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A REGULARITY OF THE WEAK SOLUTION GRADIENT OF THE DIRICHLET PROBLEM FOR DIVERGENT FORM ELLIPTIC EQUATIONS IN MORREY SPACES

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ABSTRACT. In this paper we prove that the gradient of the weak solution of the Dirichlet problem for divergent form elliptic equations, with the known term belongs to the Morrey spaces, is the element of the weak Morrey spaces.

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1. INTRODUCTION

Recently, the regularity properties of the Dirichlet problem

$$\begin{cases} Lu = f, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where f belongs to some various Morrey spaces, have been studied by some authors (for example, see [2, 3, 5]). Here Ω is bounded domain in $\mathbb{R}^n$, $H_0^1(\Omega)$ is the Sobolev spaces, and L is divergent form elliptic operator defined in $H_0^1(\Omega)$.

By using the assumption that f belongs to the Morrey spaces $L^{1,\lambda}(\Omega)$ for $0 < \lambda < n-2$, Di Fazio [3] showed that the weak solution of (1.1) is the element of the weak Morrey spaces $wL^{p_\lambda,\lambda}(\Omega)$, where $\frac{1}{p_\lambda} = 1 - \frac{2}{n-\lambda}$. For more sharp result, in the sense of inclusion between Morrey spaces, Di Fazio [4] also proved that the weak solution of (1.1) is the element of the weak Morrey spaces $wL^{q_\lambda,\lambda}(\Omega)$, where $\frac{1}{q_\lambda} = \frac{1}{2} - \frac{1}{n-\lambda}$, by taking f from the Morrey spaces $L^{2,\lambda}(\Omega)$ for $0 < \lambda < n-2$.

For the case $n-2 < \lambda < n$ and f is in the Morrey spaces $L^{1,\lambda}(\Omega)$, Cirmi et. al [2] showed that the weak solution (1.1) is bounded essentially in Ω and its gradient belongs to the Morrey spaces $L^{p,\lambda}(\Omega)$, for some $n-2 < p \leq \lambda$.

Di Fazio has not investigated the regularity of the weak solution gradient of (1.1). In this paper, we continue his works, which are different from that one by Cirmi et. al in case of parameter λ. We prove that the the weak solution gradient of (1.1) belongs to the weak Morrey spaces $wL^{p_\lambda,\lambda}(\Omega)$, where $\frac{1}{p_\lambda} = 1 - \frac{1}{n-\lambda}$, by assuming f is in the Morrey spaces $L^{1,\lambda}(\Omega)$ for $0 < \lambda < n-2$.

2. DIRICHLET PROBLEM AND MORREY SPACES

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^n$ with $n \geq 3$. These assumptions are always assumed for Ω. For $a \in \Omega$ and $r > 0$, we define

$$B(a,r) = \{y \in \mathbb{R}^n : |y-a| < r\},$$

and

$$\Omega(a,r) = \Omega \cap B(a,r) = \{y \in \Omega : |y-a| < r\}.$$

For $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the Morrey space $L^{1,\lambda}(\Omega)$ is the set of all functions $f \in L^1(\Omega)$ which satisfies

$$\|f\|_{L^{1,\lambda}} = \sup_{a\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int_{\Omega(a,r)} |f(y)| dy \right).$$

Meanwhile, the weak Morrey space $wL^{p,\lambda}(\Omega)$ is the set of all measurable functions f defined on Ω which satisfies

$$\sup_{a\in\Omega, t>0} \left(\frac{\sup_{t>0} t\,|\{x \in \Omega(a,r) : f(x) > t\}|^{\frac{1}{p}}}{r^{\frac{\lambda}{p}}} \right) < \infty.$$

For $q = 1, 2$, let $W^{1,q}(\Omega)$ is denoted the Sobolev spaces. The closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$ is denoted by $W_0^{1,q}(\Omega)$. We consider the following second order divergent elliptic operator

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{i,j} \frac{\partial u}{\partial x_i} \right), \tag{2.1}$$

where $u \in W_0^{1,2}(\Omega)$,

$$a_{i,j} \in L^\infty(\Omega), \qquad i,j = 1, \ldots, n,$$

and there exists $\nu > 0$ such that

$$\nu|\xi|^2 \leq \sum_{i=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2,$$

for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and for almost every $x \in \Omega$. We also assume a regularity condition of the coefficients $a_{i,j}$ of the operator L, that is,

$$|a_{i,j}(x) - a_{i,j}(y)| \leq \omega(|x-y|), \qquad \forall x, y \in \Omega,$$

where $\omega : (0,\infty) \longrightarrow (0,\infty)$ is non-decreasing, satisfies

$$\omega(2t) \leq C\omega(t)$$

for a constant $C > 0$ and for all $t > 0$, and

$$\int_0^\infty \frac{\omega(t)}{t} dt < \infty.$$

Let $f \in L^{1,\lambda}(\Omega)$. We are interested in investigating the following Dirichlet problem

$$\begin{cases} Lu = f, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{2.2}$$

where L is defined by (2.1).

The function $u \in W_0^{1,2}(\Omega)$ is called the weak solution of equation (2.2) if

$$\int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \phi(x)}{\partial x_j} dx = \int_\Omega f(x)\phi(x) dx, \tag{2.3}$$

for all $\phi \in C_0^\infty(\Omega)$.

3. TOOLS

Theorem 3.1 (Grüter and Widman, [6]). *There exists a unique function $G : \Omega\times\Omega \longrightarrow \mathbb{R}\cup\{\infty\}$, $G \geq 0$, such that for each $y \in \Omega$ and any $r > 0$*

$$G(\cdot, y) \in W^{1,2}(\Omega \backslash B(y,r)) \cap W_0^{1,1}(\Omega), \tag{3.1}$$

and for all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial G(x,y)}{\partial x_i} \frac{\partial \phi(x)}{\partial x_j} dx = \phi(y). \tag{3.2}$$

Furthermore, there exists a positive constant $C_0 = C(n,\nu)$ and $C_1 = C_1(n,\nu,\omega,\Omega)$ such that

$$G(x,y) \leq C_0 \frac{1}{|x-y|^{n-2}}, \tag{3.3}$$

and

$$|\nabla G(x,y)| \leq C_1 \frac{1}{|x-y|^{n-1}} \tag{3.4}$$

for all $x, y \in \Omega$, with $x \neq y$.

The function G in Theorem 3.1 is called the Green function for L and Ω. Fix $y \in \Omega$. According to (3.1), $G(\cdot, y)$ has a weak derivative in Ω, which is denoted by $\frac{\partial G(x,y)}{\partial x_i}$. Therefore

$$\int_\Omega \frac{\partial G(x,y)}{\partial x_i}\phi(x)dx = -\int_\Omega G(x,y)\frac{\partial \phi(x)}{\partial x_i}dx, \tag{3.5}$$

for all $\phi \in C_0^\infty(\Omega)$.

Let M be the Hardy-Littlewood maximal operator, defined by

$$M(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\, dy,$$

for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Theorem 3.2. [1] *Let $0 < \lambda < n$, $a \in \Omega$, and $r > 0$. If $f \in L^{1,\lambda}(\Omega)$, then there exists a positive constant C, which is independent from a and r, such that*

$$\sup_{t>0} t\, |\{x \in \Omega(a,r) : M(f)(x) > t\}| \le C r^\lambda \|f\|_{L^{1,\lambda}}. \tag{3.6}$$

The first proof of Theorem 3.2 was given by [1]. For more simple and elegant proof of this theorem, we refer to [7].

4. An Integral Operator and Main Result

From now on, we always assume that $0 < \lambda < n-2$. Let G be the Green function for L and Ω. For $f \in L^{1,\lambda}(\Omega)$, we define

$$u(x) = \int_\Omega G(x,y)f(y)dy \tag{4.1}$$

for every $x \in \Omega$. Next we define

$$u_i(x) = \int_\Omega \left| \frac{\partial G(x,y)}{\partial x_i} f(y) \right| dy, \tag{4.2}$$

for every $i = 1, \dots, n$ and $x \in \Omega$.

We note that the function u which is defined by (4.1) is the unique weak solution of (2.2). This fact can be seen in [3].

Theorem 4.1. *There exists a constant $C > 0$ such that*

$$t^{\frac{\lambda-n}{\lambda-n+1}}\, |\{x \in \Omega(a,r) : |u_i(x)| > t\}| \le C r^\lambda \|f\|_{L^{1,\lambda}}^{1-\frac{\lambda-n-1}{\lambda-n+1}}, \tag{4.3}$$

for every $a \in \mathbb{R}^n$, $r > 0$, and $t > 0$.

Proof. Let $x \in \Omega$ and $\delta > 0$. By virtue of (3.4), we first estimate

$$\int_\Omega \frac{|f(y)|}{|x-y|^{n-1}}dy = \int_{\Omega(x,2\delta)} \frac{|f(y)|}{|x-y|^{n-1}}dy + \int_{\Omega\setminus B(x,2\delta)} \frac{|f(y)|}{|x-y|^{n-1}}dy = I_1 + I_2.$$

We bound I_1 by using the Hedberg estimation, that is,

$$I_1 \le C(n)\delta M(f)(x).$$

Now we compute the bound of I_2 as follows,

$$
\begin{aligned}
I_2 &= \int_{2\delta \le |x-y|} \frac{|f(y)\chi_\Omega(y)|}{|x-y|^{n-1}} dy = \sum_{k=1}^{\infty} \int_{2^k\delta \le |x-y| < 2^{k+1}\delta} \frac{|f(y)\chi_\Omega(y)|}{|x-y|^{n-1}} dy \\
&\le C(n) \sum_{k=1}^{\infty} \frac{1}{(2^{k+1}\delta)^{n-1}} \frac{(2^{k+1}\delta)^\lambda}{(2^{k+1}\delta)^\lambda} \int_{2^k\delta \le |x-y| < 2^{k+1}\delta} |f(y)\chi_\Omega(y)| dy \\
&\le C(n,\lambda) \|f\|_{L^{1,\lambda}} \delta^{\lambda-n+1} \sum_{k=1}^{\infty} \left(\frac{2^\lambda}{2^{n-1}} \right)^k = C(n,\lambda) \|f\|_{L^{1,\lambda}} \delta^{\lambda-n+1}.
\end{aligned}
$$

Therefore

$$
\int_\Omega \frac{|f(y)|}{|x-y|^{n-1}} dy \le C(n,\lambda) \left[\delta M(f)(x) + \delta^{\lambda-n+1} \|f\|_{L^{1,\lambda}} \right], \tag{4.4}
$$

by using the estimations of I_1 and I_2. We choose

$$
\delta = \left(\frac{M(f)(x)}{\|f\|_{L^{1,\lambda}}} \right)^{\frac{1}{\lambda-n}}
$$

to minimize the right hand side of (4.4). Then

$$
\int_\Omega \frac{|f(y)|}{|x-y|^{n-1}} dy \le C(n,\lambda) M(f)(x)^{\frac{\lambda-n+1}{\lambda-n}} \|f\|_{L^{1,\lambda}}^{\frac{\lambda-n-1}{\lambda-n}}. \tag{4.5}
$$

Note that, according to (3.4) and (4.5), then

$$
|u_i(x)| \le C \int_\Omega \frac{|f(y)|}{|x-y|^{n-1}} dy \le C M(f)(x)^{\frac{\lambda-n+1}{\lambda-n}} \|f\|_{L^{1,\lambda}}^{\frac{\lambda-n-1}{\lambda-n}}, \tag{4.6}
$$

for every $x \in \Omega$, where $C = C(n, \lambda, C_1)$. Let $a \in \Omega$ and $r > 0$. For every $t > 0$, we have

$$
|\{x \in \Omega(a,r) : |u_i(x)| > t\}| \le \left| \left\{ x \in \Omega(a,r) : M(f)(x) > C t^{\frac{\lambda-n}{\lambda-n+1}} \|f\|_{L^{1,\lambda}}^{\frac{\lambda-n-1}{\lambda-n+1}} \right\} \right|.
$$

Now we use Theorem 3.2 to obtain

$$
\begin{aligned}
&\left| \left\{ x \in \Omega(a,r) : M(f)(x) > C t^{\frac{\lambda-n}{\lambda-n+1}} \|f\|_{L^{1,\lambda}}^{\frac{\lambda-n-1}{\lambda-n+1}} \right\} \right| \\
&\le C \frac{r^\lambda \|f\|_{L^{1,\lambda}}}{C t^{\frac{\lambda-n}{\lambda-n+1}} \|f\|_{L^{1,\lambda}}^{\frac{\lambda-n-1}{\lambda-n+1}}} = C \frac{r^\lambda \|f\|_{L^{1,\lambda}}^{1-\frac{\lambda-n-1}{\lambda-n+1}}}{t^{\frac{\lambda-n}{\lambda-n+1}}},
\end{aligned}
$$

where $C > 0$ is independent from a, r, and t. This means

$$
t^{\frac{\lambda-n}{\lambda-n+1}} |\{x \in \Omega(a,r) : |u_i(x)| > t\}| \le C r^\lambda \|f\|_{L^{1,\lambda}}^{1-\frac{\lambda-n-1}{\lambda-n+1}}
$$

for every $a \in \Omega$, $r > 0$, and $t > 0$. ∎

Lemma 4.2. $u_i \in L^1(\Omega)$ *for every* $i = 1, \ldots, n$.

Proof. Let $a \in \Omega$. Since Ω is bounded, then we can choose $r > 0$ such that $\Omega \subseteq B(a, r)$. This means $\Omega = \Omega(a, r)$. By the Cavalieri Principle, we have

$$\begin{aligned}
\int_\Omega |u_i(x)|dx &= \int_{\Omega(a,r)} |u_i(x)|dx \\
&= \int_0^\infty |\{x \in \Omega(a,r) : |u_i(x)| > t\}|dt \\
&= \int_0^{|\Omega(a,r)|} |\{x \in \Omega(a,r) : |u_i(x)| > t\}|dt \\
&\quad + \int_{|\Omega(a,r)|}^\infty |\{x \in \Omega(a,r) : |u_i(x)| > t\}|dt.
\end{aligned}$$

Note that

$$\int_0^{|\Omega(a,r)|} |\{x \in \Omega(a,r) : |u_i(x)| > t\}|dt \le \int_0^{|\Omega(a,r)|} |\Omega(a,r)|dt = |\Omega(a,r)|^2 < \infty.$$

Using Theorem 4.1 and the fact $\frac{n-\lambda}{\lambda-n+1} + 1 < 0$, then

$$\begin{aligned}
\int_{|\Omega(a,r)|}^\infty |\{x \in \Omega(a,r) : |u_i(x)| > t\}|dt &\le Cr^\lambda \int_{|\Omega(a,r)|}^\infty t^{\frac{n-\lambda}{\lambda-n+1}} dt \\
&= Cr^\lambda |\Omega(a,r)|^{\frac{n-\lambda}{\lambda-n+1}+1} < \infty.
\end{aligned}$$

Therefore

$$\int_\Omega |u_i(x)|dx \le |\Omega(a,r)|^2 + Cr^\lambda |\Omega(a,r)|^{\frac{n-\lambda}{\lambda-n+1}+1} < \infty.$$

This proves the lemma. ∎

Lemma 4.3. *If u is defined by* (4.1)*, then the weak derivatives of u is given by*

$$\frac{\partial u(x)}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\int_\Omega G(x,y)f(y)dy\right) = \int_\Omega \frac{\partial G(x,y)}{\partial x_i} f(y)dy,$$

for every $i = 1, \dots, n$.

Proof. Let ϕ be an arbitrary element of $C_0^\infty(\Omega)$. We claim that $\frac{\partial G(x,y)}{\partial x_i} f(y)\phi(x) \in L^1(\Omega \times \Omega)$. This is because

$$\begin{aligned}
\int_\Omega \int_\Omega \left| \frac{\partial G(x,y)}{\partial x_i} f(y)\phi(x) \right| dydx &= \int_\Omega |u_i(x)||\phi(x)|dx \\
&\le \max_{x\in\Omega} |\phi(x)| \int_\Omega |u_i(x)|dx < \infty,
\end{aligned}$$

which is concluded from Tonneli's theorem and Lemma 4.2. Therefore we can use Fubini's theorem and (3.5) to obtain

$$\begin{aligned}
\int_\Omega \left(\int_\Omega G(x,y)f(y)dy\right) \frac{\partial \phi(x)}{\partial x_i} dx &= \int_\Omega f(y) \left(\int_\Omega G(x,y) \frac{\partial \phi(x)}{\partial x_i} dx\right) dy \\
&= -\int_\Omega f(y) \left(\int_\Omega \frac{\partial G(x,y)}{\partial x_i} \phi(x)dx\right) dy \\
&= -\int_\Omega \left(\int_\Omega \frac{\partial G(x,y)}{\partial x_i} f(y)dy\right) \phi(x)dx.
\end{aligned}$$

The proof is complete. ∎

The following is our main theorem. This shows that the gradient of the weak solution of (2.2) belongs to the weak Morrey spaces.

Theorem 4.4. *If* u *is defined by* (4.1)*, then* $|\nabla u| \in wL^{p_\lambda,\lambda}(\Omega)$ *where* $\frac{1}{p_\lambda} = 1 - \frac{1}{n-\lambda}$.

Proof. It is enough to proof that $\frac{\partial u(x)}{\partial x_i} \in wL^{p_\lambda,\lambda}(\Omega)$ for every $i = 1, \ldots, n$. According to Lemma 4.3, (4.2), and Theorem 4.1, we have

$$\begin{aligned} t^{\frac{\lambda-n}{\lambda-n+1}} \left| \left\{ x \in \Omega(a,r) : \left| \frac{\partial u(x)}{\partial x_i} \right| > t \right\} \right| &\leq t^{\frac{\lambda-n}{\lambda-n+1}} \left| \{ x \in \Omega(a,r) : |u_i(x)| > t \} \right| \\ &\leq C r^{\lambda} \|f\|_{L^{1,\lambda}}^{1-\frac{\lambda-n-1}{\lambda-n+1}}. \end{aligned}$$

Therefore this theorem is proved. ∎

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