{\infty} \frac{\overline{O}_{2n-1}}{(2n-1)(4n-3)} = \frac{G}{2} + \frac{9\zeta(2)}{16} - \frac{\pi\ln 2}{8},$$
$$\sum_{n=1}^{\infty} \frac{\overline{O}_{2n-1}}{(2n-1)(4n-1)} = \frac{G}{2} - \frac{3\zeta(2)}{16} + \frac{\pi\ln 2}{8}.$$

This inspires us to consider the case whereby we set the function $r(n)$ in Proposition 2.1 to be equal to the reciprocal of a product of linear polynomials with distinct leading coefficients. Applying our splitting technique, this leads us to non-trivial results, as suggested below.

Example 4.7. *Let us highlight the interesting evaluations whereby*

$$\sum_{n=0}^{\infty} \frac{H_n^{(2)}}{(n+1)(2n+1)} = \frac{\pi^2\ln(2)}{3} - \frac{3\zeta(3)}{2}$$

and

$$\sum_{n=0}^{\infty} \frac{H_n^{2}}{(n+1)(2n+1)} = \frac{5\zeta(3)}{2} - \frac{\pi^2\ln(2)}{3} + \frac{8\ln^3(2)}{3}. \tag{4.13}$$

The evaluation of this former series requires, according to our splitting method, the symbolic evaluation of the antiderivative for

$$-\frac{\ln(x)\left(\ln(1-x) + 2\sqrt{x}\tanh^{-1}\left(\sqrt{x}\right)\right)}{(x-1)x}, \tag{4.14}$$

giving us a very complicated expression involving polylogarithmic expressions such as

$$\mathrm{Li}_3\left(e^{2\tanh^{-1}(\sqrt{x})}\right),$$

and similarly for (4.13).

We note that an infinite series of the form in (2.2), up to an index shift, with a rational function factor given by a linear polynomial quotiented by a quartic denominator, was evaluated by Michael Vowe in [33] via a proof mainly oriented around telescoping arguments, whereas Chu proved the same result in [10], highlighted as a Corollary to an application of the modified Abel lemma that also involves telescoping; as it turns out, the splitting method upon which our article is based may be applied directly to obtain the same result, which further emphasizes the versatility of Proposition 2.1, and how the integration technique given by this Proposition may be used in a very systematic way, with regard to sums as in (2.2).

4.6. **Cubic harmonic sums.** In general, evaluating series involving cubic harmonic numbers is much more difficult compared to the evaluation of series involving H_n^2 as a factor in the summand for $n \in \mathbb{N}$. We note that a family of cubic harmonic sums is evaluated in [16] using a somewhat similar approach compared to our splitting method, i.e., by using series involving $H_n H_n^{(2)}$ together with integral identities in order to evaluate Euler-type sums involving H_n^3. However, our splitting method may be applied to generalize results from [16]. For example, by indefinitely integrating

$$-\frac{\big(2x+(x-1)\ln^2(1-x)-2(x-1)\ln(1-x)\big)\ln(x)}{2(x-1)x^2} \tag{4.15}$$

and

$$-\frac{(\ln(1-x)-2)\ln^2(1-x)((x-2)\ln(1-x)-2x)}{2x^3}, \tag{4.16}$$

a direct application of our splitting method gives us that

$$\sum_{n=0}^{\infty} \frac{H_n^3}{(n+1)(n+2)} = 4\zeta(3) + 1 + \frac{\pi^2}{3},$$

and an identity for a family of sums with quadratic denominators generalizing this result is proved in [16, p. 430], but our integration method may be applied to much more general cubic harmonic sums, as below:

$$\zeta(3) - \frac{23}{32} = \sum_{n=0}^{\infty} \frac{H_n^3}{(n+1)(n+2)(n+3)}.$$

We note that applications of the modified Abel lemma as given in [8] (see Theorem 3.1) may also be applied to prove the above evaluations involving Apéry's constant.

We also refer the interested reader to the following references concerning cubic harmonic sums: [10, 17, 18, 30, 36]. A particular source of motivation in the application of our methods comes from the field of number theory, making a particular note of the reference [2], in which the closed-form evaluation

$$\zeta(3) = \frac{1}{9} \sum_{n=1}^{\infty} \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{2^n} \tag{4.17}$$

is highlighted as a main result and proved using integral identities for Stirling numbers. As it turns out, our splitting method may be applied in a direct way to greatly improve upon such results, e.g., by separately evaluating the series obtained by expanding the above summand. There actually is already a known evaluation for the g.f. for cubed harmonic numbers, and *Mathematica* is actually able to evaluate

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{2^n},$$

but we can rigorously prove such results and systematically generalize such results using Proposition 2.1. To begin with, we claim that the power series

$$\sum_{n=0}^{\infty} y^n H_n H_n^{(2)}$$

admits a polylogarithmic form. In this direction, we start with the g.f. for the harmonic sequence, so that

$$\sum_{n=0}^{\infty} \frac{y^n H_n \left(x^n \ln(x)\right)}{1-x} = -\frac{\ln(x)\ln(1-xy)}{(x-1)(xy-1)}, \tag{4.18}$$

with *Mathematica* being able to provide an explicit evaluation for the antiderivative of the right-hand side, with respect to x. In order to evaluate the power series

$$\sum_{n=0}^{\infty} y^n H_n^3,$$

we begin by differentiating the g.f. for harmonic numbers. According to the usual moment formula for $H_n^2 + H_n^{(2)}$, following our splitting method, we need to evaluate the antiderivative for the following, with respect to x:

$$\sum_{n=0}^{\infty} y^n H_n \left(nx^{n-1}\ln^2(1-x)\right) = -\frac{y\ln^2(1-x)(\ln(1-xy)-1)}{(xy-1)^2}. \tag{4.19}$$

Mathematica is able to indefinitely integrate the right-hand side with respect to x, and the required limit formulas are easily seen to hold. So, for example, we may determine that

$$\sum_{n=1}^{\infty} \frac{H_n^3}{2^n} = \zeta(3) + \frac{\pi^2\ln(2)}{3} + \frac{\ln^3(2)}{3},$$

and we may similarly evaluate the other two series obtained by expanding the summand of Batir's series in (4.17).

5. CONCLUSION

We very much encourage further applications of our splitting method. For the time being, we offer a brief survey of extant results on the application of identities involving expressions as in $H_{pn}^2 \pm H_{pn}^{(2)}$, since our methods, as we have demonstrated, are extremely powerful when it comes to such identities, motivating the exploration as to how such methods may be applied in conjunction with the results put forth in the references provided below.

Equivalent forms of the moment identity in (1.3) and variants of this moment formula are often included in Sofo's work in the study of harmonic sums; in this regard, we begin by recording the identity

$$\int_0^1 x^{m-1}\ln^2(1-x)\,dx = \sum_{n=0}^{m-1}(-1)^n\binom{m-1}{n}\frac{2}{(n+1)^3} \tag{5.1}$$

indicated in [25], in which identities for $\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+k)}$ and $\sum_{n=1}^{\infty}\frac{H_n^2}{n\binom{n+k}{k}}$ are given; the same identity in (5.1) is also noted in [12], in which the aforementioned Au-Yeung series is generalized. Also, we make a particular note of the identity whereby

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^2 + H_n^{(2)}}{\binom{n+k}{k}} = k\int_0^1\int_0^1 \frac{x(1-x)^{k-1}\ln^2(1-y)}{(1+xy)^2}\,dx\,dy,$$

given by Sofo in 2015, in [24], in which the evaluation of

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^2}{\binom{n+k}{k}} \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^2}{n\binom{n+k}{k}}$$

is investigated. We note that Sofo [23] also used the moment formula in (1.3) along with a similar kind of splitting approach as suggested in Section 3 to prove an explicit identity for $\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q+1}}$; we also suggest that the interested reader review the results in [19, 21, 22] involving quantities of the form $H_n^2 + H_n^{(2)}$. For additional publications on integral identities as in (1.3) and/or on series involving factors as in $H_n^2 \pm H_n^{(2)}$, we refer the reader to [14, 15, 20, 27, 37], along with [29, §6].

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